

Exploring Mathematics Teachers' Pedagogical Content Knowledge in the Context of Knowledge of Students

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Abstract

Functions are one of the basic topics taught in mathematics curriculum at Secondary school level requiring knowledge from the students' past, and uniting mathematical topics. Mathematics teachers have both their own learning experience of functions, as well as their own teaching experience, leading to the question of what level of student knowledge teachers have related to teaching functions. In this way thirteen mathematics teachers participated in this current research. The research investigated student knowledge determined to be part of pedagogical content knowledge (PCK) by Shulman (1986) and accepted by a variety of researchers as a component of PCK. The teachers participating in the study may be counted as successful in knowing the preliminary knowledge of students and the difficulties and misconceptions experienced by students; however they were identified to be deficient in terms of knowing how to correct the errors and misconceptions experienced. It also appeared that school type and experience were not important factors. These results have significant implications for developing teaching practices and professional development of teachers.

Keywords: functions, mathematics teachers, pedagogical content knowledge, student knowledge

1. Introduction

Shulman is the one of the researchers who produces foundational research on teachers and teacher's knowledge. Shulman (1986) analyzed teacher's knowledge by creating a theoretical framework based on the categories of content knowledge (organization and amount of knowledge in the teacher's mind), pedagogical content knowledge (a type of content knowledge related more to how the topic may be taught) and curriculum knowledge (including material used in teaching and the order of topics). Later Shulman (1987) stated that the knowledge basis of teaching comprised 7 categories (content knowledge, general pedagogical knowledge (pedagogy), curriculum knowledge (syllabus), knowledge of students, knowledge of the educational system, knowledge of educational targets, values, history and philosophy, pedagogical content knowledge (educational content knowledge)). The characteristic required in teachers of pedagogical content knowledge (PCK) was first mentioned in the literature by Shulman (1986) and later the pedagogical content knowledge of teachers was investigated by many researchers. It was determined that PCK has a central role in the development of teachers (Ball, 1990; Borko et al. 1992; Chick & Baker, 2006; Hill, Ball & Schilling, 2008; Smith, 2007).

A variety of research into mathematical content knowledge and mathematical pedagogical content knowledge (An, Kulm & Wu, 2004; Daehler & Shinohara, 2001; Jones & Moreland, 2004; McDuffy, 2004; Sánchez & Llinares, 2003; Stacey et al., 2001) has determined different knowledge components required in a teacher (Cited in Aksu and Konyalıoglu, 2015).

Fennema and Franke (1992) classified the components necessary in a teacher to teach mathematics:

1. Mathematical knowledge
- Content knowledge
    - Nature of mathematics
    - Cognitive organization of knowledge
2. Knowledge of mathematical presentation (Illustration and Representation)
3. Student Knowledge
- Knowledge of Student Cognitive Development
4. Teaching Knowledge and Decision-making

Before Shulman (1986) and later by other researchers (Fermandez et al., 1995; Geddis et al., 1993; Grossman, 1990; Hasweh, 2005; Loughran et al., 2006; Magnusson et al., 1999; Marks, 1990; Shulman, 1987; Smith & Neale, 1989; Tamir, 1988), knowledge about students, accepted as a component of PCK, was one of the most important components of teacher knowledge (Cited in Aksu, 2013).

According to Shulman (1986), pedagogical content knowledge:

> It includes, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations-in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 9)

Additionally, An, Kulm and Wu (2004) stated that student knowledge includes the teacher’s awareness of preliminary knowledge of students while learning a topic, the operational and conceptual knowledge gained, forms of learning, and knowing difficulties and misconceptions experienced while learning the concepts. As a result, as student knowledge (understanding of students) is accepted as an important part of pedagogical content knowledge or teacher knowledge, it becomes necessary to determine what level of student knowledge teachers possess.

Functions are among the basic topics in mathematics on the Secondary school mathematics curriculum requiring knowledge the students acquired in primary and middle school, and are an important topic unifying mathematical topics. As a result this study investigated teachers’ student knowledge on the topic of functions.

