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Mathematical Modeling: An Important Concept in Mathematics Education

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Abstract
The purpose of this article was to demonstrate the essence and relevance of mathematical modeling as an important classroom practice that promote mathematical proficiency in mathematics education for all students. Research have shown that mathematical modeling education play a major role in everyday teaching and learning of mathematics. Additionally, mathematical modeling encourages the development of mathematical practices, processes, and skills useful for today's world. Furthermore, research indicates that students can make significant mathematical and socio-cultural improvements in mathematics when exposed to mathematical modeling. However, mathematical modeling proficiencies and competencies have been under emphasized in most K–12 schools in the United Sates. This article therefore investigated the role of mathematical modeling and how to implement modeling practices in mathematics education. Strategies for teaching modeling and examples of authentic modeling tasks are illustrated in this article to highlight the relevance and importance of mathematical modeling. It is argued that mathematical modeling education is a powerful tool for developing students' quantitative reasoning, productive disposition, problem-solving skills, and modeling competencies.
Keywords: mathematical modeling; mathematics education; modeling practices; modeling process; modeling task; models; problem-solving

1. Introduction

Researchers, professional organizations, and mathematics standards have underscored the need and relevance of mathematical modeling and especially in school mathematics (Blum & Borromeo Ferri 2009; English, 2004, Gainsburg, 2008; Lesh & Doerr 2003; National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010; Pollak, 2003). Recent studies signify that students can make important mathematical and socio-cultural improvements from working on authentic modeling tasks (Doerr & English, 2003; English & Watters, 2004). The NCTM specified that instructional programs from K–12 should enable students to use representations to model and interpret physical, social, and mathematical phenomena (NCTM, 2000). Additionally, the Common Core State Standards for Mathematics (CCSSM) emphasizes the use of mathematical modeling: "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (NGA Center & CCSSO, 2010, p. 7).

Mathematical modeling education is a powerful vehicle for K–12 students to develop important mathematical practices, problem-solving competencies, and 21st century skills (Asempapa, 2015; Lesh & Doerr, 2003). For this reason, K–12 schools are the educational environment where students should begin a meaningful development of modeling processes and practices (Langrall, Mooney, Nisbet, & Jones, 2002; Lehrer & Scauble, 2003). Consequently, this article discusses how mathematical modeling as an educational endeavor is important in mathematics education. Specifically, the article examines how mathematical modeling is alike and different from mathematical problem-solving. Furthermore, the article explores how teachers can implement mathematical modeling tasks in their classrooms. The article concludes with the notion that mathematical modeling is essential and relevant in students' education—the challenges of teaching mathematics through mathematical modeling are worth the benefits that stimulate student learning.

2. Mathematics Education and Modeling

Mathematics education is a field of scientific inquiry that is supposed to address the teaching and learning of mathematics. According to Klien (2003), the prescriptions for the future of mathematics education in the United States were articulated early in the 20th century by one of the nation's most influential education leaders, William Heard Kilpatrick. The importance of mathematics education has never been greater than now and the near future. Training students to become adept users of mathematics and to appreciate its usefulness is of paramount importance. Researchers in mathematics education are primarily concerned with the tools, methods, and approaches that facilitate the practice of mathematics education (Sierpinska et al. 1993).

The object of study in mathematics education include "the teaching of mathematics; the learning of mathematics; problem-solving; applications; teaching and learning situations; the relations between teaching, learning, and mathematical knowledge; societal views of mathematics and its teaching; or the system of education itself" (Sierpinska et al. 1993, p. 275). However, Sriraman and English (2010) viewed mathematics education as a design science where researchers draw on several practical and disciplinary perspectives needed for solving the complex problems of learning and teaching as they occur in our environments. Among the more pragmatic aims of mathematics education is the improvement of teaching practice as well as students' understanding and achievement in mathematics through problem-solving (Sierpinska et al. 1993).

