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Economic Trend Resistant$2^{n-(n-k)}$ Designs of Resolutions III and IV Based on Hadamard Matrices

Ahlam Guiatni prof. Hisham Hilow
Mathematics Department-The University of Jordan-Amman –JORDAN

Abstract
This article utilizes the Normalized Sylvester-Hadamard Matrices of size 2^kx$2^k$and their associated saturated orthogonal arrays OA(2^k, 2^k - 1, 2, 2) topropose analgorithmbased on factor projection (Backward/Forward) for the construction of three systematic run-after-run$2^{n-(n-k)}$ fractional factorial designs: (i) minimum cost trend free $2^{n-(n-k)}$designsof resolution III (2^{k-1}≤n≤2^k– 1 – k)by backward factor deletion (ii) minimum cost trend free $2^{n-(n-k)}$ designsof resolution III (k+1≤n≤ 2^{k-1}– 2+k) by forward factor addition (iii) minimum costtrend free $2^{n-(n-k)}$ designsof resolution IV (2^{k-2}≤n≤2^{k-1}-2) ,where each $2^{n-(n-k)}$design is economic minimizing the number of factor level changes between the 2^ksuccessive runs and allows for the estimation of all factor main effects unbiased by the linear time trend,which might be present in the 2^ksequentially generated responses. The article gives for each $2^{n-(n-k)}$design: (i) the defining contrast displaying the design's alias structure(ii) the k independent generators for sequencingthe design's $2^{n-(n-k)}$ runs by the Generalized Fold over Scheme and (ii) the minimum total cost of factor level changes between the $2^{n-(n-k)}$ runs of the design. Proposed designs compete well with existing systematic$2^{n-(n-k)}$designs (of either resolution) in minimizing the experimental costandin securing factors' resistance to the non-negligible time trend.
Keywords: Sequential fractional factorial experimentation; Time trend free systematic run orders; Generalizedfoldover scheme for sequencing experimentalruns; The total cost of factor level changes between successive runs; The Normalized Sylvester –Hadamard Matrices; Orthogonal Arrays and factor projection; Design resolution and the alias structure.
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1. Introduction
Experiments are carried out in all fields:industrial, educational, agricultural, medical, etc.,where experimentation has led to many innovations and discoveries. Experiments investigate generally the effect of one or several factors on an outcome by manipulating the experimental runs(i.e. humans ,animal, trees, etc.) with these factors, where some multi-factor experiments are called factorial experiments allowing the investigation of the effect of several factors andtheir interactions. Factorial experiments are symmetric or asymmetric, where full factorial 2^k experiments are symmetric and more economical than other full p^kfactorial experiments (p>2). Factorial 2^k experiments are mainly used at the start of an experimental investigation in order to identify the most significant factors without the interest ofcharacterizingthese effect precisely (linear, quadratic, etc). However, factorial 2^k experiments grow in size and complexity as the number of factors get larger, where experimentation becomes costly and unmanageable.Therefore, fractional factorial 2^{k-p} experiments or orthogonal arrays (regular or non-regular) are substitutes in early stages of factorial experimentation , since theyare more economical requiring less experimentation effort and are less costly if high order factor-interactions are negligible.

Full 2^k or fractional 2^{k-p}factorial experiments are often conducted randomly. However, randomization of allruns of full or fractional factorial experiments may result in large number of factor level changes between runs, rendering experimentation costly and /or impractical, especially if these experiments involve factors with hard-to-vary levels, like for instance oven temperature.Therefore, full 2^k or fractional factorial 2^{k-p} experiments involving difficult-to-vary factors should be carried out sequentially run- after- run or block of runs after block (i.e. not randomly but systematically) .

The experimental cost when carrying out full 2^k or 2^{k-p}fractional factorial experiments sequentially run-after-run involve both the cost of changing factor levels between successive runs as well as the measurement cost of each experimental run. For more on this cost issue, See [25]. We will however concentrate on minimizing the former cost, where we will assume equal cost for changing levels of allthe k the two-level factors (A_1 , A_2 , A_3 , …, A_k) of these 2^kor2^{k-p}experiments .We will also aim at achieving factor's resistance to the time trend, which might be present in the sequentially generated 2^k or 2^{k-p}responses,which may bias factor effects. This time trend effect could be smooth of linear /quadratic form or it could be stochastic ofvarying serial correlations. The former time trend form (i.e. the polynomial) will be adopted in this research.

