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# The Persistent Difficulty of Early Fraction Ideas in Early Secondary School Mathematics

Young Rae Kim, Mi Sun Park

Department of Curriculum and Instruction, College of Education and Human Development,

Texas A&M University-San Antonio, One University Way, San Antonio, TX 78224, United States

**Abstract**

This study explored the nature of difficulties of seventh and eighth grade students who struggled building their conceptual understanding of early fraction ideas, in particular ordering fractions. The participants engaged in a sequence of lessons that involved the use of fraction circles. The intervention of four weekly class sessions was adapted from the Rational Number Project (RNP) curriculum that has been created for and refined through teaching experiments in the RNP research since 1979. Pre and post group interviews were conducted with each student group for a sufficient identification of the nature of the students' difficulties. This study identified the whole number dominance strategies used by the students for ordering fractions before and even after the intervention. The students also revealed minimal use of informal ordering strategies that involve more conceptual than procedural understanding of the concept of initial fraction ideas. Considering the short intervention, there was subtle (but meaningful) evidence for a positive influence of the fraction circle model developed within the RNP on students' developing understanding of early fraction ideas. This study suggests the need of a remedial intervention for early secondary students showing the persistent difficulty with early fraction ideas. Students need to be given enough time with not only concrete models but also appropriate usage of language to support a complete understanding of how to use the models.

**Keywords:** Rational Number Project (RNP), fraction circle model, initial fraction ideas, ordering fractions

## 1. Introduction

Fraction concepts are one of the most challenging topics in elementary and early secondary school mathematics (National Mathematics Advisory Panel [NMAP], 2008; Siemon, Beswick, et al., 2015). Research studies indicate that many children have difficulties with the concept of early fraction ideas (Behr, Harel, Post, & Lesh, 1992; Carpenter, Coburn, Reys, & Wilson, 1976; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Hansen, Jordan, & Rodrigues, 2017; Mack, 1995; Post, 1981; Siemon, Beswick, et al., 2015). These difficulties result in a lack of understanding about fraction concepts and fractional operations, which can lead to significant difficulties in algebra (Behr, Lesh, Post, & Silver, 1983). It is critical that the nature of these difficulties of early secondary students who struggle with the concept of early fraction ideas be explored, and that an earlier intervention be provided so that they have access to algebra and opportunities to be successful in mathematics.

This study explored the nature of difficulties of seventh and eighth grade students who struggled to build their conceptual understanding of early fraction ideas, in particular, ordering fractions. This study involves a remedial intervention to help seventh and eighth grade students who struggle with the concept of early fraction ideas, in particular, ordering fractions. The student group interviews were conducted before and after a remediation program using fraction circles to explore the students' understanding of early fraction ideas and the intervention effects.

## 2. Research Review

Proficiency with rational number concepts, and conceptual and procedural understanding of fraction operations is critical to the foundation of algebraic understanding (Behr et al., 1983; NMAP, 2008). Results of the 2009 National Assessment of Educational Progress (NAEP) assessment showed that 77% of the nation's eighth graders were not proficient on their cumulative mathematics score, and the average score of the number properties and operations strand was below the proficient level (U.S. Department of Education, 2010). By the time American students are in eighth grade algebra or pre-algebra mathematics, they have been exposed to fractions for several years. In pre-algebra and algebra curriculum, rational number concepts are extended to more general algebraic concepts, and incomplete understandings of fractions can lead to significant difficulties in algebra (Behr et al., 1983).

The difficulties eighth grade students have with fractions are of significant concern because staggering mathematics attrition rates have been demonstrated beginning at pre-algebra and algebra levels. When students disengage from mathematics, they limit educational and economic opportunities for themselves, and the mathematical competitiveness of the United States declines (NMAP, 2008). It is therefore critical that efforts be taken to improve the way students who struggle with rational number concepts and fractional operations are supported so that they have access to algebra and opportunities to be successful in mathematics. In particular, an earlier intervention is necessary for students who struggle with the concept of early fraction ideas.

### *2.1 Difficulty with The Concept of Early Fraction Ideas*

#### 2.1.1 Overgeneralization of whole number thinking

One factor related to the difficulty of the concept regarding early fraction ideas is student overgeneralization of whole number thinking to the field of rational numbers. Research has shown that knowledge of whole numbers dominates students thinking when they start working with fractions. Many students consider either denominators or numerators, while ignoring the comparative relationships between numerator and denominator. Students also commonly calculate the difference between numerator and denominator to compare fractions (Behr & Bright, 1984; Behr et al., 1992; Behr, Wachsmuth, Post, & Lesh, 1984; Mack, 1995; Post, Wachsmuth, Lesh, & Behr, 1985). Researchers have pursued it as a major goal to help students overcome the overgeneralization of whole number thinking and build their conceptual understanding of early fraction ideas (Cramer, Behr, Post, & Lesh, 2009a; Cramer, Wyberg, & Leavitt, 2009b).