However, problem-solving experiences that students' typically meet in school mathematics lack modeling processes and practices needed for today's world. Anecdotal experiences show that most U.S. mathematics teachers hardly use mathematical modeling in their classrooms. Additionally, studies have shown that modeling practices and activities only play a minor role in everyday teaching of mathematics in most K–12 schools (Doerr & English, 2003; Langrall et al., 2002; Pollak, 2003; Zawojewski, 2010. This is not unique in the case of the United States. In most classroom practices all over the world, mathematical modeling still has a far less prominent role than is desirable (Blum & Borromeo Ferri, 2009; Boaler, 2001). Among most teachers, a popular interpretation of mathematical modeling concerns the use of representations to illustrate a mathematical concept. This idea may originate from teachers' understanding of NCTM's (2000) representation standard: students should "use representations to model and interpret physical, social, and mathematical phenomena" (p. 67) and the CCSSM fourth standard for mathematical practice: model with mathematics (NGA Center & CCSSO, 2010). Although, the CCSSM addresses mathematical modeling as a mathematical practice for all grades, it is limited to only high schools as a conceptual category (NGA Center & CCSSO, 2010).

3. Mathematical Modeling and Problem-Solving

Research into mathematics and statistics education has clearly shown that both fields are particularly adapted to the constructivist theory of learning (Noddings, 1990). A fundamental strategy in constructivist theory is problem-solving and models. Although experts have not agreed on a common conceptual definition of problem-solving, it has been widely accepted in both mathematics and statistics education. Mathematical modeling although different from problem-solving is related to both mathematics and statistic education because models and problem-solving—key components in mathematical modelling—are utilized in both situations.

Mathematical modeling is different from problem solving in two main ways. First mathematical modeling always involves an open-ended task– a task with high cognitive demand, with multiple points of entry, and relates to real-life experience (Blum & Borromeo Ferri, 2009). Most problem-solving tasks are school-based context and modeled after some structured rules–consist of rules for computing answers to questions (Zawojewski, 2010). Second, in solving modeling tasks, one need to come up with assumptions (hypotheses) and the solution involves an iterative process– hence the modeling process. However, this is not the case in most problem-solving tasks in K–12 schools.

The existing approaches to problem-solving tasks include instruction that assumes the required concepts and that procedures must be taught first and then practiced through solving routine "story" problems" (Sriraman & English, 2010). Normally these approaches do not engage students in genuine critical thinking. A rich and powerful alternative to mathematical problem-solving is mathematical modeling. Mathematical modeling is descriptive solution to real-world problems composed of one or more representations. Mathematical modeling treats problem-solving as integral to the development of an understanding of any given mathematical concept or process (Lesh & Zawojewski, 2007).

3.1 What are Models and Mathematical Modeling?

Throughout the literature on mathematical modeling and models, experts have offered many definitions for the concept "model." According to English, Fox, and Watters (2005), models are conceptual systems used to construct, interpret, explain, and mathematically describe a situation. Alternatively, Lesh and Fennewald (2010) explained that a "model is a system for describing (or explaining or designing) another system for some specific purpose" (p. 7). Models are meaningful conceptual systems that foster conceptual understanding and mathematization (Lesh & Doerr, 2003). However, if the behavior predicted by our model does not reflect what we see in the real world, then the model has to be changed and not the world (Dym, 2004).

Researchers in mathematics education community have offered different definitions for mathematical modeling. Although there is no one agreed-upon definition of mathematical modeling, it is a process where one identifies a situation in the real world, makes certain assumptions and choices, and then uses a mathematical model to obtain a solution that can be translated back into the real world. According to Swetz and Hartzler (1991), mathematical modeling requires translations between reality and mathematics, where students are challenged to study a situation using models and testing that the solution makes sense in the context of the real-world situation. Alternatively, Pollak (2003) explained that mathematical modeling involves a situation in the real-world, making certain assumptions, using a mathematical model to obtain a mathematical formulation, and then applying mathematical techniques to the formulation to get results reasonable in the real-world. Thus, mathematical modeling is a real-life task, which involves mathematical practices and processes such as critical thinking, high cognitive demand, and communication.