The mathematical knowledge required for teachers to teach mathematics was defined in two basic categories by Ball et al., (2008) as common content knowledge and specialized content knowledge (numbers, processes and models and functions and algebra). Wilson (1994) stated that how the function topic was perceived among the basic topics of mathematics was important for how a teacher teaches it. For example, if the teacher defines the function concept within the process chain, this teacher will probably include more activities like algebraic and arithmetic operations while teaching functions. Another teacher, who may consider the function concept as underlain by models of real situations with the aid of graphs will focus on the use of activities such as mathematical illustrations of algebraic processes (like applications of functions in daily life) during teaching.

Lucas (2006) compared the teaching experiences of 8 teachers and 10 pre-service teachers with compound functions in terms of pedagogical content knowledge. The teachers gave similar answers to controversial questions. Especially, when participants dealt with mechanical/functional/arithmetic dimensions of compound functions they showed they had low levels of conceptual knowledge. According to the results of this study, teaching experience did not affect the content knowledge of teachers in teaching compound functions. As experience increased, it was expected that teachers’ mathematical knowledge of teaching would increase so this result was contrary to logic. To explain this result it was emphasized that the educational system does not sufficiently emphasize mathematical knowledge for teaching but focuses on mathematical content. As a result the solution is to focus more on how to teach rather than what to teach.

Even (1993) researched the correlation between function concept knowledge and pedagogical knowledge for teaching of Secondary school student teachers in two stages. This study included 152 student teachers from different universities and two open-ended questions were asked in interviews (what comes to mind first when you say function? and how do you teach the function topic?). According to the results of this study, student teachers had deficient knowledge of the conceptualization of functions and it was found that the limited knowledge of functions affected how they would teach in the future.

Bayazıt and Aksoy (2010) studied pedagogical content knowledge about functions of two teachers. In their results, the teachers had similar thoughts about the difficulties encountered by students when learning functions and on diagnosing misconceptions that developed and understanding their mental causes. However, they displayed different teaching approaches to resolving these difficulties and misconceptions.

When the research is investigated, common misconceptions of students about functions include; difficulties matching each element of a domain and elements in range sets, whether a graph of a function is a regular and continuous curve or line (Vinner, 1983), reducing inverse process rules in the inverse function concept (Eisenberg, 1991; Even, 1992) and considering x and y variables as algebraic equations (Tall & Baker, 1992).

This study investigated student knowledge of mathematics teachers about functions. The student knowledge in this study was investigated under the categories of;

- Knowledge of preliminary knowledge of students
- Knowledge of difficulties and misconceptions experienced by students
- Knowledge of how to correct errors and misconceptions

2. Method

It was decided that the most appropriate method for the structure of the study was the qualitative research approach. Qualitative research allows the opportunity for in-depth consideration of the information and meaning of obtained data (Creswell, 2013). The resource for the study comprised written interview questions. The study findings were analyzed by two researchers together using descriptive analysis methods.

2.1 Participants

The study participants comprised mathematics teachers employed at different schools in different states. The information relating to participants is given in the table below.

Table 1. *Demographic Information of Participants*

	Gender	Years of Service	Type of School
T 1	Female	5	General Secondary School
T 2	Female	4	General Secondary School
T 3	Male	5	General Secondary School
T 4	Male	8	General Secondary School
T 5	Male	16	General Secondary School
T 6	Female	6	Science Secondary School
T 7	Male	15	Science Secondary School
T 8	Male	12	Science Secondary School
T 9	Male	7	Vocational Technical Secondary School
T 10	Male	6	Vocational Technical Secondary School
T 11	Male	5	Vocational Technical Secondary School
T 12	Female	5	Vocational Technical Secondary School
T 13	Male	32	Vocational Technical Secondary School

2.2 Data Collection Tool and Analysis

To investigate the student knowledge of mathematics teachers about functions in the study, the researchers prepared a written interview form. In developing the interview form, the interview forms from primarily pedagogical content knowledge studies were investigated. Additionally function questions prepared related to pedagogical content knowledge was investigated. After the interview form was prepared, a pilot study was completed with two teachers separate from the study group. The questions with comprehension difficulties during the pilot application were re-composed. Later the researchers completed the interview form. Responses to interview questions were analyzed descriptively and for content. During the analysis of research data, the responses given by teachers to each question were organized according to the aim of the research.