#### 2.1.2 Rational number subconstructs

Another factor related to the difficulty of learning fractions is that rational number concepts involve distinct subconstructs: part-whole, measure, ratio, quotient, and operator. For students to develop a complete understanding of fraction concepts, they are required to understand different interpretations of each subconstruct as well as the relationships between the interpretations (Behr, Harel, Post, & Lesh, 1993; Kieren, 1976, 1988; Nesher, 1986; Post, Behr, & Lesh, 1982; Vergnaud, 1983). An early and fundamental subconstruct of rational numbers is the part-whole interpretation that represents the partitioning of a continuous or discrete object into equal sized parts. Another subconstruct of rational number is the interpretation as a measure focusing the concept of unit, for example a point on a number line. The subconstruct of ratio requires students to understand comparative relationships between two quantities and the quotient subconstruct is the interpretation of a rational number as a division of two quantities. Rational numbers can also be interpreted as an operator that transforms other objects.

In particular, the part-whole subconstruct is often considered the fundamental interpretation for rational numbers and basic to the all other interpretations (e.g., Behr et al., 1983; Kieren, 1988; Post et al., 1982). This subconstruct is based on a student's ability to partition continuous objects or a set of discrete objects into equal-sized parts. It provides students with "pre-concept" activity for equivalence and order relations of fractions and for algebraic operations on rational numbers (Post et al., 1982). Behr et al. (1983) contend that an incomplete understanding of this subconstruct can be the root of many student difficulties in algebra.

The fraction circle model has been shown to be a strong model for helping students construct understanding and create mental images of the part-whole construct of fractions (Cramer & Henry, 2002; Cramer, Post, & delMas, 2002). A significant part of this study focuses on the use and effectiveness of the fraction circle model and the mental images that students construct to help them compare and order fractions.

### *2.2 Fraction Circle*

Concrete manipulative models are physical representations that support the construction of mental images and understanding of rational number concepts. English and Halford (1995) posit that children's understanding depends on their abstraction of appropriate mental models from external representations and students need assistance in connecting the underlying concepts and procedures with models. In particular, students will encounter difficulties in conceptual and procedural understanding if they fail to connect mental models within the rational number system to their mental models with the integer number system. English and Halford (1995) recommend that as children model problems in a new number system, they be encouraged to discuss their actions and solutions and their connections to the original context of a problem. Interaction with teachers and peers can assist in the formation of strong and appropriate connections among concepts and models.

The fraction circle model used in the Rational Number Project (RNP) research and curriculum has been shown to be a powerful model for the creation of images and understanding of the part-whole construct of rational numbers and ordering and estimation processes (Cramer & Henry, 2002; Cramer et al., 2002; Cramer & Wyberg, 2009). The fraction circle model is easily constructed from students' prior knowledge of whole numbers, and the concrete manipulative aids are expected to help them build mental representations for fractions that support their understanding of the relative sizes of fractions and their generalizations about fraction size.

The sequencing and understanding of concepts, models and procedures is critical to developing a rich and deep understanding of these concepts (English & Halford, 1995; Cramer & Henry, 2002; Cramer et al., 2002). The strengths and limitations of concrete and pictorial models must be considered when selecting and utilizing models in instruction. Cramer and Wyberg (2010) suggest that the use of multiple models may be the most effective strategy when working with students as they develop conceptual and procedural understanding of fractions. Once the manipulation of a representation becomes automatic, there is a limited amount of conceptual learning that can be acquired from that representation. The introduction and use of different manipulatives can create the necessary cognitive dissonance required for insight and understanding (Behr et al., 1983).

*2.3 Rational Number Project Curriculum*

The Rational Number Project (RNP) curriculum has been created for and refined through teaching experiments in the RNP research since 1979. The RNP curriculum supports the use of multiple representations of rational numbers and structures student learning activities around the use of different models for connections among different representations. The following are four core beliefs that the curriculum embodies:

(a) Children learn best through active involvement with multiple concrete models, (b) physical aids are just one component in the acquisition of concepts: verbal, pictorial, symbolic and real-world representations also are important, (c) children should have opportunities to talk together and with their teacher about mathematical ideas, and (d) curriculum must focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer et al., 2009a).