Modeling as an iterative process contains the following phases: (a) understanding the phenomenon, (b) constructing a physical representation or model, (c) mathematizing the phenomenon and performing computations, (d) interpreting results, (e) validating them in context of the real world, and (f) disseminating them through discussions and in writing (NGA Center & CCSSO, 2010). Mathematical modeling presents students with realistic problem-solving experiences that involve critical thinking skills. Modeling tasks requires strategizing, using prior knowledge, and testing and revising solutions in a real-world context (Lesh, Doerr, Carmona, & Hjalmarson, 2003). Thus, the process of formulating and improving a mathematical model to represent and solve real-world problems describes mathematical modeling. On a similar note, mathematical modeling is the process of translating between the real world and mathematics in both directions (Blum & Borromeo Ferri, 2009). The modeling process is a building link between mathematics as a way of making sense of our physical or social world and mathematics as a set of formal structures and representations (English et al. 2005).

3.2 Mathematical Modeling Tasks in Mathematics Education

Mathematical modeling connects to mathematics education through modeling tasks. Modeling tasks are real-life applications and of high cognitive demand. All tasks are not created equal–different tasks require different levels and kinds of student thinking (NCTM, 2014; Stein, Smith, Henniningsen, & Silver, 2009). Students learning gains are greatest in classrooms in which instructional tasks are of high cognitive demand and students learn to explain their thinking and reasoning (Stein et al. 2009). In order to present student's natural ways of modeling, the article discusses two examples of modeling tasks.

The first modeling task called "Lighthouse" is related to mathematics education and was adapted from Blum and Borromeo Ferri (2009, p. 48). *In the bay of Sandusky, in Ottawa County, Ohio, directly on the coast, a lighthouse called "Marblehead Lighthouse" was built in 1819, measuring 15.0 m in height. Its beacon was meant to warn ships that they were approaching the coast. How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution*. The second modeling task called the "Basketball Game" is associated to mathematics education and was adapted from Lakoma (2007, p. 390). *Two boys - Jack and Mark - try to score at a basketball target. They both have the same frequencies of success: 50%. They decided to play a game: each will throw the ball until he fails to score. When one fails, the other takes over the throwing. The first who scores a hit is a winner. Jack always starts first. What are the chances for winning for these boys? Do you think the game is fair*?

Both tasks are real-life situations, with high cognitive demand; promote reasoning and critical thinking, and problem-solving. Consequently, the modeling tasks pose a challenge in constructing a mathematical model for the real-world situations. Therefore, with an explicit modeling process, students are able to understand the process and develop strategies to solve the modeling task(s). There are variations in the modeling process but they are closely related. This article discusses two that are mostly common. They are Blum and Borromeo Ferri's (2009) modeling process and the CCSSM modeling process.

3.3 The Modeling Process versus Problem-Solving Process

A primary goal of mathematics teaching and learning is to develop the ability to solve a wide variety of complex mathematics problems (Stanic & Kilpatrick, 1988). It is strongly believed that the most efficient way for learning mathematical concepts is through problem-solving. NCTM (2000, 2014) recommended problem-solving as the focus of school mathematics and emphasized problem-solving and quantitative reasoning in mathematics instruction. Thus, mathematics instruction should be designed so that students experience mathematics as problem-solving. The four-step process that forms the basis of any serious attempt at problem-solving as shown in Figure 1 include: (a) understand the problem; (b) devise or make a plan; (c) carry out the plan; and (d) look back (Pólya, 1957).