3. Results

This section includes analysis of data obtained from the interview forms. Initially mathematics teachers were asked how many years they had been teaching functions, how students perceived this topic and what preliminary information was required to learn the topic. The answers to these questions are presented in the table below.

As observed in the table, the majority of teachers sufficiently taught functions. They showed they had necessary ideas about the topic. Nearly all teachers were of the opinion that the topic of functions was not difficult for students. Only two teachers stated that the topic was difficult for students. T3 gave the following explanation based on preliminary knowledge to explain why the topic was not difficult:

T3: It is not a difficult topic for a standard student with primary school education (who has gained the outcomes in the syllabus).

Many of the teachers gave the necessary preliminary knowledge required to understand functions as being sets and nearly half gave the answer of correlations. Some teachers stated topics such as operational skills, exponential-rooted numbers and logic.

Table 2. *Results from the first three questions*

	Years of Teaching	Degree of difficulty for students	Preliminary Knowledge Required
T1	5	Partially difficult	Coordinate system (for function graphs), sets (for function varieties)
T2	4	Some sections difficult	Sets, equations, inequalities
T3	5	Varies depending on preliminary knowledge of the student	Sets, Cartesian multiplication process skills
T4	8	Partially difficult	Sets, coordinate system, Cartesian multiplication, correlation
T5	16	Not difficult	Sets, correlation, first degree equations, Cartesian multiplication
T6	6	Partially difficult	Variable concept, Cartesian multiplication, exponential-rooted numbers, rational numbers
T7	12	Not difficult	Numerical operation skills
T8	10	Not difficult	Factoring
T9	2	A difficult topic	Ordered pairs, Cartesian multiplication and correlations
T10	3	Not difficult	Exponential-rooted numbers, rational numbers, correlations
T11	2	Partially difficult	Sets, correlations, equations, numbers
T12	5	Not difficult	Sets, Cartesian multiplication and correlations
T13	25	A difficult topic	Logic and processes

The teachers' answers to common errors, difficulties or misconceptions about functions encountered among students are categorized below.

Table 3. *The teachers' answers about students' difficulties*

Categories	**Teachers**
Unable to show algebraic equations on tables or graphs and unable to read graphs	T1, T5, T6,T13
Unable to distinguish function types (compound, unit, inverse)	T1, T5, T13
Unable to determine domain, range and image sets (state what x and y represent in y=f(x))	T1, T2, T3, T6, T8, T9, T11, T12
Lack of knowledge about operational skills, value calculation and insertion	T2, T4, T5, T7, T10

Eight mathematics teachers stated that students experienced problems with domain, range and image sets of functions. T2 made the following explanation related to this:

T2: In questions like "how much is f(3) if f(x-5)=2x-1?" students write 3 directly instead of x. They aren't distinguish the range from the image set.

Some teachers stated that students encountered difficulties due to lack of preliminary knowledge, like calculation skills and value calculation.

Later opinions were sought from teachers on how to resolve these errors. At this point teachers focused on topics such as good focus on basic concepts related to functions, developing operational skills of students, consolidating preliminary knowledge, giving concrete examples and using models.

T3: In the introduction section to functions and subtopics, creating student awareness related to function concepts, and doing work to assimilate these concepts will increase levels of preparedness in students.

T5: As much as possible, concrete examples should be given. However, concrete examples should not be exaggerated, it is necessary not to bypass the core of functions, in other words processes.

