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Students' Misconceptions of the Limit Concept in a First Calculus Course

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Abstracts

Misconceptions of the limit concept were examined in 130 pre-engineering students in Dilla Universities. Questionnaire and Interview were deigned to explore students understanding of the idea of a limit of a function and to explore the cognitive schemes for the limit concept.The study employed a quantitative-descriptive or survey design. The empirical investigation was done in two phases. A questionnaire on the idea of a limit was given to 130 students during the first phase. During the second phase 14 interviews were conducted. Then, the results indicated that students in the study see a limit as unreachable, see a limit as an approximation, see a limit as a boundary, view a limit as a dynamic process and not as a static object, and are under the impression that a function will always have a limit at a point. Regarding the relationship between a continuous function and a limit were: Students think that a function has to be defined at a point to have a limit at that point. A function that is undefined at a certain point does not have a limit; Students think that when a function has a limit, then it has to be continuous at that point. Other misconceptions were: The limit is equal to the function value at a point, i.e. a limit can be found by a method of substitution, when one divides zero by zero, the answer is zero, Most of the students know that any other number divided by zero is undefined. The study concluded that many students' knowledge and understanding rest largely on isolated facts, routine calculation, memorizing algorithm, procedures and that their conceptual understanding of limits, continuity and infinity is deficient. The outstanding observation was that students see a limit as unreachable. This could be due to the language used in many books to describe limits for example 'tends to' and 'approaches'. Another view of a limit that the students have is that a limit is a boundary point. This could be because of their experience with speed limits, although that could always be exceeded. Lecturers ought to become aware of their students' understanding and possible misconceptions. Diagnosing the nature of students' conceptual problems enables lecturers to develop specific teaching strategies to address such problems and to enhance conceptual understanding.Finally, the study suggested that concepts such as limit, involves a construction process, students build on and modify their existing concept images. Lecturers, in teaching the topic of limit, could develop concepts first before embarking on techniques in problem solving. Students need to conceptualize first before applying the formula.

Keywords: Limit Concept, Misconceptions; Limits of functions; Concept Image; Concept Definition

I. INTRODUCTION

Reference [28] shows that; there is much concern about the large numbers of students taking Calculus and the rote, manipulative learning that takes place. Research into the understanding of Calculus has shown a whole spectrum of concepts that cause problems for students. In particular, student difficulties with the abstract concepts of rate of change, limit, tangent and function are well documented. These concepts involve mathematical objects or processes specific to Calculus. The general tendency now is for less emphasis on skills and greater emphasis on the understanding of the underlying concepts.

In an article, *The History of Limits,* the authors state that the idea of" limit" is the most fundamental concept of Calculus. Every major concept of Calculus namely derivative, continuity, integral, convergence or divergence is defined in terms of limits. In fact "limit" is what distinguishes at the most basic level what we call Calculus, as the mathematics of change, variation, related rates and limits, from the other branches of mathematics (algebra, geometry and trigonometry).

Ref. [4] agrees that the mathematical concept of limit holds a central position which permeates the whole of mathematical analysis –as a foundation of the theory of approximation, of continuity and of differential calculus and integral calculus. He is of the opinion that this mathematical concept is a particularly difficult idea, typical of the kind of thought required in advanced mathematics.

Ref. [4] further mentions that one of the greatest difficulties in teaching and learning the idea of a limit does not only lie in its richness and complexities, but also in the extent to which the cognitive aspects cannot be generated purely from the mathematical definition. The distinction between the definition and the concept itself is didactically very important. Remembering the definition of a limit is one thing, but acquiring the fundamental conception is another.

The limit concept has long been considered fundamental to an understanding of Calculus and Real Analysis. Recent studies have confirmed that a complete understanding of the limit concept among students is comparatively rare [29]. There is general agreement in the literature that students have trouble with the idea of limit, whether it is in the context of functions and continuity or of sequences and series ([3]; [7]; [23]; [29]). Moreover, many of the difficulties encountered by students in dealing with other concepts (continuity, differentiability, integration) are related to their difficulties with limits. The reason for this may be due to inappropriate and weak mental links between knowledge of limits and knowledge of other calculus concepts such as continuity, derivative and integral. Well-constructed mental representations of the network of relationships among calculus concepts are essential for a thorough understanding of the conceptual underpinnings of the calculus [3].

Reference [29] states that conceptions of limit are often confounded by issues of whether:

- A function can reach its limit,
- A limit is actually a boundary,
- Limits are dynamic processes or static objects
- Limits are inherently tied to motion concepts.

These issues give rise to incomplete or alternative conceptions of limit. Williams is of the opinion that these alternative conceptions of limit relate closely to the view of limiting processes held by the mathematical community prior to Cauchy's rigorous delta-epsilon definition of limit.

Reference [29] shows that; most students completing a course in calculus have a pre-rigorous understanding of limit and very few ever achieve full understanding of the rigorous definition. Ref. [5] confirms students' difficulties when they say that most students have little success in understanding this important mathematical idea. Although for many students such pre-rigorous understanding may suffice, Ref. [21] suggested that such informal models of limit could lead to more serious misunderstandings and interfere with future learning.

There are unavoidable sources of naive misconceptions inherent to the idea of a limit ([7]; [23] ;). One is the influence of *language or words*, in which certain terms remind us of ideas that intrude into students' attempts to represent mathematical concepts. These terms are phrases such as 'tends to', 'approaches' or 'gets close to'. When these phrases are used in relation to a sequence *approaching* a limit, they invariably carry the implication that the terms of the sequence *cannot equal the limit.*

For example, in the sequence 1; ½; ¼; ...., the $\lim_{n\to\infty}\frac{1}{n} = 0$, although the last term will never be equal to zero, In addition to the words, there are the *ideas* that these words conjure up, which have their origins in earlier experiences. These ideas should not be part of a mental representation of the mathematical concept. Another source of difficulties is the sheer complexity of the ideas. These ideas cannot appear "instantaneously in complete and mature form", so that "some parts of the idea will get adequate representations before other parts will". It would therefore be necessary for each student to gradually build mental representations of the limit concept.

The following types of misconceptions are also described in literature ([29]; [26]; [14]; [5]).

- Confusion over whether a function can reach its limit,
- Confusion over whether a limit is actually a boundary,
- Confusion whether limits are dynamic processes or static objects and
- Whether limits are inherently tied to motion concepts.

A. Statement of the problem

Limit is a basic idea in calculus. "Without limits calculus simply does not exist. Every single notion of calculus is a limit in one sense or another." [19]. Instantaneous velocity is the limit of average velocities; the slope of a tangent line to a curve is the limit of the slope of secant lines; an infinite series is the limit of a finite sum; the area of a circle is the limit of areas of inscribed polygon as the number of sides increase infinitely. In the formal teaching of calculus the stated limits are obtained by methods of differentiation and integration, which the fundamental theorem of calculus refers to as reverse processes.