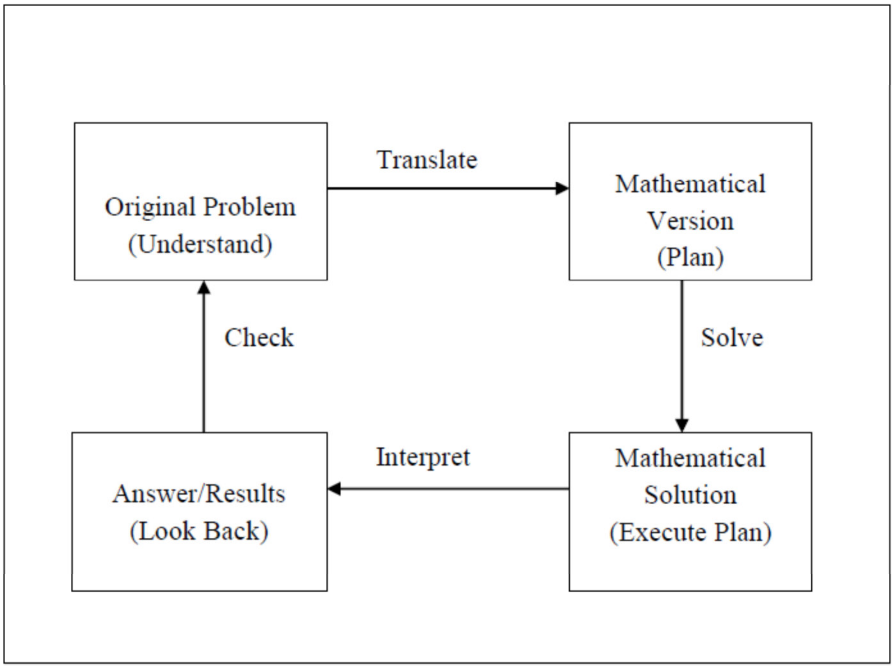

Figure 1. A problem-solving process (adapted from Pólya, 1957)

Clearly, the linear nature of the models used in several textbooks does not promote the spirit of Polya's problem-solving process and teaching students to think critically. By their nature, traditional models present problem-solving as a procedure to be memorized, practiced, and they lead to answer getting (Wilson, Fernandez, & Hadaway, 1993). A deficiency of problem-solving is the lack of problem posing or problem formulation. Problem-solving tasks that students' usually encounter move from a given situation to an end where the "givens" and the "steps" are clearly identified (Zawojewski & Lesh, 2003). Fortunately, the modeling process begins with a real-world situation that promotes critical thinking. The seven steps of the modeling process have well been known and used successfully for the past few decades as a heuristic for teaching mathematical modeling (Blum & Borromeo Ferri, 2009). The modeling process as illustrated in Figure 2 include: "1) understanding the task; 2) simplifying/structuring; 3) mathematizing; 4) working mathematically; 5) interpreting; 6) validating; and 7) presenting/exposing" (Blum & Borromeo Ferri, 2009, p. 46). An advantage of mathematical modeling over problem-solving is the real-life connections, high cognitive demand of task, and problem posing.

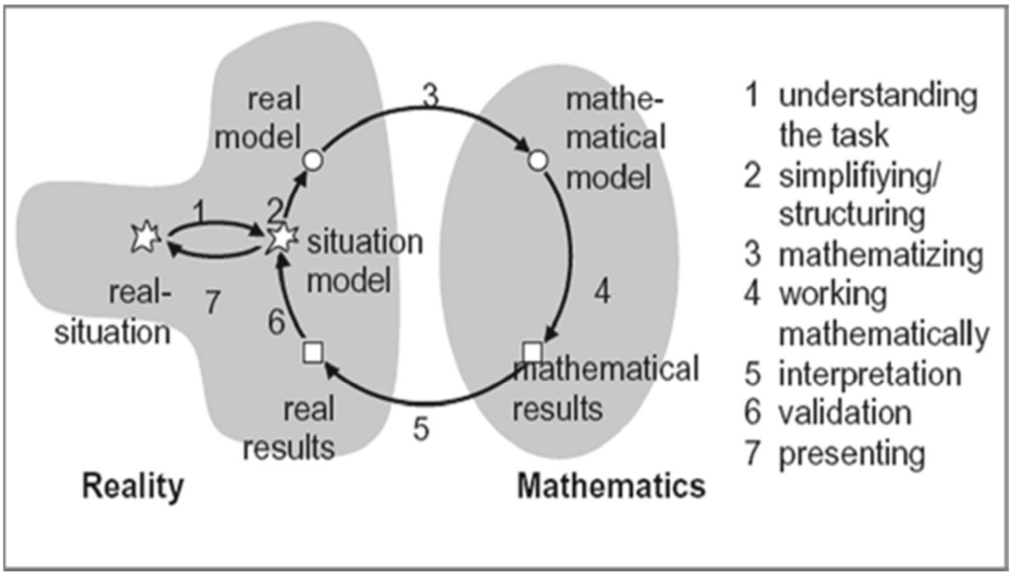

Figure 2. A mathematical modeling process (adapted from Blum and Liess, 2007)