For run-after-run full 2^kfactorial experimentation,there are2^k! run orders (i.e. permutations) while for run-after-run 2^{k-p} fractional factorial experimentationthere are 2^{k-p}! run orders, where not all these run orders(permutations) of either experimentation scheme are economical with regard to the number of factor level changes nor all are resistant to the time trend effect. One of the 2^k! run orders of the full 2^k factorial experiment

in k two-level factors (A_1 , A_2 , A_3 , …,A_k) is the well-known systematic standard order:
(1), a_1, a_2 , a_1a_2 , a_3 , a_1a_3 , a_2a_3 , $a_1a_2a_3$, a_4, a_1a_4 , a_2a_4, $a_1a_2a_4$, a_3a_4, $a_1a_3a_4$, $a_2a_3a_4$, $a_1a_2a_3a_4$,a_5, ….,$a_1a_2a_3$……a_k(1.1) where the respective number of level changes for these k factorsbetween the 2^ksuccessive runsare:(2^k-1),(2^{k-1}-1),(2^{k-2}-1) ,(2^{k-3} -1) ,… , ($2^{k-(k-1)}$-1) , giving a total cost of factor level changes [(2^k-1)+(2^{k-1}-1)+(2^{k-2}-1) +(2^{k-3}-1) +… + ($2^{k-(k-1)}$-1)]=(2^{k+1} – k-2),which is not minimal (i.e. costly).The minimal total cost is (2^k- 1) ,where only one factor level change is made between any two successive runs of the 2^k runs. The standard orderin (1.1) is not only costly [i.e. being not minimal], it is also not time trend resistant since none of the k main effects A_i (i =1,2,…,k)is trend free(i.e. orthogonal) under this run order , where the dot product of each main effect column A_i (i =1,2,…,k) with the column of runs order vector (1 up to 2^k) is not zero. This dot product is often referred in the language of this research as the Time Count statistic. For more on this statistic,see [15].

2. Literature review and description of the research problem

Research on sequencing the 2^k runs of the full 2^k factorial experiment run after run has concentrated on finding run orders better than the standard order (1.1) in : (i) minimizing the number of factor level changes between successive runs and/or in (ii) securing factors' resistance to the time trend effect. Similarly, research on sequencing the 2^{k-p} runs of the 2^{k-p} fractional factorial experiment has concentrated on these two main objectives, where different algorithms exist for sequencing the 2^k runs of the full 2^k factorial experiment and also different algorithms exist for sequencing the 2^{k-p} runs of the 2^{k-p} fractional factorial experiment for achieving either of these two optimality criteria .

Systematic full or fractional factorial experimentation started with the works of [11],[13],[14] and [15],where small size full or fractional two-level factorial experiments (in at most 32 experimental runs and in at most 5 two-level factors)have been sequenced to achieve either or both of the above two optimality criteria. For a brief review on run-after-run full 2^k factorial experimentation (excluding the one block at a time scheme), we start with the work of [9] who proposed to sequence all 2^k runs using an algorithmic approach called the Generalized Foldover Scheme (GFS) which employs k independent run generators to sequence all 2^kruns, where all the k factor main effects (A_1 , A_2 , A_3 , …, A_k) are robust to the polynomial time trend and where the number of factor level changes between successive runs is nearly minimum totaling (2^k + 11), which isabove the minimal (2^k-1) by only twelve . Different sets of independent GFS generators sequence all 2^kruns differently, where some GFS generator sets achieve one of these optimality criteria while other GFS generator sets achieve both.However, [9] did not characterize these different sets of generators but employed aparticular GFS generator setto achieve factors' time trend resistance regardless of minimizing factor level changes.It is worth to note that theGFS approach does not recover all 2^k ! possible run orders of the full 2^k factorial experiment but only a subset of them, yet it produces good run orders in terms of the above optimality criteria. The GFS approach fixes the first run to be the null run (1)=0000…0 and also it fixes the k run generators to be located at 2nd,3rd,5th,9th,….[(k-1)+1]stin the entire sequence of the 2^k runs.