Without proper grasp of the limit concept, a very important branch of mathematics known as analysis would also not exist. At an elementary level, analysis deals with the notions of real number, function, limits of numerical sequences and functions, and continuity. At an advanced level these extend to analysis of several variables, complex analysis, and functional analysis [1]. The implication of this description is that the concepts of infinitely large and infinitely small are also important in analysis courses.

Despite the importance of the idea of limit in calculus, students continue to hold incomplete and alternative conceptions of it even after careful instruction. However, "this does not prevent them from working out exercises, solving problems and succeeding in their examinations." [4].

From the concerns and issues raised in the preceding paragraphs, carrying out research by studying how engineering students at undergraduate level conceptualize the idea of limit may help in suggesting ways to alleviate the problem. The knowledge derived from this research study will have contribution to the mathematics education literature.

I became aware of students' problems with the idea of a limit from my experience of teaching calculus for first year students Dilla University. In Calculus I/ applied mathematics I course in most case the first topics is about limit .This included the limit of a function at a point, a limit as x approaches infinity , one side limit and the relationship between limits and continuity or something related to these topics. Little time, about four hours, was allocated to these aspects related to limits. This was just a brief preparation for the derivative, differentiation and integration which took up a large part of the course and the lecturing time.

As the objective of the courses outline dictates that at the end of the unit, the students could calculate limits in a variety of problems, but in most cases they found it very difficult to explain the idea of a limit in their own words. So that I became curious and wanted to investigate the idea of limits further. I wanted to determine why students find this idea so complicated; whether the complexity is inherent to the nature of the idea itself; whether it is due to the way in which it is taught or whether it is due to any other reasons.

A possible explanation for the identified problem is that the idea of a limit is often taught in isolation. Students see the idea as a concept on its own without any relationship to other Calculus ideas. Students do not realize the important role that limits play in the study of Calculus at University level in Differential Calculus, Integral Calculus and continuity.

They cannot explain why the idea of a limit is fundamental to Calculus. They are unable to link limits to the broader field of Calculus. The motivation for this investigation thus was to determine the conceptions that students have of the idea of a limit and to try and reveal the specific difficulties and misconceptions concerning limits.

The following key problems or questions related to the idea of a limit **were** needed to be investigated in this research. They were:

- How do students understand the idea of a limit?
- What are their difficulties to understand the limit concept?
- How do they relate the continuity/discontinuity of a function at a Point to the existence of a limit at that point?
- What kind of misconceptions do they form about the idea of limit?

B. Objectives of the study

The aim of this investigation was to identify difficulties and misconceptions that pre-engineering students of Dilla and Hawasa Universities have of the limit concept. The focus will be on the limit of a function. I want to determine whether these difficulties and misconceptions are different to or in accordance with those found in the literature.

In order to achieve this aim, the following specific objectives were formulated:

- To Investigate how students perceive and understand the limit concept,
- Explore how they relate the continuity/ discontinuity of a function to the existence of the limit.
- To investigate their difficulties on limit and preliminary concepts, and
- To analyze the gathered data in order to determine Misconceptions that students have.

C. Significance of the study

The important role that limits play in Calculus is acknowledged by most educators and educationists, but it is a fact that the inherent nature of limits is quite complex. Most students find it difficult to understand this idea.

Therefore, the identification of the students' difficulties and misconceptions will have value in the following ways:

- It might lead to a better understanding of the students' thought processes and the quality of learning that takes place.
- Knowledge of misconceptions can be employed in planning more effective teaching strategies and methods.
- It can also be used to present richer learning experiences to the students.
- The identification of misconceptions can create worthwhile opportunities to enhance learning.
- Minimizing misconceptions and providing better understanding of the idea of a limit.

II. LITERATURE REVIEW

D. The Historical Development Of Limits

For many centuries, the ideas of a limit were confused with vague and sometimes philosophical ideas of infinity i.e. infinitely large and infinitely small numbers and other mathematical entities. The idea of a limit was also confused with subjective and undefined geometric intuitions the history of limit. Ref. [4] is of the opinion that the idea of a limit was introduced to resolve three types of difficulty:

- Geometric problems, for example, the calculations of area, 'exhaustion' and consideration of the nature of geometric lengths;

- The problem of the sum and rate of convergence of a series;
- The problems of differentiation that come from the relationship between two quantities that simultaneously tends to zero.

The term *limit* in our modern sense is a product of the late 18^{th} and early 19^{th} century Enlightenment in Europe. It was developed as a means of putting the differential and integral calculus on a rigorous foundation. Until this time, there were only rare instances in which the idea of a limit was used rigorously and correctly. Our modern definition is less than 150 years old. The idea of a limit remains one of the most important concepts (and sometimes one of the most difficult) for students of mathematics to understand the history of limit

In the beginning, when Newton and Leibniz were developing calculus, their work was based on the idea of ratios and products of arbitrarily small quantities or numbers. Newton called them fluxions and Leibniz referred to them as differentials. These ideas were built on the work of Wallis, Fermat and Descartes, who had found a need in their work for some concept of the 'Infinitely small' *The Platonic Realms.*

The root of these issues poses the following question: "How close can two numbers be without being the same number?" Another question is: "How small can a number be without being zero?" The effective answer given by Newton and others was that there are infinitesimals. They are thought of as positive quantities that are smaller than any non-zero real number. Such a concept seemed necessary, because the differential calculus relied crucially on the consideration of ratios, both of whose terms were vanishing simultaneously to zero.

Despite the success of Newton's and Leibniz's methods as well as the enthusiasm of mathematicians such as Lagrange and Euler, the idea of infinitesimal became more and more difficult to support during the 18^{th} century. The main complaint was that it is impossible to imagine in any concrete way an object that is infinitely small. Without a firm theoretical basis for infinitesimals, mathematicians could not be completely confident in their methods. Such a theoretical basis did not seem forthcoming, despite nearly two centuries of effort by the mathematical community of Europe.

A new way of thinking about ratios of vanishing quantities was being introduced by the French mathematician, D'Alembert, namely the method of limits. His formulation was nearly identical to that in use today, although it relied heavily on geometrical intuitions. For example, D'Alembert saw the tangent to a curve as a limit of secant lines, as the end points of the secants converged on the point of tangency and became identical with it 'in the limit'. This is exactly how the derivative is motivated by calculus courses world-wide today. However, considered purely as a geometrical argument, without a numerical or functional foundation, this idea of limiting secant lines is subject to age-old objections of the sort exhibited by Zeno's paradoxes "*The Platonic Realms"*

Another great French mathematician, Cauchy, provided the rigorous formulation of the limit concept that would meet all objections. Cauchy's definitions of the derivative and the integral as limits of functions transformed our understanding of the calculus. It opened the door to a rich period of growth and innovation in mathematics *'The Platonic Realms'*

E. Epistemological Obstacles In The Historical Development

It is useful to study the history of the limit concept to locate periods of slow development and the difficulties which arose during its development. This may indicate the presence of epistemological obstacles.