Alternatively, the CCSSM summarized the modeling process in a six-step model to work for both mathematical and statistical situations. The CCSSM modeling process as shown in Figure 3 involves: (a) identifying variables in the situation and selecting those that represent essential features; (b) formulating a model by creating and selecting representations that describe relationships between the variables; (c) analyzing and performing operations on these relationships; (d) interpreting the results in terms of the original situation; (e) validating the conclusions and then either improving the model or, accepting the model; and (f) reporting on the conclusions with reasons (NGA Center & CCSSO, 2010). The modeling process aims among other things at providing students with a "better comprehension of mathematical concepts, teaching them to formulate and to solve specific situation-problems, awaking their critical and creative senses, and shaping their attitude towards mathematics" (Blum, 2002, p. 161). There are some variations in the modeling process described above, however, there are two common things to notice about the models. First, as described in Figure 2 and Figure 3, it is essentially a looping process and there is problem posing. Second, it is a process that students generally find difficult to undertake, despite the fact that the process is simple to state, the steps are logical, and the language non-technical.

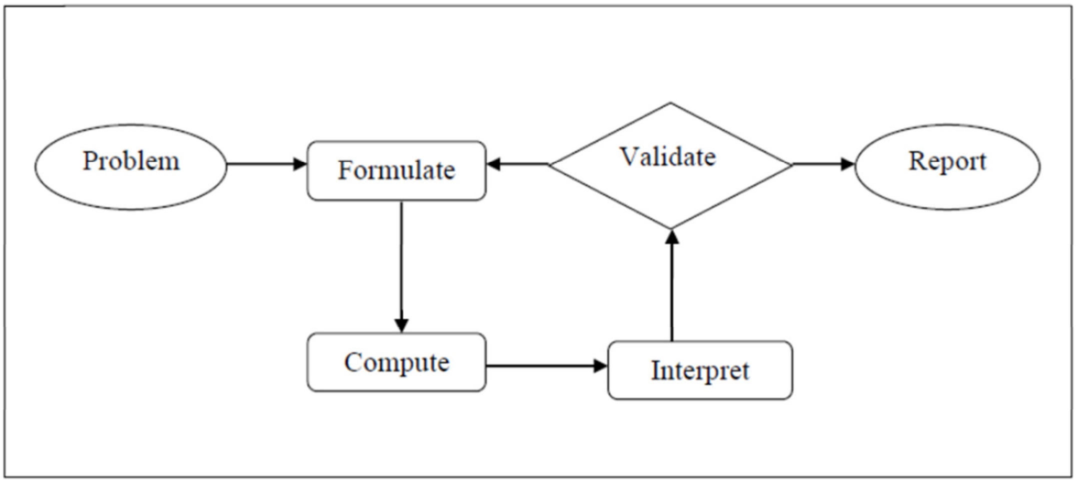

Figure 3. A modeling process adapted from the CCSSM (NGA Center & CCSSO, 2010).

3.4 Benefits of Mathematical Modeling Tasks

For mathematics to be meaningful and relevant to students, experiencing modeling activities at the K–12 is a necessity (English, 2007). Modeling activities create opportunities for learners to perceive mathematics as useful and applied, rather than abstract and isolated (Greer, Verschaffel, & Mukhopadhyay, 2007). It is increasingly recognized that modeling provides students with a "sense of agency" in appreciating the potential of mathematics as a critical tool for analyzing important issues in their lives, their communities, and the society as a whole (Greer et al. 2007). Another benefit is that when students get the opportunity to engage in modeling tasks, they become more engaged in the learning and consequently their achievement gains in mathematics improves (Boaler, 2001; Pollak, 2003). Modeling engages and supports students' interest in mathematics and makes mathematics more meaningful and relevant (English & Watters, 2004; Pollak 2003). Mathematical modeling fosters among students conceptual understanding, competencies, creative and innovative abilities, and comprehension of the socio-cultural role of mathematics (Blum, 1995). Thus, modeling activities are fruitful grounds for students' representational and socio-cultural development of mathematics.

