A Unique Common Fixed Point Theorem for Four Maps under Contractive Conditions in Cone Metric Spaces

S.Vijaya Lakshmi and J.Sucharita
Department of Mathematics, Osmania University, Hyderabad-500007, Andhra Pradesh, INDIA..
Vijayalakshmi.sandra@gmail.com and sucharitha _gk@yahoo.co.in

Abstract

In this paper, we prove existence of coincidence points and a common fixed point theorem for four maps under contractive conditions in cone metric spaces for non continuous mappings and relaxation of completeness in the space. These results extend and improve several well known comparable results in the existing literature.

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1. Introduction and preliminaries.

In 2007 Huang and Zhang [3] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequentl, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3,4] and the references mentioned therein). Recently, Abbas and Jungck [1] have obtained coincidence points and common fixed point theorems for two mappings in cone metric spaces .The purpose of this paper is to extend and improves the fixed point theorem of [6].

Throughout this paper, E is a real Banach space, N = {1,2,3,……} the set of all natural numbers. For the mappings f,g :X→X, let C(f,g) denotes set of coincidence points of f , g , that is C(f,g):={z$\in$X : fz = gz }.

We recall some definitions of cone metric spaces and some of their properties [3].

Definition1.1. Let E be a real Banach Space and P a subset of E .The set P is Called a cone if and only if :

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a,b \in R$, $a,b \geq 0$,$x,y \in P$ implies $ax + by \in P$;

(c) $x \in P$ and $-x \in P$ implies $x = 0$.

Definition1.2. Let P be a cone in a Banach Space E , define partial ordering '$\leq$'on E with respect to P by $x \leq y$ if and only if $y-x \in P$ .We shall write x<y to indicate $x \leq y$ but $x \neq y$ while X<<y will stand for $y-x \in$ Int P , where Int P denotes the interior of the set P. This Cone P is called an order cone.

Definition1.3. Let E be a Banach Space and $P \subset E$ be an order cone .The order cone P is called normal if there exists L>0 such that for all $x,y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L \|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition1.4. Let X be a nonempty set of E .Suppose that the map

d: $X \times X \rightarrow E$ satisfies :

(d1) $0 \leq d(x,y)$ for all $x,y \in X$ and

$d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space .

It is obvious that the cone metric spaces generalize metric spaces.

Example1.1. ([3]) . Let $E = R^2$, $P = \{(x, y) \in E$ such that : $x, y \geq 0\} \subset R^2$,
$X = R$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant .Then (X,d) is a cone metric space.

Definition1.5. Let (X,d) be a cone metric space .We say that $\{x_n\}$ is

(a) a Cauchy sequence if for every c in E with $0 \ll c$, there is N such that
for all n , m > N , $d(x_n, x_m) \ll c$;

(b) a convergent sequence if for any $0 \ll c$,there is N such that for all
$n > N$, $d(x_n, x) \ll c$, for some fixed $x \in X$.

A Cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma1.1. ([3]) .Let (X,d) be a cone metric space , and let P be a normal cone with normal constant L .Let $\{x_n\}$ be a sequence in X .Then

(i). $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.

(ii). $\{x_n\}$ is a cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n,m \to \infty)$.

Definition 1.6.([5]). Let f, g: X→X. Then the pair (f , g) is said to be
(IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

(2). Common fixed point theorem

In this section, we obtain existence of coincidence points and a common fixed point Theorem for four maps on a cone metric space.

The following theorem is extends and improves Theorem 2.1[6].

Theorem2.2. Let (X,d) be a cone metric space and P a normal cone with normal constant L. Suppose that the mappings S,T ,I and J are four self –maps on X such that $T(X) \subset I(X)$ and $S(X) \subset J(X)$ and $T(X) = S(X)$ and satisfy the condition

$$\|d(Sx,Ty)\| \leq k\|d(Ix,Jy)\| \text{ for all } x,y \in X. \qquad (2.1)$$

Where $k \in [0,1)$ is a constant .