Ref. [4] defines an epistemological obstacle as knowledge which functions in a certain domain of activity and becomes well-established. Then it fails to work satisfactorily in another context where it malfunctions and leads to contradictions. It then becomes necessary to destroy the original insufficient knowledge and to replace it with a new concept that operates satisfactorily in the new domain.

Epistemological obstacles occur both in the historical development of scientific thought and in educational practice. They have two essential characteristics:

- They are unavoidable and essential constituents of knowledge to be acquired,
- They are found, at least in part, in the historical development of the concept [4].

There are four major epistemological obstacles in the history of the limit concept according to [4], these are:

- The failure to link geometry to numbers,
- The idea of the infinitely large and infinitely small,
- The metaphysical aspect of the idea of limit and
- Is the limit attained or not?

The idea of a limit is difficult to introduce in mathematics because it seems to have more to do with metaphysics or philosophy. This metaphysical aspect of limit is one of the principal obstacles for today's students. This obstacle makes the comprehension of the limit extremely difficult, particularly because it cannot be calculated directly using familiar methods of algebra and arithmetic. The question whether the limit is actually reached or not, is still alive in the minds of today's students. This is another obstacle to the idea of a limit.

F. The Nature Of The Idea Of A Limit

The idea of a limit is first defined informally and then formally (rigorously) before different views on its nature are discussed. As mentioned in Chapter 1, the focus of this investigation is on the limits of functions.

G. An Informal Definition Of Limit

If the values of a function $f(x)$ approach the value L as x approaches c we say that f has limit L as x approaches c and write,

$$\lim_{x \to c} f(x) = L$$

H. The formal definition

The limit of $f(x)$ as x approaches c is the number L if the following criterion holds: Given any radius $\epsilon > 0$ about L there exists a radius $\delta > 0$ about c such that for all x,

$0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$

The idea of a limit signifies a progression to a higher level of mathematical thinking which [23] calls, advanced mathematical thinking. This progression involves a difficult transition. It begins where concepts have an intuitive basis founded on experience to one where they are specified by formal definitions and their properties reconstructed through logical deductions. During this transition (and long after), earlier experiences and their properties as well as the growing body of deductive knowledge, exist simultaneously in the mind. This produces a wide variety of cognitive conflict that can act as an obstacle to learning.

[22] Quotes Cornu who sees the idea of a limit as the first mathematical concept that students meet where one does not find a definite answer by a straightforward mathematical calculation. Abstract ideas such as limit, could be conceived operationally as processes or structurally as objects. The dual character of mathematical concepts that have both a procedural and a structural aspect was investigated by many researchers. Ref. [20] used the word reification to describe the gradual development of a process becoming an object. Ref. [10] postulated a theory of how concepts start as processes which are encapsulated as mental objects that are then available for higher-level abstract thought.

There is general agreement in the literature that process or operational conceptions must precede the development of structural object notions ([5]; [23]). Various authors have the tendency to set up a dichotomy between dynamic or process conceptions of limit and static or formal conceptions. The latter is normally identified with the formal $\epsilon - \delta$ definition [5].

There is general agreement that process or operational conceptions must precede the development of structural or object notions [20]. There are, however, two different views of the relationship students have to a process conception of limit, also called a dynamic conception. Some authors seem to indicate that a dynamic conception is easy and natural for students to develop ([23]; [29]; [21]). According to this view, the main difficulty is for students to pass from a dynamic conception to a formal understanding of limits. There is even a suggestion that students' dynamic ideas hinder their movement toward developing a formal idea.

Other authors seem to feel that developing a strong dynamic conception is necessary for a formal understanding. A formal understanding must build on the student's dynamic conception. In this view, the difficulty comes in constructing the dynamic idea and this difficulty is the obstacle to understanding ([7] ; [5]).

The nature of the idea of a limit is relatively complex. Epistemological obstacles occur because of the nature of the concept itself. The models that students have of the idea of a limit are discussed in greater detail in the next section. These models can have certain misconceptions which are the focal point of this investigation.

I. Spontaneous Conceptions Of Limits

The teaching of most mathematical ideas does not begin on virgin territory. In the case of limits, the student already has a certain number of ideas, images and knowledge, from daily experience, before any teaching on this subject commences. Ref. [4] describes these conceptions of an idea that occur prior to formal teaching, as spontaneous conceptions. During a mathematics lesson these spontaneous ideas do not disappear, but they mix with newly acquired knowledge, are modified and adapted to form the student's personal ideas.

In the case of the idea of a limit, the words 'tends to' and 'limit', have a significance for the students before any lesson begins. The students continue to rely on these meanings after they have been given a formal definition. Investigations have shown many different meanings for the expression 'tends towards'. Some of them are: 'to approach' (eventually staying away from it), 'to approach ... without reaching it', 'to approach ... just reaching it and to resemble' (without any variation).

The word limit itself can have many different meanings to different individuals at different times. Most often it is considered as an 'impassable limit'. It can also be: an impassable limit that is reachable, a point that one approaches, without reaching it (or reaching it), the end or finish and an interval [4].

J. Misconceptions Of The Idea Of Limit

Many authors distinguish between concept images and concept definitions ([27]; [23]; [7]; [4]).

The term concept image is used to describe the total cognitive structure that is associated with a concept. This includes all the mental pictures and associated properties and processes. This concept image is built up over the years through all kinds of experiences and is subject to change. The concept definition, on the other hand, is seen as a form of words used to specify that concept [21].

For each individual, a concept definition generates its own concept image. There is no single idea of limit in the minds of students. The concept images of limit contain factors that conflict with the formal concept definition. This gives rise to several kinds of misconceptions. Ref. [4] mentions the following example: "It is clear that the initial teaching of limits tends to emphasize the process of approaching a limit, rather than the concept of the limit itself. The concept imagery associated with this process, contains many factors that conflict with the formal definition ('approaches but cannot reach', 'cannot pass', 'tends to', etc.) Students develop images of limits and infinity which relate to misconceptions concerning the process of 'getting close' or 'growinglarge' or 'going on forever'" (p 156).

The following types of misconceptions are described in literature ([29]; [26]; [14]; [5]).

- Confusion over whether a function can reach its limit,
- Confusion over whether a limit is actually a boundary,
- Confusion whether limits are dynamic processes or static objects and
- Whether limits are inherently tied to motion concepts.

Ref. [26] and [14] mention more common misconceptions. Students think that 'limits' simply entail substituting the value at which the limit is to be found, into the expression. They often think that limits are only encountered when trying to ascribe a value to a function at a point where the function is undefined. Students often think that function values and limits are the same. Students talk of a limit not being defined at a point, when it is the function that is not defined at the point. Students think only about the manipulative aspects and do not focus on the idea of the limit. Ref. [3] argued that formula might make calculations easier, but did not promote understanding. He advises that the development of students' conceptual understanding should go hand in hand with the development of their manipulative skills.