T6: Visuals should be used, similar problems should be solved. Lots of examples used. The topics they are stuck on should be determined, and erroneous solutions should be compared with correct solutions.

Another question asked of teachers was whether they used different representations or models while teaching functions; if so, what type of models did they use. Below, some findings belonging to some teachers who used models are given.

T10: As the subject progressed, I explained the idea that the function can be thought as a machine that for each input returns a corresponding output. I emphasized that "everything may be a function in our lives" in classroom. For instance:

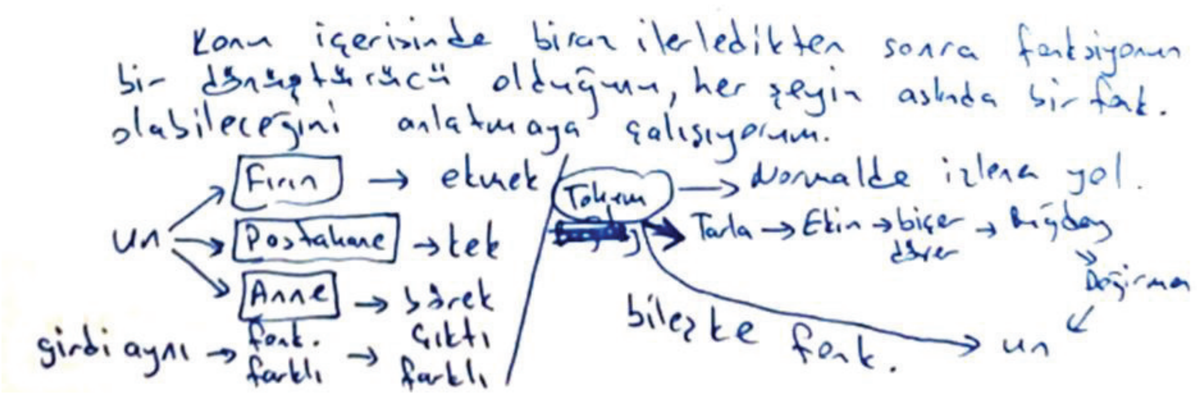


Figure 1. T10 uses model while teaching function

T13: I use models like maps of Turkey, regions, and mincing machines, neighboring.

T6: The function machine model is the model I use most. While giving the definition of a function, I create relationships between the student, school, home, and cafe. I accept the students as being elements in a domain. Imagining this makes it easier.

T5: Yes I am trying to do something different. For example:

**Function can be described as a metaphor of Courier Company*

**When I was teaching the graph of a function, I draw analogy: curve in Cartesian plane like a snake and I ask students how far the snake away from the vertical and horizontal axis.*

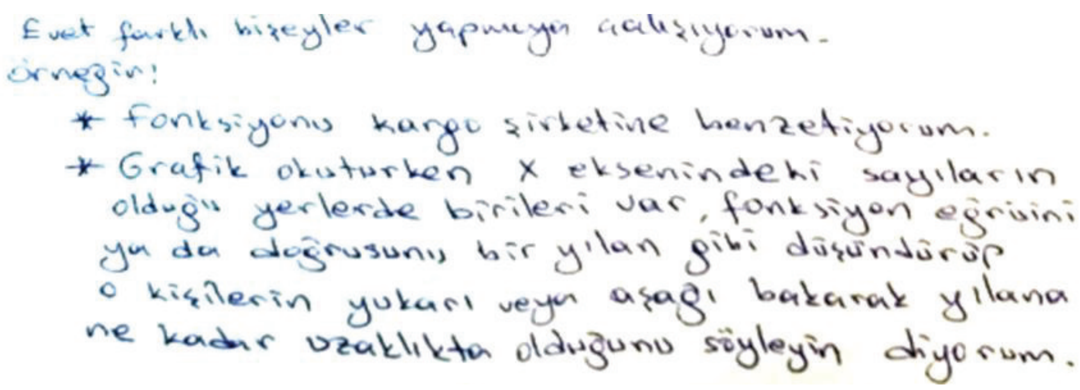


Figure 2. T5 uses model while teaching function

T4: At the moment in books we use input-output (factory) or table images.