RNP research has suggested that students should be given experience with multiple representations, as well as experience with multiple interpretations of fractions (e.g., Behr et al., 1983; Cramer et al., 2009a, 2009b; Post et al., 1982).

RNP research has shown the use of concrete models over long periods of time is critical to the help students develop mental images needed to work conceptually with fractions. In particular the fraction circle model has stood out as the most powerful model supporting student construction of mental images and concepts of fraction size, ordering and estimation as well as informal strategies for fraction operations (Cramer & Henry, 2002). More information of the RNP curriculum and research can be found on the project's website at this address:

http://www.cehd.umn.edu/rationalnumberproject/default.html

*2.4 Representations of Understanding*

Hiebert and Carpenter (1992) define mathematical understanding as connections within a structured internal network. A mathematical concept is understood if its mental representation is part of a dynamic and complex network, and "the degree of understanding is determined by the number and the strength of the connections" in the network (Hiebert & Carpenter, 1992, p. 67). Although networks of knowledge are psychological constructs, the internal representations can be revealed through external representations. Lesh, Post and Behr (1987) identify five distinct types of external representations and emphasize "translations among them, and transformations within them" are important for inferences into student understanding (Lesh et al., 1987, pp. 34).

Figure 1 shows the multiple representations translation model. Lesh, et al. (1987) describe how understanding is revealed through this model three ways. A student demonstrates understanding through recognizing the embodiment of an idea in different representational systems, through flexible manipulations of an idea within a given representational system, and through accurate translations among representational systems (p. 36). The relationship between internal and external representations is significant because the external representations with which a student interacts affects the student's internal representations and the way a student interacts with external representations reveals something about the student's internal representations (Hiebert & Carpenter, 1992, p. 66). The framework outlined in Hiebert and Carpenter's (1992) theory of understanding and the Lesh Translation Model serve as the conceptual framework of this study.

It is commonly accepted that students should learn mathematics with conceptual understanding that is rich in relationships (Hiebert & Lefevre, 1983; Hiebert & Carpenter, 1992; Silver, 1983). Conceptual knowledge is a complex web of knowledge in which pieces of knowledge are connected to the web. This type of knowledge is built through the construction of relationships among and within concepts as new information is assimilated into the network. Procedural knowledge in mathematics includes the formal language and representations of mathematics, as well as algorithms and rules for completing mathematical tasks. Procedural knowledge is structured hierarchically and can be learned meaningfully when linked to conceptual knowledge (Hiebert & Lefevre, 1983; Hiebert & Carpenter, 1992).

Mathematical procedural errors are often produced when rules are applied in ways that distort the syntactic guidelines of a problem and violate relevant mathematical concepts (Hiebert & Lefevre, 1983). When students learn procedural mathematics without conceptual understanding, mathematics is viewed as individual rules that are to be memorized in isolation rather than concepts connected into a structure of understanding. When mathematics is reduced to a set of rules and procedures, students struggle to use their mathematics knowledge flexibly and appropriately because critical connections among and between concepts and procedures are missing.

Conceptual understanding is generative in that is helps students construct new understanding and use procedures flexibly and appropriately, it promotes remembering by strengthening connections between knowledge and reduces the amount of individual bits of knowledge to be remembered, and it enhances transfer between and among concepts and procedures. Further, conceptual understanding positively influences student beliefs about mathematics by creating connections that allow students to view mathematics as more than a collection of procedures and rules that are memorized in isolation (Hiebert & Lefevre, 1983; Hiebert & Carpenter, 1992).