The conclusion can be drawn that the idea of a limit is complex. Students have difficulty in understanding this important idea. Ref. [5] mention that they have not found any reports of success in helping students overcome their difficulties. Some authors report that even using technology has not been successful in doing so.

K. Prerequisites For The Understanding Of Limits

In order to understand limits and other calculus concepts, students ought to have a global interpretation of the behaviour of functions and their graphs [15]. This includes the idea of the continuity of functions. Students also need to be familiar with graphs i.e. the interpretation of given graphs as well as the sketching of a graph when its algebraic formula is given. Another important related aspect is the idea of infinity.

- ✓ **Global understanding of functions**

The function concept is one of the most fundamental in all of mathematics and one for which students seldom develop a satisfactory understanding. Reasons students have difficulty with the concept seem to centre round its complexity and its generality. The concept has many facets and associated sub-concepts which can be stated at different levels of abstraction [9].

Ref. [11] mentions that various researchers in mathematics education have been interested in students' understanding of functions. These researchers have studied the sub-concepts of function (domain, range, representation and correspondence), representations used for function (graphs, rules, tables and arrow diagrams) and ways in which students use and conceptualize functions.

This research has shown that the working definition of function held by most students is that of function as a rule of correspondence, represented by a formula. Students are strongly committed to this view and believe that a function must have the same rule of correspondence over its entire domain. Functions defined differently on different parts of the domain (piecewise functions) present great difficulty. A number of these researchers have noted that students interpret functions in a point-by-point, or local, way rather than globally. Tasks that encourage students to interpret graphs qualitatively may help build a global interpretation.

In this discussion, the focus is primarily on the 'limit of a function' and the 'continuity of a function at a point'.

✓ **The continuity of functions**

The issue of continuity has become one of practical as well as theoretical importance. Continuous functions are the functions we normally use in the equations that describe numerical relations in the world around us. They are the functions we use to describe how a body moves through space or how the speed of a chemical reaction changes with time. It is important to know when continuity is called for, what it entails and how to test for it [12]. A function $y = f(x)$ that can be graphed throughout its domain with one continuous motion of the pen (that is without lifting the pen) is an example of a continuous function. A function is continuous if it is continuous at each point of its domain. The continuity test can be used to test for continuity at a point.

L. *The Continuity Test*

A function $y = f(x)$ is continuous at $x = c$ if and only if it meets all three of the following conditions:
1. $f(c)$ *exists* (f *is defined at* c) 2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$) 3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

To test for continuity at endpoints, the appropriate one-sided limit is used [12]. For any function $y = f(x)$ it is important to distinguish between continuity at $x = c$ and having a limit as $x \to c$. The limit is where the function is headed as $x \to c$. Continuity is the property of arriving at the point where $f(x)$ has been heading when x actually gets to c [12].

III. THE TEACHING AND LEARNING OF THE IDEA OF A LIMIT

The teaching and learning of this important, but complex idea ought to be planned and implemented with great care. The problem-centered approach based on a constructivist perspective seems to be an effective way to teach and learn the idea of a limit.

M. A Constructivist Perspective On Teaching And Learning

The central idea of the constructivist theory is that mathematical knowledge cannot be transferred ready-made from one person to another. It ought to be constructed by every individual learner. This theory maintains that students are active meaning-makers who continually construct their own meanings for ideas communicated to them. This is done in terms of their own, existing knowledge base. This suggests that a student finds a new mathematical idea meaningful to the extent that connections are established between the new idea and his/her existing knowledge [2].

Mathematical learning is a constructive process that requires individual cognitive activity. This individual cognitive activity cannot be controlled and manipulated by a lecturer as if it is a physical phenomenon. This view of learning is in contrast to the view that a student is a passive receiver of knowledge and that misconceptions can be deleted from memory by taking in clear explanations from the person giving the explanations. Due to this constructive process, the mathematical knowledge (contained in the explanations) can be modified during the process of transmission [2].

Learning a mathematical idea, such as limit, involves a construction process. This implies that students build on and modify their existing concept images. The concept image consists of all the mental pictures together with the set of properties that an individual associates with a given concept. The concept images which an individual constructs through his own activities may differ in various respects to the formal mathematical concepts. This leads to the formation of alternative conceptions or misconceptions [2].

N. A Problem-Centered Learning Approach To Teaching

A problem-centered approach to mathematics teaching is based on the acceptance that students construct their own knowledge. This approach attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions. The construction of mathematical knowledge is firstly an individual and secondly a social activity [18].

In the problem-centered approach, problem solving plays a dual role according to [13]. The development of problem solving skills plays a central role on the one hand and problem solving is the dominant learning type on the other hand. Students learn mathematics through solving problems. They do not learn new mathematics when they solve problems by purely utilizing existing knowledge.

To learn through the solution of problems, students need to be exposed to problems which are new to them. The knowledge that develops through this process should not have been available beforehand. This, however, does not neglect the fact that there is still some mathematical knowledge that students cannot discover by themselves, namely the physical knowledge and the social knowledge.

The role of the teacher is no longer that of transmitter of knowledge to the students, but rather a facilitator of their learning. This facilitating role that the teacher plays puts a lot of pressure on his/her organizational skills as well as subject didactical knowledge.

Social interaction plays an important role in the problem-centered approach. Ref. [18] mentions the following three purposes of social interaction in the classroom:

- Social interaction creates the opportunity for students to talk about their thinking and encourages reflection;
- Students learn not only through their own constructions but also from one another and
- Through classroom social interaction the students also interact with the teacher.

The teacher therefore plays an important role in selecting appropriate problems that have to be solved. The teacher also has to create a classroom culture that is conducive to learning. Ref. [18] Stress the fact that the amount and quality of learning that takes place, depend on the classroom culture and on the expectations of both the teacher and the students.

Ref. [14] warns that most students need to be eased into the difficult concepts of calculus, especially the idea of a limit. He advocates methods that are generally intuitive rather than rigorous, especially in the beginning. He states that nothing is guaranteed to build up an antipathy to calculus more than a 'long, formal, dry treatment' of the idea of a limit as step one. If in addition, the $\in -\delta$ definition is used, the limit notation will more than likely trigger a mental block whenever it is encountered in future. When teaching first courses in calculus, it must be kept in mind that it took the most brilliant minds of the mathematical community centuries to arrive at this rigour [14].

IV. THE ROLE OF TECHNOLOGY IN TEACHING AND LEARNING LIMITS

The research on the role of technology in teaching and learning limits is very limited. Ref. [23] reports that teaching the idea of a limit using the computer has, on the whole, fared badly.