If $S(X) = T(X)$ is a complete subspace of X, then {S,I} and {T,J} have a coincidence point in X. More over,if {S, I} and {T, J}are (IT)-Commuting then, S,T ,I and J have a unique common fixed point.**Proof.** For any arbitrary point x_0 in X, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Jx_{2n+1}$$
$$\text{and} \quad y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}, \quad \text{for all } n = 0,1,2,\ldots..$$

By (2.1),we have

$$\begin{aligned}\|d(y_{2n}, y_{2n+1})\| &= \|d(Sx_{2n}, Tx_{2n+1})\| \\ &\leq k\|d(Ix_{2n}, Jx_{2n+1})\| \\ &\leq k\|d(y_{2n-1}, y_{2n})\|\end{aligned}$$

Similarly, it can be show that

$$d(y_{2n+1}, y_{2n+2}) \leq k\, d(y_{2n}, y_{2n+1}).$$

Therefore, for all n ,

$$\|d(y_{n+1}, y_{n+2})\| \leq k\|d(y_n, y_{n+1})\| \leq \ldots\ldots\ldots \leq k^{n+1}\|d(y_0, y_1)\| .$$

Now, for any m>n,

$$\|d(y_n, y_m)\| \leq \|d(y_n, y_{n+1})\| + \|d(y_{n+1}, y_{n+2})\| + \ldots\ldots\ldots + \|d(y_{m-1}, y_m)\|$$

$$\leq [k^n + k^{n+1} + \ldots..+ k^{m-1}]\, \|d(y_1, y_0)\|$$

$$\leq \frac{k^n}{1-k} \| d(y_1, y_0) \|.$$ From (1.3), we have

$$\| d(y_n, y_m) \| \leq \frac{k^n}{1-k} L \| d(y_1, y_0) \|.$$

Which implies that $\| d(y_n, y_m) \| \to 0$ as $n, m \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

Let us suppose that S(X) is complete subspace of X . Completeness on S(X) implies existence of $z \in S(X)$ such that $\lim_{n\to\infty} y_{2n} = Sx_{2n} = z$.

$$\lim_{n\to\infty} Jx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z. \quad (2.2)$$

That is , for any $0 << c$, for sufficiently large n ,we have $d(y_n\ y) << c$.

Since $z \in T(X) \subseteq I(X)$, then there exists a point $u \in X$ such that $z = Iu$.

Let us prove that $z = Su$. By the triangle inequality ,we have .

$$\| d(Su,z) \| \leq \| d(Su,Tx_{2n+1}) \| + \| d(Tx_{2n+1}, z) \|$$
$$\leq k \| d(Iu, Jx_{2n+1}) \| + \| d(Tx_{2n+1}, z) \|.$$

Letting $n \to \infty$, we get

$$\| d(Su,z) \| \leq k \| d(z,z) \| + \| d(z, z) \|.$$
$$\leq k(0) + 0 = 0.$$ That is $Su = z$.

Therefore $z = Su = Iu$, that is u is a coincidence point of S and I. (2.3)

Since $, z = S(X) \subseteq J(X)$, there exists a point $v \in X$ such that $z = Jv$ .We shall show that $Tv = z$. We have

$$\| d(Tv, z) \| \leq \| d(Su,Tv) \|$$
$$\leq k \| d(Iu, Jv) \|$$
$$\leq k \| d(z, z) \| = 0.$$

Implies $Tv = z$.

Therefore $z = Tv = Jv$, that is v is a coincidence point of T and J. (2.4)

From (2.3) and (2.4) it follows

$$Su = Iu = Tv = Jv\ (=z). \quad (2.5)$$

Since (S,I) and (T,J) are (IT) – commuting .

$$\| d(SSu, Su) \| = \| d(SSu, Iu) \|$$
$$= \| d(SSu, Tv) \|$$
$$\leq k \| d(ISu, Jv) \|$$
$$= k \| d(SIu, Su) \|$$
$$= k \| d(SSu,Su) \| < \| d(SSu,Su) \|,$$ (since $k<1$) ,a contradiction

$SSu = Su(= z)$.

$Su = SSu = SIu = ISu$.

That is $SSu = ISu = Su\ (= z)$.

Therefore $Su = z$ is a common fixed point of S and I. (2.6)

Similarly, $Tv = TTv = TJv = JTv$

Implies TTv = JTv = Tv(= z).

Therefore Tv (=z) is a common fixed point of T and J. (2.7)

In view of (2.6) and (2.7) it follows S,T ,I and J have a common fixed point namely z.

Uniqueness , let z_1 be another common fixed point of S , T , I and J .

Then $\| d(z, z_1) \| = \| d(Sz, Tz) \|$

$\leq k \| d(Iz , Jz_1) \|$

$\leq k \| d(z, z_1) \|$

$< \| d(z, z_1) \|$. (since $k<1$).

Which is a contradiction,

Implies $z = z_1$.

Therefore S, T, I and J have a unique common fixed point .

Remark2.1. If S = T and I = J then the theorem reduces to the theorem 2.1 of Stojan Radenovic [6] with I(X) complete ,which is an improvement of Theorem 2.1.of [6]. Since in this paper J(X) is complete which is a super space of I(X)

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