f(x) = 5x+2

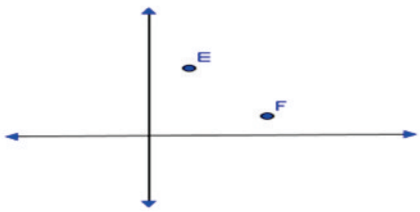
Path (time) = 5 . (time) + 2

Path (t) = 5.t + 2

Teachers generally used models like factories and function machines, and were identified as seeing functions as a type of transformation.

Finally, ideas about errors of students related to functions were obtained in two questions at the end of the written interview. The response given for these two questions are given below.

Problem 1: A student is asked to draw the function graph passing through points E and F shown in Figure 3. The student draws the graph in Figure 4. When the student is asked if there is another answer to this question, they say "no".



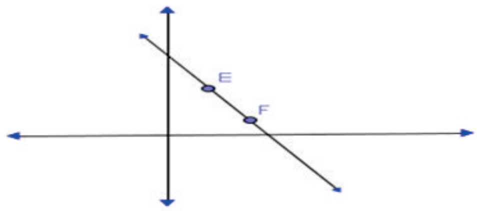


Figure 3. Two points on coordinate plane *Figure 4*. Drawing graph by a student

What do you think the student may have been thinking while answering the question? What may have caused the student to have such a misconception or misunderstanding? In fact, they were asked to direct this question to the students during a lesson. A student came to the board and drew a straight line through these two points on the coordinate system. When asked if they could draw a different graph the student said *"no"*. At the same time it was identified that some students in the class said that more, in fact an infinite number of graphs may be drawn.

Nearly all the teachers asked this question said that the misconception was due to the idea that students had been previously taught; *"only one line passes through two points"*. Only the linear function type came to the student's mind here. They stated this was the cause of the misconception.

What type of mathematical ideas should be presented in order for students to find the correct answer? An example of the answer a teacher gave to this question is listed below.

T10: Marking the road with pen may be requested from students through ways between two different cities on the map. These two points can be reached by using different ways. Therefore, many graph of function can be drawn by using these two points. The target of this is to reach infinity.

T4: They could draw a graph of the variation of temperature within a day, etc.

Rest of the teachers made similar comments of T6.

T6: I can show the graph of a polynomial functions such as x^2 and x^3 having always a smooth curve in order to compare each other. Different type of function which can pass through these points, can be shown.

In general in this situation teachers showed examples of different types of functions and showed solution paths for drawing graphs.

The second question and the answers given by teachers are presented below.

Problem 2: A student is requested to find the inverse of the function $f(x) = x^2$ and gives the answer $f^{-1}(x) = \sqrt{x}$

The teachers were asked these questions about the problem. What do you think the student was thinking as they answered? What may have caused the student to have such a misconception or misunderstanding? Some of the responses of the teachers are given below.

T1: They experienced learning difficulties related to the inverse function concept. They thought that changing the places of the x and y variables are the solution for the inverse function of x. They thought that exponential numbers and rooted numbers are inverse of each other.

T6: It's caused by insufficient knowledge of root equations. They may not have thought of the effect of domain and range concepts on the question.

T8: They don't understand domain and range. They didn't understand one-to-one and onto functions. There's a problem with correlation.

T10: They may have considered that an exponential equation is the inverse of a rooted equation. They didn't think that when negative numbers are squared they give positive results. As a result when they took the inverse they only thought of $\sqrt{x}$ *and didn't consider* $-\sqrt{x}$.

Later teachers were asked "what type of mathematical ideas should be presented in order for students to find the correct answer?"

T10: In $x^2 = 4 \ what \ is \ x$ *?*

We could find the answer to that. Or what is 4 the square of? We could research the question. So we could explain it's necessary to find two different answers.