Ref. [4] is of the opinion that the computer may very well play a significant role in providing an environment where the student may gain appropriate experiences to construct the limit concept. However, such approaches are very likely to contain their own peculiar epistemological obstacles. It is necessary to reflect on student experiences in the new environment to see precisely what is learnt and in what form the knowledge is stored in the memory.

Ref. [4] further states that the interaction with the computer may involve programming. The individual can construct computer processes that may permit the acquisition and mastery of the corresponding mathematical ideas. It may involve pre-prepared software to enable the student to experience carefully selected environments that model the idea of a limit. It is also possible to imagine a kind of computer 'toolbox' for the learning of the mathematical idea: a computer environment that will permit students to manipulate objects and to construct knowledge.

Various other approaches are possible. In a context such as that of studying limits, it is vital that the computer software is designed within a teaching strategy based on the careful analysis of the idea that must be acquired.

Ref. [15] mentions that an increasing number of students is using graphic calculators as they become more available and inexpensive. Claims are widespread that:

- the use of graphic calculators will allow for more conceptual approaches;
- students will understand the relationships between symbolic algebra and graphical representations more readily;
- Students will now be able to solve problems that were previously inaccessible to them because of the formal mathematics required for their solution.

It is evident that more research is needed to determine the specific role that technology plays in learning about limits.

V.METHODOLOGY

Quantitative-descriptive designs, also known as a survey, were used on the areas of concepts in limit of a function. The aim is to obtain insight in to the students' Understanding of limit in order to identify their misconceptions and difficulties.

O. Research Design

Survey research was preferred in this study, since in survey research the investigator selects a sample of respondents and administers a questionnaire and / or conduct interview to collect information on the variables of interest. The data that were gathered used to describe characteristics of a certain population .survey are generally used to find out more about people's beliefs, attitudes, values, understandings as well as other types of information. They are frequently used in education, politics, business, government and psychology because accurate information can be obtained for large numbers of people with a small sample. Most surveys describe the incidence, frequency and distribution of the characteristics of an identified population ([17]; [8]).

In this survey, students' understandings of the idea of a limit were determined in order to identify their misconception.

The survey also allows that the information can be collected by means of questionnaire at a relatively low cost. The questionnaire can distributed to the whole group/sample and collected immediately after completion. All respondents have to answer the same questions and the anonymity of each participant can be ensured. Heid (http://www.maa..org/saum/maanotes49/109.htm) stresses the fact that students' answers on tests do not always show their true level of understanding. Sometimes they understand more than their answers indicate. Other times, despite using the correct words, they do not understand what they write. So that the use of in-depth interview is of great value in this regard. This content-based type of interviews is of great value in this regard. This content _ based type of interview is not just an oral test or quiz but rather a way to dig more deeply into the complexities of students' mathematical understanding.

In particular cross-sectional data collections were used. Since the data were be collected at one point in the time. Cross_ sectional is essential in survey method to gather data at a particular point in time with the intention of describing the nature of existing conditions can be compared or determining the relationships that exist between specific events.

Thus, in this study a quantitative descriptive design (survey) were used and cross_ sectional data collection will also be employed to address objectives in the study.

P. Sampling

Multi stages Stratified random sampling technique with SRS were used to select **130 students** from both Universities for a test.

Out of these students (sample) sleeted for the questionnaire; 14 students were selected for interview purposefully. Purposeful sampling here was not mean to achieve population validity. The intent was to achieve a variety of in-depth understanding of selected individuals, not to select that will represent accurately. Ref. [6] Stated, the logic and power behind purposeful selection of informants is that the sample should be information rich.

Students with unusual, interesting or incomplete response will be chosen. High achievers as well as low- achievers will also be included. The reason for this is to explore possible differences between rich and poor students' understanding of a limit.

Q. Data Collection Instruments

The following were instruments used in the study:

1) The Questionnaire

The purpose of the questionnaire was to determine how the students understand the idea of a limit of a function. The focuses were both on the informal definition and epsilon-delta definition. I were developed my own questionnaire with the following objectives in mind:

- To determine the student's understanding of a limit;
- To determine the student's understanding of the continuity or discontinuity of a function at a point;
- To find out how the students understand the existence or non-existence of a limit at a point of continuity/discontinuity;
- To determine how the students deal with the graphing of a function with a discontinuity at a point.

The questionnaire was consisting of questions and statements. Open questions as well as closed questions were included. Students were also asked to sketch the graph of a function with a discontinuity at a point.

2) In-Depth Interviews

The aim of these content-based interviews was to find out more about each student's personal understanding of limits in a way that the questionnaire did not necessarily reveal. In other words, to determine the nature of each student's concept image of a limit. It was a way of accessing what he/she thought in the absence of additional purposeful teaching. It was also to assess the breadth and depth of his/her mathematical understandings in general.

The objectives for the interviews were very similar to those mentioned in the questionnaire. They were:

- To assess the student's graphical understanding of a limit,
- To assess the student's symbolic understanding of a limit,
- To determine the student's understanding of the continuity or discontinuity of a function at a point,
- To determine how the latter influences the students' understanding of the existence or non-existence of a limit at that point.

R. Data Collection Procedures

The investigations were done in two phases. In the first phase a questionnaire were used to gather data from sample students on their understanding of limit. The response were recorded and analyzed. It was administered to the sample group as a whole by arranging program out of their lectures. Fixed times were given to complete the items. The students were not allowed to communicate with each other; each student owns work and thoughts were required.

During the second phase interviews were conducted with 14 purposefully selected students. The interview will be completed in one day that was arranged after two weeks of the questionnaire. On the interview I used good note taking and good listening. The response were later transcribed and analyzed according to the nature of a limit.

S. Method Of Data Analysis

The data were analyzed in order to identify the misconceptions that these students have of the idea of limit. The data will be analyzed by making use of open coding. Open coding is the part of analysis that pertains specifically to the naming and categorizing of phenomena through the close examination of data. During open coding, the data were broken down into discrete parts, closely examined and compared for similarities and differences.

Conceptualizing the data were the first step in analysis. It involves taking apart an observation or response and giving each discrete incident or idea a name. This name was stands for, or represents a phenomenon. This was done by comparing incident with incident so that similar phenomenon can be given the same name.

Once particular phenomena have been identified in the data, the concepts are grouped around them. The process of grouping concepts that seem to pertain to the same phenomenon is called categorizing. The categories are then named. The categories are further developed in terms of their properties [8]. In addition to these some statistical tools like percentage and averages were used to determine the number of students who fall in a certain category and this intern was help me to make generalization.

VI. PRESENTATION AND DATA ANALYSIS

The questionnaire was administered to 130 students after their lecture hours in both universities. The students were informed about the questionnaire a week before, but not on the nature of its content. The reason for this was that the researcher did not want them to revise what they had done earlier. They had to rely on what they could remember.

Question 1

The number of true and false responses to the five questions in **Question 1** is shown below.