Another major contributionis the work of [5],who utilized the layout of the full 2^k factorial experiment [under the standard order(1.1)]in (2^k -1) columns representing all k main effects (A_1 , A_2 , A_3 , …, A_k) and their interactions,then applied the Interaction-Main effects Assignmentto assign k independent interaction columns as new k main effects,hence generating new run ordersrobust to the polynomial time trend but in extremely large number of factor level changes (i.e. large experimentation cost). This assignment approach does not recover all possible 2^k! run orders of the full 2^k factorial experiment but rather a subset of them, where some run orders can be generated by the GFS approach.

The four algorithms of [2], [3], [8] and [12] sequence the 2^kruns of the full 2^k factorial experiment in minimal total of number of factor level changes [i. e.(2^k-1)] maintaining only one factor level change between any two successive runs of the entire 2^kruns. Hence, minimality of factor level changes is not uniquely achievable,where different algorithms may achieve different minimal run orders,yet not all these minimal run orders are time trend resistant.Algorithms [2] and [3] can be sequenced by the GFS approach while algorithms [8] and[12]can not. Also, algorithms of [2] and[8] have increasing pattern of factor level changes for the k factors A_i (i =1,2,…,k),hence assigning hard-to-vary factors to the design's first factor columns in minimal level changes,whereas algorithms [3] and [12] have decreasing patternsassigningthesefactors to the last columns of the design. Reference [18] has conducted a comparison among the four runs sequencing algorithms: [8], [12], [5], and [9] for thefull2^k factorial experiment with regard to the above two optimality criteria as well as regarding the possibility of the usage of the GFS to sequencing the 2^kruns and also regarding the characterization of the pattern of factor level changes(monotonic or not) .

Now for a brief survey on algorithms for sequencing the 2^{k-p} runs of the more economic2^{k-p} fractional factorial experiments run after run(i.e. not block after block), we start with reference [9] who provided three sets of independent GFS generators for sequencing runs of three 2^{k-i}designs (i=1,2,3) in respective minimum total cost of factor level changes:2(2^k-1),(2^{k-1}-1) and (2^{k-2} +13) but regardless offactors'time trend resistance. The GFS set for each 2^{k-i} design has i generators (i=1,2,3).Defining contrasts displaying the design's alias structure were provided for each 2^{k-i} designbut higher levels of fractionation(i.e. i>3) were not considered and the pattern of factor level changes was not characterized.Reference[10] provided a small catalog of GFS sequenced 2^{k-p} fractionated experiments [k<16 and p<8], where all factor main effects are robust against the polynomial time trend and where the total number of factor level changes arekept minimum . The (k-p) independent run generators for each 2^{k-p}design and the total cost of factor level changes were provided but neither the resolution nor the defining contrast were given nor the pattern of factor level changes was characterized. Reference [6] utilized the standard order of the full 2^k experiment in (1.1) laying out all main effects A_i (i =1,2,…,k) and their interaction columns in increasing number of level changes [from 1 up to (2^k-1)] then constructed two types of $2^{n-(n-k)}$designs:minimum cost $2^{n-(n-k)}$ designs of resolution III (2^{k-1}≤n≤2^k-1)and minimum cost $2^{n-(n-k)}$ designs of resolution IV (2^{k-2}≤n≤2^{k-1})but regardless of factors' time trend resistance.However, neither the defining relations nor the GFS generator sets were reported nor the minimal total cost of factor level changes wascomputed. [19] elaborated on the work of [6] and constructed minimum cost trend free $2^{n-(n-k)}$ designs of resolution IV but without providing the GFS generators.

[4]employed an algorithm based on the GFS approach to sequence runs of symmetric orthogonal arrays OA(N,n,q,3) of resolution III in minimum number of factor level but regardless of factors' time trend resistance, where factors have prime number of levels greater than 2 and where the number of factors is constrained to [(N/q-1)/(q-1) +1 ≤n≤(N-1)/(q-1)] to ensure runs non-duplication. Defining relations were not provided, norprovision was made for the total cost of factor level changes. [1] constructed half fractions (i.e. $2^{(k+1)-1}$) from the full 2^k factorial experiment having itsk factorsA_i (i =1,2,…,k) laid out in minimaltotal number of factor level changes [i.e. (2^k -1)], then incorporated an additional factor $A_{k+1}=A_1A_2A_3…A_k$represented by the interaction of all the k factorsA_i (i =1,2,…,k),where the total number of level changes for all (k+1) factors is in increasing pattern totaling =[$1+2+2^2+2^3+…+2^{k-1}+2^{k-1}$] =$2(2^{(k-1)}-1)$,yet not all these (k+1) factorsare time trend free. Higher levels of fractionation (i.e. $2^{(k+i)-i}$, i>1) were not considered and the GFS approach can not be applied to recover the run order of these $2^{(k+1)-1}$half fractions.