T11: The function should be explained well with graphs and charts. One-to-one and onto should be mentioned.

The teachers generally stated the necessity to emphasize one-to-one and onto situations in answering this question.

4. Discussion and Conclusion

This study investigated the student knowledge component of pedagogical content knowledge of mathematics teachers with at least two years of occupational experience. The majority of participating teachers stated that functions were not a difficult topic for students but this was linked to preliminary knowledge the students brought with them. The study found that teachers stated the preconditions for better learning of functions by students were sets and correlations. In the Secondary School Mathematics (Classes 9, 10, 11 and 12) Curriculum that has been applied since the 2011-2012 Academic year by the Educational Board of the Ministry of National Education, the topic of functions is located in the 9th class Logic and Sets learning area after correlations, functions and operating topics. The teachers' thoughts on this are in line with the syllabus. Some teachers stated that preconditions are processing knowledge and numbers, clearly necessary for all topics in mathematics. Additionally, the reason for this thought may be considered to be teachers evaluating functions from functional and arithmetic dimensions. This idea is in accordance with the finding mentioned in Lucas (2006) "*teachers consider mechanical/functional/arithmetic dimensions of functions and this shows they have low levels of conceptual knowledge"*.

The common errors of students made about functions identified by the teachers complied with the student errors identified in the literature (Eisenberg, 1991; Even, 1992; Tall & Bakar, 1992; Vinner, 1983). Additionally a study by Bayazıt and Aksoy (2010) mentioned similar situations for students of teachers. This situation shows that teachers participating in the study were sufficient at identifying student errors and misconceptions. There was no noteworthy finding on how to correct errors and misconceptions. In general consolidation, solving lots of problems and use of concrete examples were recommended. When teachers were asked about models used, they generally gave the input-output machine used in the curriculum. This situation shows teachers used the curriculum and lesson books as resources for teaching. DeMarois, McGowen and TalI (2000a, 2000b) determined that the function box (machine) may be an appropriate start for the concept of functions (Cited in Akkoç, 2006). They observed it was insufficient to present an original model, whereas a study by Delice, Aydın and Kardeş (2009) identified the most appropriate mathematical topics for visualization were spatial geometry in 1st place with 11.2% and functions in 2nd place with 6.7%.

In the example given in Problem 1, teachers stated that students were stuck on the previously learned concept of "*only one line can pass through two points"*. This situation is in parallel with findings in Vinner (1983) and Bayazıt and Aksoy (2010). As a solution they emphasized the necessity to show different types of functions. In the curriculum teachers are asked to use dynamic software to draw function graphs. However, no teacher participating in this study mentioned computer-supported learning for graphs.

For the erroneous inverse function given in Problem 2, teachers stated that student erred at the operational level. They thought that the student was deficient in preliminary knowledge needed for process skills. They showed domain and range concepts and one-to-one and onto concepts as causes of misunderstanding. For a solution they stated the need to focus on these concepts. This situation is in accordance with findings in the study by Bayazıt and Aksoy (2010).

The results of the study show that teachers are successful at identifying misconceptions of students about the topic of functions and errors they may make. Additionally it appeared that the type of school of employment and experience had no effect on identification of errors. To correct these identified errors, the solutions recommended by teachers remained superficial. In other words, explanations could not be given at both pedagogical content knowledge level and technological pedagogical content level. On the use of models and representations, they remained bound to the curriculum; it was concluded they did not try to produce unique or different representations.

In this study, conclusions were reached about three items of the student knowledge component of pedagogical content knowledge, underlain by research by Shulman (1986). The teachers participating in the study may be counted as successful in knowing the preliminary knowledge of students and the difficulties and misconceptions experienced by students; however they were identified to be deficient in terms of knowing how to correct the errors and misconceptions experienced. Again, it appeared that school type and experience were not important factors. This reveals that during training after degree, in other words during in-service training, there is a need to complete activities developing pedagogical content knowledge.

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