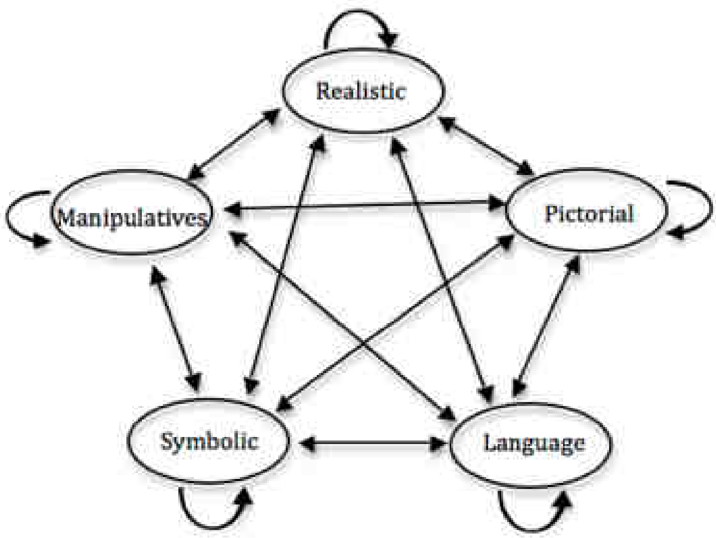



Figure 1. Lesh Translation Model

## 3. Research Goals and Research Questions

The purpose of this study is to explore (a) the nature of difficulties that seventh and eighth grade students who struggle building their conceptual understanding of early fraction ideas, in particular ordering fractions, have and (b) the influence of fraction circle model designed within the RNP Curricula (Cramer et al., 2009a, 2009b) on their developing understanding of early fraction ideas, in particular ordering fractions. The following questions guided the research:

1. What is the nature of difficulties that seventh and eighth grade students in remedial mathematics classes have with the concept of early fraction ideas, in particular, ordering fractions?

2. How does the fraction circle model designed within the Rational Number Project Curriculum influence students' conceptual understanding of early fraction ideas, in particular, ordering fractions?

## 4. Methodology

This study is a teaching experiment design research. The researchers performed the role of the teachers in the facilitation of student and problem interactions. This methodology is appropriate because the intention of the study is to explore the influence of concrete and pictorial models on student understanding (Lesh & Kelly, 2000). We pursued a qualitative methodology to address students' difficulties with the concept of early fraction ideas, in particular, ordering fractions, and how the fraction circle model influences students' conceptual understanding of early fraction ideas, in particular, ordering fractions.

### *4.1 Research Site*

The setting of this research study was a suburban 6th-8th grade middle school in the U.S Midwest with close to 750 students. The demographics of the school were 79% white, 8% Hispanic, 6% black, 6% Asian and less than 1% American Indian. Thirty percent of the school's student population received free or reduced price lunch. The school did not make Adequate Year Progress (AYP) in Mathematics. In order to improve mathematics performance, the school implemented supplemental mathematics courses for seventh and eighth grade students, called STEM (Science, Technology, Engineering, and Mathematics) Mathematics Courses. These courses were mixed ability courses and utilized computer tutorial software focusing on arithmetic concepts and procedures that are fundamental to success in algebra.

### *4.2 Participants*

The participants in this study were four 7th grade students (Group A: four females) and four 8th grade students

(Group B: two males and two females) pulled out from supplemental mathematics courses. The students were identified as being at the sixth grade mathematics level or below and have shown particular deficiencies in their understanding of fractions in their class work and district-wide assessments, the MAP Tests (see Table 1). In reporting from this study pseudonyms were used to protect students' identity.

Table 1. Student MAP data sheet

| Student Groups | Student Pseudonym | MAP | N/O | Alg | DA/P | Geo/M | Growth | Grade |
|---|---|---|---|---|---|---|---|---|
| Group A | Sasha | 226 | 226 | 224 | 219 | 234 | 2 | 6 |
| | Mary | 222 | 223 | 215 | 223 | 227 | 2 | 5 |
| | Kristie | 225 | 234 | 217 | 234 | 214 | 3 | 6 |
| | Jennifer | 221 | 215 | 229 | 224 | 216 | -3 | 5 |
| Group B | Anna | 227 | 225 | 232 | 228 | 225 | 5 | 6 |
| | Leena | 225 | 225 | 226 | 224 | 224 | 7 | 6 |
| | Keven | 220 | 215 | 222 | 220 | 224 | 17 | 5 |
| | Tony | 220 | 212 | 221 | 225 | 223 | 16 | 5 |

* MAP = MAP Math Overall Score, N/O = Number & Operation Score, Alg = Algebra Score, DA/P = Data Analysis & Probability Score, Geo/M = Geometry & Measurement Score, Growth = Change In Scores, Grade = Math Grade Level

*4.3 Intervention*

The intervention consisted of a sequence of lessons adapted from the Rational Number Project: Initial Fraction Ideas (Cramer et al., 2009a) curriculum. The complete RNP curriculum can be found at the following address: http://www.cehd.umn.edu/ci/rationalnumberproject/rnp1-09.html. The lessons involved the use of fraction circles and focused on ordering fractions. Each group of students engaged in a weekly class session for 35 minutes of class time. The researchers worked with each group of students as they engaged in the lessons and activities for four weekly class sessions (see Table 2).