Extending the scope of the interaction main-effect assignment of [1], reference [3] has provided an algorithm based on the reverse foldover scheme to generate full 2^k factorial experimentin minimal number of factor level changes [i.e. (2^k -1)] then applying the interactions-main effects assignment to create additional two-level factors for the construction of a small catalog of systematic 2^{k-p}designs (4 ≤k≤9 and 1≤p≤5), where all factor main effects are linear trend free but regardless of the minimality of the cost of factor level changes. Defining contrasts were given for each systematic2^{k-p}design but no provision was made for the total cost of factor level changes.These trend free 2^{k-p} designscan not however be sequenced by the GFS approach.

[24] proposed an algorithm based on parity check matrices of binary linear codes to find the GFS independent run generators forsequencing runs of regular orthogonal arrays (i.e. 2^{k-p} designs) so that their main effects are time trend freebut regardless of minimality of factor level changes.No catalog is reported and also no provision is made on how to construct these parity check matrices.The algorithmwas however illustrated using some examples from special binary linear codes, namely Reed Muller codes, cyclic codes and BCH codes. Finally , [22] represented experimental runs of regular 2^{k-p}designs as graph vertices then applied Travelling Salesman Algorithm to locate graph paths (i.e. run orders) of minimal distance without regard to factors' time trend resistance.These minimallysequenced 2^{k-p}designs(4 ≤k≤15 and 1 ≤p≤11) cannot however be sequenced by the GFS approach ,since many of theserun orders do not start with the null treatment (1)=(000…0000.Defining contrasts were provided but neither the factors'pattern of level changes nor the total cost of factor level changes werereported.

Having completed this literature review and having seen that it is not yet complete especially forfractional factorial experimentation (regular or non-regular),where it lacks systematic 2^{n-k}fractional factorial experiments of resolutions III and IV inminimum cost of factor level changes and resistant to the time trendbut without limiting either the number of factors nor the fractionation level.Therefore, this article addresses this problem utilizing the Normalized Sylvester –Hadamard Matrices of order $2^k$and their associated saturated orthogonal arrays OA(2^k , 2^k -1, 2, 2) to construct by factor projection threetypes of systematic $2^{n-(n-k)}$ fractional factorial designs: (i) minimum cost trend free $2^{n-(n-k)}$ designsof resolution III (2^{k-1}≤n≤2^k -1-k) by backward factor deletion (ii) minimum cost trend free $2^{n-(n-k)}$ designsof resolution III (k+1≤n≤ 2^{k-1} -2+k) by forward factor addition (iii) minimum costtrend free $2^{n-(n-k)}$ designsof resolution IV [2^{k-2}≤n≤2^{k-1}-2],where each $2^{n-(n-k)}$ design (of either resolution) is economic in minimum number of factor level changes and allows for the estimation of all main effects A_i (i =1,2,…,n) unbiased by the linear time trend.Theoretical reference for thisconstruction will be based on results in [18], [20] and [21].

The rest of this paper proceeds as follows: Section 3 introduces Hadamard matrices and their subclass the Normalized Sylvester-Hadamard matrices of order 2^k then the section examines orthogonality of their columns to the time trend factor. Section 4 discusses (through factor projection)the relationship between the Normalized Sylvester-Hadamard matrices of order 2^k and their associated saturated orthogonal arrays OA(2^k,2^k-1,2,2) with full $2^k$and fractional 2^{k-p} factorial experiments,where various illustrative factor projections will be given when k=4 . Sylvester -Hadamard matrices of order 2^k and their associated saturated OA(2^k,2^k-1,2,2) are then utilized in Section 5 for the construction of the three proposed minimum costtrend free $2^{n-(n-k)}$ fractional factorial designsbythe factor projection process. Section 6 gives a brief discussion and conclusion about run-after-run fractional 2^{n-p}factorialexperimentation**.**

3. Sylvester-Hadamard matrices of size 2^kx$2^k$and time trend resistance of their 2^k columns.

This section introduces Hadamard matrices and their subclass the Normalized Sylvester-Hadamard matrices of size 2^kx2^kthen examines time trend resistance of their 2^kcolumns .