TABLE I:

Item number True False No response

Question Number	Statement Type	True	False	No Response
1.1	Boundary	71	59*	0
1.2	Dynamic-theoretical	107	23*	0
1.3	Unreachable	77*	45	8
1.4	Approximation	82	42*	6
1.5	Dynamic-practical	74*	47	9

* Correct answer of the item

Some common mistakes observed were:

- Twenty-one of the students (54.6%) indicated that a limit is a number past which a function cannot go. This indicates that they view a limit as a boundary. This is indicated in item1.1.
- Eighty two percent of the students (107 out of 130) thought that a limit describes how the value of a function moves as the value of x moves towards a certain point (item 1.2). These students probably associate a limit with the function value for a specific value of x.
- In item 1.3, 34% disagreed with the statement that a limit is a number that the function value gets closer to but never reaches. The other 59% of the students viewed the limit as unreachable.
- Eighty-two students (63%) were incorrect by saying that a limit is an approximation that can be made as accurate as you wish in item1.4.
- Forty seven (30%) of the students did not agree that the limit of a function can fail to exist at a point as asked in item 1.5. These students are of the opinion that there must be a limit at a certain point on a function.

Question 2

In this question students were asked to give their own explanation of the limit of a function. Students experienced difficulty in answering this question as indicated by the low response rate of 52, 4 %. Some of the incorrect responses are as follows:

- Student 59: " *When s approaches the number L, x approaches zero* "
- Student 82: " *The function is continuous at x = L* "
- Student 26: "*When x approaches s, then L gets very big". This student probably associates a limit with the idea of infinity*
- Student 6: " *It means that either it converges or diverges* "
- Student 17: " *The limit shows how the function is going to be* "
- Student 25: " *A function gives a certain value for every value for x* "
- Student 2: " *Approaches infinity* "

Eleven students (numbers 9, 33 and 48) tried to answer the question by simply writing the wording of the question down. One of the more sensible answers was "When x approaches *s*, it will result the limit, *L*."

Question 3

In this question the students were asked to circle the number of the statement(s) that must be true. The responses of the 42 students to this question are represented in tabular form below:

TABLE II:

Item No	No of Respondent	%
3.1	45	34
3.2	20	15
3.3	39	30
3.4*	24	18

*indicate the correct answer

- Only 25 students (19%) chose 3.4, which is the correct answer. They realized that none of the other options must be true.
- Forty-five (34%) of the students encircled item number 3.1. This revealed that they view a limit as a substitution process.
- Twenty students chose 3.2 as an answer (15%). They were of the opinion that a function must be continuous at a point to have a limit at that point.
- Thirty -nine of the students (30%) encircled 3.3 thus indicating that a function must be defined at a point to have a limit at that point.
- One student (no. 37) encircled two items and student 78 encircled three items. They were under the impression that a function must be defined and continuous at a point to have a limit and that the limit is equal to the function value at the point.

Question 4

TABLE III:

Summary of the 42 answers given in each item of this question:

Item Number	Correct	Incorrect	No Response
4.1	18	76	36
4.2	40	43	47
4.3	35	47	48
4.4	24	72	34
4.5	22*	42	66
4.6	26	49	55
4.7	32	52	46

* As seen on the table the correct answer to item 4.5 is $-\infty$. Only 22 students had this answer but the remaining students either gave wrong answer or non respondent.

Common wrong answers are:

- Zero, -2, 3 and -3 in item 4.1
- In item 4.2, thirty-eight (29%) of the students had 3 or - 3 as answers
- Fifty two students thought that the limit of $f(x)$ at $x = 3$ equals the function value of f at $x = 3$ in item 4.7.

The students experienced difficulty with the discontinuity at $x = 3$ as is clear from the small number of correct responses. They did not realize that the function value of $f(3) = 3$ and not two and that the limit of $f(x)$ as *x* approaches three is actually two.

Question 5

In this question students had to sketch the graph of a function and thereafter answer questions on the graph. The results are indicated in the table:

TABLE IV:

Item Number	Correct	Incorrect	No Response
graph	18	60	52
5.1	32	76	22
5.2	81	35	14
5.3	54	57	19
5.4	14	44	72
5.5	16	46	68

Typical incorrect answers that occurred:

- 39 or 30% of the respondents said that $f(x)$ becomes zero at the point $x = 3$ instead of being undefined at the point.
- 35 (27%) of the students thought that the graph is continuous at the point $x = 3$.
- Fifty seven students were of the opinion that $f(x)$ does not have a limit at $x = 3$.
- Forty four students wrote 0, 2 or 3 in item 5.4.
- In item 5.5, forty two students answered 0, 2 or 3.

The students experienced difficulty in sketching the graph of $f(x)$. thirty-two of the students (40%) did not even try to sketch the graph and consequently could not answer the questions on the graph with ease. Some of the students had zero as the y-intercept instead of one. Others had the slop of the graph incorrect. Some of them just ignored the point $x = 3$ because they did not know how to deal with the discontinuity at that point. In order to try something, some of them made $y = 0$ which resulted in an x - intercept at $x = 3$.

T. Representation and Analysis of the Data from the Interviews

In this section the students were not asked to calculate limits algebraically, because they were able to find most limits in that way. The questions were aimed at finding out how they understand the idea of a limit, how they think about limits and function values and how they understand the effect of a discontinuity at a point on the existence of the limit at the same point. The interview was conducted with 14 purposefully selected students.
(See Appendix B for the interview schedule)

Excerpts from interviews on question 1

In question 1, students were asked to comment on the symbolic expression $\lim_{x\to\infty}\frac{1}{x} = 0$. A rough sketch of the graph of the function was available for use by the students. Almost all 14 students were could not remember that the function is called a hyperbola. Some of the responses to the question: "What happens to the function when x approaches infinity?" were:

S_9 : *f(x) gets closer to zero. It can't be zero, it just reaches zero.*
S_1 : *When x gets very big, it becomes smaller .*
S_7 : *If x approaches infinity, f(x) also approaches infinity .*
S_{11}: *The graph approaches the x-axes, but never touches it.*

The students realized that if x becomes very big, the function value will reach zero, although it will never be zero itself. They all used the value of $f(x)$ and did not think that the limit will actually be zero. The explanation was that if you divide one by numbers that get bigger and bigger, the answer will eventually reach zero. They all mentioned that the limit approaches zero, but never reaches zero. This shows that they viewed a limit as a **dynamic process** and **not as a static object** similarly to what [3] and [29] concluded from their research on limits.

On the question whether we can write $f(\infty) = 0$, nine of the fourteen students answered yes. Two of the responses were as follows:

S_{14} : *No, well umm, it depends on the formula. Infinity is a symbol representing numbers that are very big, numbers that you can't really write.*

S_6: *Yes. If you divide one by numbers getting bigger and bigger, x becomes smaller and smaller and will eventually reach zero.*

This indicated that they see the symbol ∞ , as a number representing big numbers and not really as a symbol that represents infinity. Student S_{14} named ∞ a symbol, but he gave the impression that you could find $f(\infty)$ and that its value depends on the formula of a specific function. In this specific case, the function reaches zero when x increases.