Table 2. Schedule for lessons and interviews

| Weekly Class | Lesson / Activity |
|---|---|
| Day 1 | Initial Interview<br>Lesson 1<br>- Introduction to Fraction Circle Model<br>- Development of verbal and symbolic representations of fractions<br>- Begin work with flexible units |
| Day 2 | Lesson 1 Review<br>- Discuss and compare student work in group<br>Lesson 2<br>- Reciprocal relationship between number of parts of unit to size of parts<br>Lesson 3<br>- Fraction equivalence<br>- $\frac{1}{2}$<br>Fraction fill activity |
| Day 3 | Lesson 2 Review<br>- Lesson 2 HW<br>- Discuss and compare student work in group<br>Lesson 4<br>- Paper folding model<br>- Comparing two concrete models<br>- Fraction equivalence |
| Day 4 | Lesson 5<br>- Ordering two fractions comparing them to $\frac{1}{2}$ bench mark<br>Lesson 6<br>- Numerical relationship between numerators and denominators of fractions equal to $\frac{1}{2}$<br>- Estimation of greater or less than $\frac{1}{2}$<br>- Fractions greater than One |
| Day 5 | Post Interview |

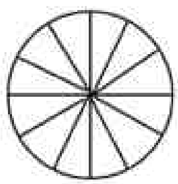
 

### *4.4 Data Collection and Analysis*

A qualitative methodology was pursued for this study. Pre and post group interviews were conducted with each student group for a dual purpose: for a sufficient identification of the nature of the students' difficulties and for an exploration of influence of the intervention. The primary sources of data for this study were transcripts of student group interviews that were video and audio-recorded, student work samples and researcher field notes. The interviews followed interview outlines from the Rational Number Project (RNP) curricula (Cramer et al., 2009a; Cramer et al., 2009b). The RNP curriculum lessons, student activities and interview protocols are designed to provide students opportunities to work flexibly within the different representations.

As mentioned before, the framework outlined in Hiebert and Carpenter's (1992) theory of understanding and the Lesh Translation Model (Lesh et al., 1987) serve as the conceptual framework used to analyze the qualitative data gathered in this study. The data analysis focused on the representations in the manipulative aids (fraction circle), pictures (fraction circles or strips), symbols, language, and real-world situations, as well as transfer among and within these representations. The data analysis was conducted by combining two coding strategies - starting with some preset categories (codes) and adding others as they become apparent from the data (Corbin & Strauss, 2008; Miles & Huberman, 1994). The list in Table 3 was used to code the data for this study.

Table 3. List of categories of student strategies for ordering fractions

| Categories | | Codes | Definition & Theoretical References |
|---|---|---|---|
| Whole Number Thinking | | WNT | The whole number knowledge dominates students' thinking when working with fractions: Students consider only or separately either denominators or numerators, ignoring the ratio subconstruct of fractions. They often calculate the difference between numerator and denominator to compare fractions (e.g., Behr & Bright, 1984; Behr et al., 1992; Behr et al., 1984; Post et al., 1985). |
| Informal Strategies | Same Numerator | SN | One of the informal strategies for ordering fractions identified by the Rational Number Project (RNP) (Cramer et al., 2009a): (When comparing fractions with the same numerator) Students reveal their understanding that an inverse relationship exists between the number of parts that a unit is partitioned into and the size of the parts. |
| | Same Denominator | SD | One of the informal strategies for ordering fractions identified by the Rational Number Project (RNP) (Cramer et al., 2009a): (When comparing fractions with the same denominator) Students compare parts of the unit that are the same size. |
| | Transitive | TS | One of the informal strategies for ordering fractions identified by the Rational Number Project (RNP) (Cramer et al., 2009a): Students use a single outside value (e.g., $\frac{1}{2}$) to compare two fractions (e.g., when comparing $\frac{3}{7}$ and $\frac{5}{9}$, i.e., $\frac{3}{7}<\frac{1}{2}<\frac{5}{9}$). |
| | Residual | RS | One of the informal strategies for ordering fractions identified by the Rational Number Project (RNP) (Cramer et al., 2009a): Students focus on the missing part(s) from the whole unit in judging the relative size of the fractions (e.g., $\frac{5}{6}$ is closer to the whole than $\frac{3}{4}$ because $\frac{1}{6}$ is less than $\frac{1}{4}$, so $\frac{5}{6}$ is bigger than $\frac{3}{4}$). |
| Procedural Strategies | Least Common Denominator | LCD | One of procedural strategies usually taught without using mental images of fractions to judge the relative sizes: Using the least common denominator (LCD) method. |
| | Division | DV | One of procedural strategies usually taught without using mental images of fractions to judge the relative sizes: Dividing numerators by denominators. |
| | Reducing | RD | Reducing fractions: Dividing the numerator and denominator of a fraction by the same non-zero number to get the fraction's lowest terms. |
| Visualization Strategies | Fraction Strips | FS | Using a physical visual representation mode: Fraction strips e.g.,  |
| | Fraction Circles | FC | Using a physical visual representation mode: Fraction circles e.g.,  |