Research shows that mathematical modeling promotes students' understanding of a wide range of key mathematical and scientific concepts and "should be fostered at every age and grade . . . as a powerful way to accomplish learning with understanding in mathematics and science classrooms" (Romberg, Carpenter, & Kwako, 2005, p. 10). In a longitudinal study involving elementary school students, the research showed significant improvement in the students' achievement and development of metacognitive and critical reasoning skills. The researchers found that the modeling tasks got the students engaged and provided opportunities for the students to express their ideas and thinking in multiple representations (English & Watters, 2004). One of the many potential benefits of the richness of modeling experiences is the possibility of students engaging in tasks that follow their interest and match their current conceptual understanding while simultaneously presenting opportunities for challenge and growth (Flevares & Schiff, 2013). Mathematical modeling brings a vital perspective to K–12 mathematics education where the real-world is not just a context to highlight the value of mathematics; rather, both the real-world and mathematics are taken seriously (Pollak, 2003). The benefits of mathematical modeling make sense to categorize under four main domains: *relevance*, *cognitive demand of task*, *critical thinking skills*, and *classroom discourse*.

4. Supporting the Teaching of Mathematical Modeling

Teaching mathematical modeling requires high quality teaching. Moreover, a teacher's pedagogical content knowledge is a significant predictor of students' achievement gains. Consequently, one of the best ways in teaching mathematical modeling is to help teachers understand what modeling is about, its importance, and how to implement modeling tasks in the classrooms. Despite the challenges associated with modeling tasks, research has proven that the teaching and learning of mathematical modeling is possible. If teaching obeys certain criteria or follows some strategies then learning mathematical modeling becomes easy. In particular, if teaching maintains a "permanent balance between (minimal) teachers' guidance and (maximal) students' independence" (Blum & Borromeo Ferri, 2009, p. 52). Moreover, teaching that encourages students without doing the thinking for them, posing purposeful questions, and supporting the students as they do their work, constitutes an effective way of creating a classroom climate for students to engage in mathematical modeling (Chapman, 2007).

Additionally, adaptive teacher interventions that give students hint on a meta-level (imagine the situation, what do you aim at, what is still missing, does the result fit the real situation) are adequate strategies to help students develop competencies in solving modeling tasks (Blum & Borromeo Ferri, 2009). Furthermore, for teachers to be effective at teaching modeling they have to be cognizant of the following; 1) teachers have to create a classroom culture that supports modeling tasks or activities; 2) teachers have to use tasks of high cognitive demand and place importance on real-world connections; and 3) teachers thinking, attitude, practice, and conception must place importance on the real-world in a way that imply the need for modeling in mathematics (Chapman, 2007).

5. Conclusion

There are many good reasons to embrace mathematical modeling in mathematics education. The comprehensive nature of modeling and its tasks make them ideal channel for advancing K–12 students learning in many directions, particularly in posing and solving problems with real-world scenarios (English et al. 2005). Modeling offers rich learning experiences and opportunities for students, and provides ingredients for building mathematically proficient students. Additionally, mathematical modeling facilitates students' collaborative problem-solving efforts as well as fosters their mathematical thinking and learning. Moreover, mathematical modeling promotes among students general creative and problem-solving attitudes, and trains students to apply mathematics to other situations. Furthermore, modeling highlights mathematical connections, addresses affective aspects of learning, and reinforces students' understanding of mathematical applications. Blum (1995), argued that mathematical modeling in schools facilitate learning, prepares students to use mathematics in different areas, develop general competencies, and comprehension of the socio-cultural role of mathematics. Therefore, future research should focus on issues and challenges in establishing modeling as a teaching methodology, and examine how teachers' knowledge about how to teach modeling affects the modeling competency of their students.

Although mathematical modeling may never supplant traditional approaches of teaching and learning mathematics, it does provide mathematics educators with the effective tool for reaching and motivating students. Modeling ensures that students have practical experience applying mathematics skills to real-life situations. Mathematical modeling offers several possibilities for promoting and establishing the importance and relevance of mathematics education. Research has shown that mathematical modeling serves many everyday situations, integrates the ideals of the 21st century learning, and emphasizes the usefulness of mathematics (Gainsburg, 2008). The use of mathematical modeling in K–12 brings back that aspect of mathematics that greatly reinforces the unity of the total mathematical experience and eliminates questions regarding "what good is this stuff?" (Pollak, 2003). The inherent nature of mathematical modeling helps K–12 students develop powerful collaborative skills that are increasingly important for today's world.

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