Excerpts from the interviews on question 2

In this question, students were asked what the value of 0.99999… is. None of, them could remember that it is called a recurring decimal number. They all said that the value is one, but explained it in interesting ways.

S_1 : *If you round off this number, it is gets close to one*

S_2 : *It is equal to one, as when you round off the numbers, but it is not right. When you use calculator, it won't give you one.*
S_3 : *It means that there are still more nines. We can say it is almost equal to one, but not exactly one.*
S_4 : *If I have to round it off, I can say it is equal to one .*
S_5: *It is gets close to one.*
S_6 : *We can say it is almost one .*
S_7: *It is equal to one, as when you round off number.*
S_8: *If you round off this thing, it is close to one.*
S_9: *It means that there are still more nines. We can say it is almost equal to one, but not really one.*
S_{10}: *If rationalizing it result one.*
S_{11}: *It is close and closer to one. So I can say it is one.*
S_{12}: *We can try to make it one.*
S_{13} : *When you use your calculator, it won't give you one* . But rounding off gives one.
S_{14} : *it is approaching to one.*

These responses, with the exception of S_4 S_{10} and S_{13}, indicated that they were thinking of the process of rounding off numbers and that the answer is close to one. This way of thinking is very similar to that in question 1 where a limit was seen as a number that a function is approaching or getting closer to, but never reaches.

Students' explanations in question 3

Question 3 dealt with the idea of a limit in a different context namely that of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. They were asked what the meaning of the limit in this symbolic expression is as well as what this expression is usually used for. The students experienced great difficulty with this question as indicated by their responses:

S_1 : *it is the derivative .*
S_2 : *I don't know exactly but it seems to find f(x).*
S_3 : *as h approach 0 ;…. Ok, f(x) becomes zero.*
S_4 : *In Calculus ,…? I do not know, maybe to find f(x).*
S_5: *We use the formula for the gradient. If h becomes zero, it is going to be x*
S_6 : *it is slope of the tangent line .*
S_7: *I do not know, maybe to find f(x).*
S_8 : *If you divide by zero, it is undefined* . so the limit is undefined .
S_9: *They give you f(x) and then you substitute f(x) into all the x's. I do not really know. If h approaches zero, it will be zero.*
S_{10}: *We use the formula for the slope. If h becomes zero, it is going to be x.*
S_{11}: *In Calculus,…in a sequence? I do not know, maybe to find f(x).*
S_{12}: *We use it to calculate the limit. In this formula we do not really care about h. It just helps us to calculate everything. If we divide by a zero, it will not be possible we cannot use a value at this point. We cancel out the h.*
S_{13} : *I would say if h approaches zero, then f(x) will approach infinity. I'm not sure where we use it.*
S_{14} : *If h becomes zero, it is going to be x.*

From these explanations it is clear that the students did not have a clear concept image of the idea of a limit in this context. They can calculate a derivative by using this formula if they are asked to do so, but they cannot describe the concept correctly in their own words.

Excerpts from the responses to question 4

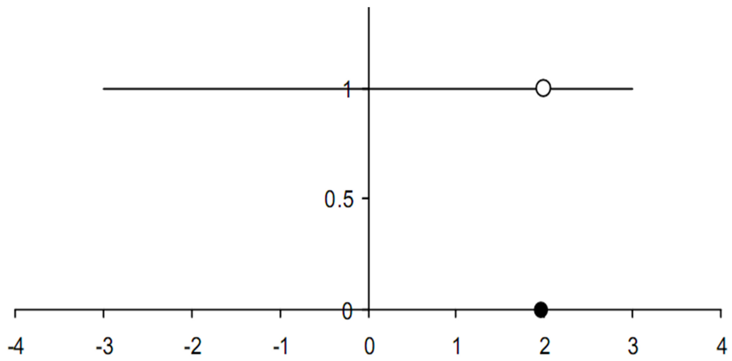
Question 4 dealt with another type of symbolic expression namely $\lim_{x\to 1} f(x) = 3$. The students were asked to describe the meaning of the expression in their own words. The focus was on whether the function has to be defined at a point to have a limit at that point and what the relationship between the function value and the limit at that point is. Some of the responses were as follows:

S_3: *It means that the limit exists at three.*
S_4*: As x approaches one, the function approaches three.*
S_9*: It says where x is one, our limit is three so as x approaches a one, and our f (x) will approach three. It just approaches the one or the three, it. Never reaches the one or the three.*
S_{12}*: Ok, it means that if x approaches one, then the graph closer and closer to three.*

Nine of the fourteen students were confident that the function must be defined at $x = 1$ to have a limit of three. If the function is undefined at that point, there will be no limit. Students S_{10} and S_{12} Were who thought that the function needn't have to be defined at the point. If it is undefined, it can still have a limit. Further, the other five students were all very confident that $f(1) = 3$. Three Students S_{14} , S_7 and S_2 were motivated their answers almost similar to as indicated below:

The limit is found by substitution, it depends on the formula you have.

Responses to question 5
In this question, the students were given the graph of a function; $f(x)$.The formula of the function was not given. They had to show where the limit of $f(x)$ as $x \to 2$ is as well as what they think $\lim_{x\to 2} f(x)$ is. Every student pointed to the point three on the y-axes. Each student motivated his/her answer by saying that if $x = 2$, the value of $f(x)$ is equal to 3.
They did not use the word limit, but rather the value of $f(x)$ at $x = 3$. All of them said that $\lim_{x\to 2} f(x) = 3$ the same reason as above.
These responses indicated that the limit and the function value were seen as being the same, i.e. $\lim_{x\to 2} f(x) = f(2)$. Bezuidenhout (2001:495) is of the opinion that this misconception may be mainly due to the use of a method of substitution to find limits algebraically. This method was also used when the idea of a limit was taught to the students at the beginning of the year.
Excerpts from interviews on question 6
In the sixth and final question of the interview, students were asked to sketch the graph of the function, $f(x) = \frac{x-2}{x-2}$ which has a discontinuity at the point $x = 2$. They had to say whether the function has a limit at $x = 2$ as well as what the value of $f(2)$ is.All the students attempted the graph by choosing values for x and calculating the corresponding $f(x)$ values. At the point $x = 2$, they had some difficulty. Students S_5, S_2 and S_{11} had just as the following graph it is good reasoning relatively:



On the question, what happens when $x = 2$, ten of them answered that $f(x)$ becomes zero. At first, they did not think of a discontinuity at that point. On the question, what $\frac{0}{0}$ is, five of them answered that it was zero. The other was correct in saying that it was undefined, although $f(2)$ remained equal to zero for some of them.
Student S_6 and S_{13}, were said that the function was undefined at $x = 2$ (although they were said that $f(x) = 0$). They explained on this by saying that the function is not continuous at the particular point and that there is no limit at $x = 2$. On the question whether the limit could be one, their response were no, because the function does not approach one. They were under the impression that there was no limit at the point because the function was undefined at the point.
Due to a hint that the function could be discontinuous at $x = 2$, the students realized that they had to make an open circle at $x = 2$ because the function was undefined at the point. Five of the students could then say that the limit at $x = 2$ is equal to one, although they still believed that the value of $f(x)$ at $x = 2$ is equal to zero.