The data were organized into the categories, and the number of times that each student strategy came up was counted to figure out general patterns in the data. The frequency table reveals a rough estimate of relative importance and subtle differences within and between the categories or subcategories (see Table 4). The type of table including frequency counts is not suited to statistical analysis, but it reveals general patterns in the data (Miles & Huberman, 1994).

Table 4. Student strategies used for ordering fractions before and after the intervention

| Student Groups | Interviews | Whole Number Thinking | Informal Strategies | | | | Procedural Strategies | | | Visualization Strategies | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | SN | SD | Transitive | Residual | LCD | Division | Reducing | FS | FC |
| Group A | Pre | 14 (43%) | 0 | 2 (6%) | 0 | 0 | 1 (3%) | 5 (15%) | 0 | 10 (30%) | 1 (3%) |
| | Post | 9 (34%) | 3 (12%) | 4 (15%) | 1 (4%) | 0 | 0 | 0 | 5 (19%) | 1 (4%) | 3 (12%) |
| Group B | Pre | 9 (75%) | 1 (8%) | 1 (8%) | 1 (8%) | 0 | 0 | 0 | 0 | 0 | 0 |
| | Post | 10 (50%) | 3 (15%) | 2 (10%) | 0 | 0 | 0 | 1 (5%) | 2 (10%) | 0 | 2 (10%) |

* SN = Same Numerator, SD = Same Denominator, LCD = Least Common Denominator, FS = Fraction Strips, FC = Fraction Circles

**5. Results and Discussion**

This study identified the whole number dominance strategies used by the students for ordering fractions before and even after the intervention. The analysis of the interview data shows that the students in both Group A and Group B frequently relied on whole number thinking for ordering fractions (see Table 4). The overgeneralization of whole number thinking to the field of rational numbers is one of reasons for why the students' difficulty with the concept of fraction order persists. The following excerpt illustrates this.

Keven: *This one is the smallest (pulls* $\frac{14}{15}$*).*

*(Emma puts* $\frac{5}{12}$ *next to it)*

*[Keven arranges the remaining:* $\frac{1}{10}, \frac{3}{5}, \frac{3}{4}$*]*

Researcher: *Do you agree, everyone?*

Tony: *Hmmm, I kind of think this one's the smallest (moves* $\frac{1}{10}$ *to position on the left) cause this has the lowest number on top.* $\frac{14}{15}$ *is the highest (moves to farthest right position). (Moves* $\frac{3}{4}$ *next to* $\frac{1}{10}$*, then* $\frac{3}{5}$ *then*

$\frac{5}{12}$*)*

Researcher: *So Tony, tell me about your order.*

Tony: *It goes from least to greatest.*

Researcher: *And what part of the fraction are you looking at?*

Tony: *The top…and the bottom. (Points to* $\frac{1}{10}$*) This one's got the smallest top of the fraction. (Points to* $\frac{3}{4}$ *and* $\frac{3}{5}$*) And these two I ordered by bottom cause they're the same on top.*

Keven: *It doesn't mean the bottom number's bigger, cause the bottoms numbers' (indecipherable) are less there is more, like 15 and the bottom of* $\frac{3}{4}$*,* $\frac{3}{4}$ *is bigger than 4…*$\frac{4}{15}$*, ah* $\frac{14}{15}$*.*

When a researcher asked four students in Group B to arrange five paper pieces with fractions on a desk in an order, Keven quickly ordered them based on the denominators. He arranged them in the opposite order of denominators from left to right, mentioning "this one ($\frac{4}{15}$) is the smallest." Emma helped him arrange the fractions putting $\frac{5}{12}$ next to $\frac{14}{15}$. The ordering was $\frac{14}{15}, \frac{5}{12}, \frac{1}{10}, \frac{3}{5}$, and $\frac{3}{4}$.