U. Students' Conceptions Of The Idea Of A Limit
In the time of interviews, students were given the opportunity to express their ideas and thoughts not only on limits, but also on infinity and continuity. Their response is summarized.

1) Summary of the identified conceptions on limits

The following conceptions on the nature of a limit were identified:
- Students see a limit as unreachable.
- Students see a limit as an approximation.
- Students see a limit as a boundary.
- Students view a limit as a dynamic process and not as a static object.
- Students are under the impression that a function will always have a limit at a point**.**

2) Conceptions regarding the relationship between a continuous function and a limit were:

- Students think that a function has to be defined at a point to have a limit at that point. A function that is undefined at a certain point does not have a limit.

- Students think that when a function has a limit, then it has to be continuous at that point.

Other misconceptions were:

- The limit is equal to the function value at a point, i.e. a limit can be found by a method of substitution.
- When one divides zero by zero, the answer is zero. Most of the students know that any other number divided by zero is undefined.
- When one rounds a number off, the number gets closer to another number, but does not equal it, ex $0.999\ldots = 1$.

VII. DISCUSSION

The identification of these conceptions points to the fact that most of these students have very limited conceptions about limits and continuity. This could be ascribed to different facts the time given to complete the lesson, the way that the concept is introduced, limited concept about pre calculus concepts and the like to indicate some.

Several researchers have experienced the difficulty that students have with limits and have investigated the conceptions that they have. The conceptions mentioned above, are in accordance with those that these researchers have found. Ref. [29] and [23] concluded that conceptions of limit are often confounded by issues of whether a function can reach its limit, whether a limit is actually a boundary, whether limits are dynamic processes or static objects and whether limits are inherently tied to motion concepts.

Ref. [26] and [3] investigated first year students' understanding of limits and found that students have in mind that the limit is about mechanical substitution. They also found that students only think about the manipulative aspects and do not focus on the concept of a limit. According to their findings, students' talk of a limit not being defined at a point when it is the function that is not defined at that point [14] confirms the fact that students think that limits simply entail substituting the value at which the limit is to be found, into the expression. He also found that students think that limits and function values are the same.

Ref. [15] go a step further by saying that students commonly believe that a limit cannot be reached and are uneasy about the mismatch between their intuitions and the answers they produce through mathematical manipulation. They seem to experience conflict between formal, precise definitions and the informal, natural language interpretations used commonly in the discourse. Ref. [23] is of the opinion that the formal definition is also fraught with cognitive problems. In this course, the students only dealt with the informal definition, the formal definition was not discussed. Ref. [15] explains that the primitive brain notices movement. The mental notion of the limit of a function is more likely to focus on the moving point than on the limit point. The limit concept is conceived first as a process and then as a concept or object. Ref. [20] calls this the dual nature of mathematical conceptions. She is of the opinion that abstract notions such as function or limit, can be conceived in two fundamentally different ways: structurally as objects, and operationally as processes. These two approaches, although apparently incompatible, are in fact complementary. The processes of learning and of problem-solving consist in an intricate interplay between **operational and structural conceptions** of the same notions. The operational conception is, for most people the first step in the acquisition of new mathematical notions. The transition from computational operations to abstract objects is a long and inherently difficult process. This could explain why most students see a limit as a process and not as an object- not enough time is allowed to make this transition.

VIII. CONCLUSIONS

The data analysis and data interpretation shed some light on the ways in which students think about limits. The identified misconceptions are in most cases similar to the ones identified by other researchers. The findings of this study show that many students' knowledge and understanding rest largely on isolated facts and procedures and that their conceptual understanding of limits, continuity and infinity is deficient. Lecturers ought to become aware of their students' understanding and possible misconceptions. Diagnosing the nature of students' conceptual problems enables lecturers to develop specific teaching strategies to address such problems and to enhance conceptual understanding.

The outstanding observation was that students see a limit as unreachable. This could be due to the language used in many books to describe limits for example 'tends to' and 'approaches'. These words are verbs or action words and as [15] describes, the action in this mathematical setting is 'getting to a limit' sets up a dynamic interpretation of a limit. These words represent a movement towards a point without ever getting there. This explains the dynamic view of the majority of students that a limit is a process and not an object. Another view of a limit that the students have is that a limit is a boundary point. This could be because of their experience with speed limits, although that could always be exceeded.

Many students think that a function must be defined at a point to have a limit at that point. If there is a point of discontinuity there could not be a limit at that point. They also think that a limit is equal to the function value at that point. This is true for continuous functions but not for piecewise functions or discontinuous functions.

IX. IMPLICATIONS

From the results of the study most of these students have very limited conceptions about limits and continuity. It was found that many students' knowledge and understanding rest largely isolated facts, routine calculation, memorizing algorithm and procedures and their conceptual understanding of limit and related concepts were deficient. Therefore based on the findings the following recommendations are given:

- ✓ Examining how we teach and how students learn is thus essential.
- ✓ Teachers/Lecturers ought to become aware of their students' understanding and possible misconception.
- ✓ Diagnosing the nature of students' conceptual problems enables teachers/ lecturers to develop specific teaching strategies to address such problems and to enhance conceptual understanding of students.
- ✓ The method of teaching should be arranged in the way that helps students to have better conception about limit concept and to construct their own knowledge.
- ✓ Teachers, in teaching the topic of limit, could develop concepts first before embarking on techniques in problem solving (Students need to conceptualize first before applying the formulae).
- ✓ Providing visualization like diagram and graphs wherever possible to help students to understand limit better way.
- ✓ Their lack of prerequisite knowledge inevitably affected students' conceptual knowledge of limit. Hence, instructors could perhaps place more emphasis on prerequisite concepts of limit in particular and Calculus in general.

Recommendation for further research

The following are points that require further investigation:

- investigate those cognitive processes which are assumed to be involved in the student's acquisition of the idea of a limit;
- investigate the role of a problem-centered approach on the learning of limits and students subsequent understanding;
- To investigate the effect that the use of technology, for example computer programs and graphing calculators might have on students' understanding of limits;
- To develop a teaching strategy that will lead to the most accurate of limits.

The investigation of these four aspects related to the teaching and learning of this important idea in calculus, ought to lead to more effective teaching strategies.

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