Without considering the numerators at all, they seemed to recognize a pattern focusing on the denominators. Even though Keven seemed to coherently show his understanding that an inverse relationship exists between the number of parts that a unit is partitioned into and the size of the parts, his focus was only stuck in the denominators: "cause the bottoms numbers' (indecipherable) are less there is more." He revealed his misconception on fraction order: as denominators are bigger, fractions are smaller. Although this reasoning is correct for considering unit fractions, Keven was not considering the numerator and denominator together as parts of a single number. Instead, he was extending the reciprocal relationship between number of parts of a unit's partition and the size of the parts through whole number reasoning.

When prompted if others agreed with Keven or not, Tony moved $\frac{1}{10}$ to the first left side and $\frac{14}{15}$ to the final right side, mentioning "cause this ($\frac{1}{10}$) has the lowest number on top. $\frac{14}{15}$ is the highest" and moved $\frac{3}{4}$ next to $\frac{1}{10}$, then $\frac{3}{5}$, then $\frac{5}{12}$. He completely changed the arrangement of the paper pieces focusing on just the numerators, ignoring the denominators. His ordering strategy indicated that fractions are larger as the numerators are bigger, without considering the denominators at all. However, when Tony explained about the ordering of $\frac{3}{4}$ and $\frac{3}{5}$, his focus moved to the denominators: "these two I ordered by bottom cause they're the same on top."

Here is another excerpt illustrating the whole number dominance strategies used by the students for ordering fractions.

Researcher: *(And) what about you? Are they ($\frac{3}{5}$ and $\frac{3}{4}$) the same or which one is less?*

Mary: *I think that three fourths is bigger.*

Researcher: *And why do you think that?*

Mary: *Because, like, there's only going to be one space left over, and in this one there's gonna be two spaces left.*

Researcher: *Kristie, how do you think, or what are you thinking about that lets you know three sevenths is less (than $\frac{8}{12}$)?*

Kristie: *Because if you look at the gap, wait that doesn't make sense, umm…*

Researcher: *What do you mean by gap?*

Kristie: *Like if you count three, four, five, six, seven, there's four spaces.*

Researcher: *Well what about eight twelfths?*

Kristie: *It has four. They're the same. I'm changing my answer.*

Researcher: *Ok, … what are you thinking (looking at Mary)?*

Mary: *I think that eight twelfths is bigger because I reduced it and got two thirds. And only one space was left over.*

When a researcher asked Mary to order the fractions, $\frac{3}{5}$ and $\frac{3}{4}$, she correctly responded that "three fourths is bigger" than three fifths. However, her reasoning was wrong. She relied on whole number thinking focusing on the differences between numerator and denominator: "Because, like, there's only going to be one space left over, and in this one there's gonna be two spaces left." She wrongly applied the "residual" strategy (see Table 3) to order the fractions just focusing on the parts that were not colored or shaded.

When prompted why Kristie wrote down "$\frac{3}{7} < \frac{8}{12}$" on her notebook, she incorrectly switched her answer: "they're the same ($\frac{3}{7} = \frac{8}{12}$)." She used subtractions to compare the fractions, and looked at the "gap" between the denominator and numerator: "there's four spaces." Mary maintained the same reasoning strategy that she used in the prior question for ordering fractions relying on the whole number thinking. She reduced $\frac{8}{12}$ to get $\frac{2}{3}$, and focused on the parts that were not shaded: "I think that eight twelfths is bigger because I reduced it and got two thirds. And only one space was left over." Even though her answer was correct, her reasoning was wrong because she did not care the comparative relationship between numerator and denominator. As mentioned before, it was a wrong use of the "residual" strategy.

In these cases, the students looked at the difference between numerator and denominator to identify equivalence and order relations of fractions. This strategy is one of whole number dominance strategies (e.g., Behr et al., 1984; Cramer et al., 2009b).

Another finding from the analysis of the interview data is that the students revealed minimal use of informal ordering strategies that reflect their use of mental images of fractions to judge the fraction's relative size (see Table 4). The informal strategies for ordering fractions have been identified by the Rational Number Project (RNP). When using the four strategies: same numerator, same denominator, transitive and residual strategies (see Table 3), students use their mental images of fractions to judge the relative size of the fractions. Thus, the informal strategies involve more conceptual than procedural understanding of the concept of initial fraction ideas (Cramer et al., 2009a). However, our interview data showed that the students infrequently used the informal strategies for ordering fractions. They seldom constructed the informal ordering strategies nor did they use mental images of fractions. An additional interesting fact shown in the Table 4 is that the students showed little reliance on the procedural strategies usually taught for ordering fractions as well as the informal strategies.

Even though the whole number thinking still dominated after the intervention, Table 4 reveals some subtle (but meaningful) evidence of improvement. The frequencies of whole number thinking (34%) and informal strategies (33%) used by the students in Group A are almost the same. In addition, the decrease of the fraction strips use (from 30% to 4%) became apparent in the data from Group A (see Table 4). Before the intervention, there was an evidence for misleading of the fraction strips use to whole number thinking. For example, the students considered the denominators as the number of "rooms" in their fraction strip model (they called the pieces divided from a whole "rooms") and finally focused on the total number of "rooms" colored according to the numerators for comparing two fractions. They considered the numerators and denominators as four independent numbers then treated them like whole numbers. Note that the term "room" is linked to whole number. The decrease of the fraction strips seems to linked to the decrease of whole number thinking (from 45% to 34%) and the relative increase of informal strategies use by the students in Group A (from 6% to 33%). This finding indicates that students should be carefully taught with even concrete models, such as fraction strips. The lack of understanding of how to use concrete models could interfere with the development of understanding of fraction concepts.

Finally, the excerpt below shows how a student in Group A used an informal strategy, the "transitive" strategy (see Table 3), to identify an order relation of fractions.

Researcher: […] Are they ($\frac{3}{7}$ and $\frac{8}{12}$) the same or is one less?

Sasha: Three sevenths is less.

Researcher: How do you know, Sasha?

Sasha: Because eight twelfths is bigger than half and three sevenths is almost half, but isn't.

Sasha clearly explained her reasoning how she compared the fractions, $\frac{3}{7}$ and $\frac{8}{12}$, by using a single outside value of $\frac{1}{2}$: $\frac{3}{7} < \frac{1}{2} < \frac{8}{12}$.

**6. Conclusion and Implications**

This study investigated the nature of difficulties of seventh and eighth grade students who struggled to build their conceptual understanding of early fraction ideas, in particular, ordering fractions, and the influence of the fraction circle model designed within the RNP Curricula on developing their understanding of early fraction ideas, in particular, ordering fractions. The analysis of interviews with the students who struggle showed that their difficulties with the concept of early fraction ideas persist. These persistent difficulties of early fraction ideas were due to a lack of the students' understanding of the part-whole and ratio subconstructs of fractions. The students mainly relied on whole number thinking for ordering fractions. The students revealed minimal use of informal ordering strategies that reflect their use of mental images of fractions to judge the fraction's relative size. Even after the four weeks of instruction, the students' overgeneralization of whole number thinking to the field of rational numbers was still the main factor related to their difficulties with the concept of ordering fractions. The whole number dominance strategies were often used by the students for ordering fractions before and even after the remediation intervention. However, there was subtle (but meaningful) evidence that the intervention using the fraction circle model had a positive influence on the students' understanding of fraction concepts: an increasing use of informal ordering strategies and a decreased reliance on whole number thinking.

A limitation of this study is the short intervention time period. The instruction for the four weekly class sessions was not enough for the students to overcome their whole number thinking. There were not any clear effects of the intervention (at least in the statistical sense) on the students' understanding of fraction concepts. This might be due to the short intervention time period. The students need more time with concrete models, such as fraction circles, to develop mental images and a conceptual understanding of fractions. This limitation provides opportunities for future research by suggesting studies to investigate the influence of a remediation intervention with enough time period (e.g., at least more than the total of 7 hours that is about three times the intervention time period of the current study) using the fraction circle model on students' understanding of fraction concepts. Further research, such as a larger-scale quantitative research, is required for generalizability.

Concrete models help students develop their own understanding of fraction concepts based on their existing knowledge, such as their knowledge of whole numbers. However, the misleading use of the fraction strips that supported their whole number thinking illustrated how an incomplete understanding of a concrete model could interfere with the concept of fraction order. Lessons and activities that involve the use of concrete models with a complete understanding of them would show promise for effective remediation processes that focus on the development of mental images regarding fractions, and ultimately, a conceptual understanding of fraction concepts. Further research is also needed to investigate how to effectively support students' complete understanding of the use of concrete models using appropriate and accurate language.

This study suggests the need of a remedial intervention for early secondary students showing the persistent difficulty with early fraction ideas. Students need to be given enough time with not only concrete models but also appropriate usage of language to support a complete understanding of how to use the models. An earlier intervention involving the use of different models for connections among multiple representations should also be considered for students to develop their conceptual understanding of fraction concepts.

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