AHadamard matrix H_m of order m is a square matrix with entries +1 and -1 such that $H_m'H_m = H_mH_m'=mI_m$wherematrixI_m is the identity.That is, all rows (columns) of the HadamardmatrixHm are orthogonal, where each row (column) has m/2 +1's and m/2 -1's . Alsoany two rows (columns) of matrix $H_m$have equal number of the four pairs: (+1,+1) ,(-1,-1),(+1,-1) , (-1,+1), namely m/4. Additional properties of Hadamardmatricesare:

i. A Hadamardmatrix$H_m$canbe changed into another equivalentHadamardmatrix by permuting its rows (columns) and/or by multiplying its rows(columns) by −1.

ii. TheHadamardmatrix$H_m$whose all its entries in the first row (column) are+1'sis called Normalized, where any Hadamard matrix $H_m$is equivalent [by property (i)] to a Normalized Hadamard matrix of the same order.All rows (columns) of the Normalized Hadamard matrixH_m(except the first) arepair-wise orthogonal.

iii. The order of any Hadamard matrix $H_m$iseither 1, 2 or m=4n, where n is a positive integer.That is, the order of all Hadamard matrices (exceptmatrices H_1 and H_2) is a multiple of 4,where a subclass of the Hadamardmatrices $H_m$are the matrices of orderpowers of2,which are themain focus in this research.

iv. TheKroneckerproduct of any two Hadamard matrices $H_m \otimes H_n = H_{mn}$is Hadamard too. In particular

$H_{2n} = H_2 \otimes H_{n=} H_n H_n$Where $H_2=$ $\begin{pmatrix} 1 & 1 \end{pmatrix}$

H_n-H_n1 $\begin{pmatrix} -1 \end{pmatrix}$

Hence, $H_2 \otimes H_2 = H_4 =$
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

The Kronecker productof the Hadamardmatrix $H_2$by itself four times is the matrix $H_{16} = H_2 \otimes H_2 \otimes H_2 \otimes H_2 = H_2 \otimes H_8 = H_4 \otimes H_4$laid out explicitly in Table (3.1) representing the Normalized Sylvester-Hadamard matrix $H_2^4$of size 16x16.

Table(3.1):The NormalizedSylvester-Hadamard matrix $H_{16} = H_2 \otimes H_2 \otimes H_2 \otimes H_2$

Run Order (i.e. Row Number)	Columns of the NormalizedSylvesterHadamard matrix H_{16}															
	C_1	**C_2**	**C_3**	**C_4**	**C_5**	**C_6**	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**	**C_{12}**	**C_{13}**	**C_{14}**	**C_{15}**	**C_{16}**
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
3	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
4	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
5	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
6	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
7	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
8	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
9	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
10	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
11	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
12	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
13	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
14	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
15	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
16	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
Number of column sign changes	**0**	**15**	**7**	**8**	**3**	**12**	**4**	**11**	**1**	**14**	**6**	**9**	**2**	**13**	**5**	**10**

A glance at matrix H_{16} in Table (3.1) revealsthat its 15 columns (except the first) are orthogonal, where the pairwise dot product between any two of these 15 columns is zero. The number of +1's and -1's in each such columns are balanced (8 +1's and 8 -1's) and the number of sign changes in these15 columns range from 1 up to 15=(2^4-1),but they are not in increasing order.

Matrix H_{16} in Table (3.1) has the further property that its four rows {2, 3, 5 and 9} are such that (i) the content of : (i) row 2 alternate between 1 and -1 eight times, (ii) row 3 alternates between the double 1,1 and the double -1,-1 four times, while (iii) row 9 consists of 8 +1's followed by 8 -1's. These 4 rows are independent and can generate the remaining eleven rows of the matrix H_{16} by dot products(twice, thrice, four times) . Specifically, row 4 is the dot product of rows 2 and 3 while the last row is the dot product of all these four generator rows. Columns of the matrix H_{16} have also this property, where the 4 columns {C_2 ,C_3,C_5,C_9} with sign changes {15, 7, 3 and 1} have the same respective alternating patterns and are also generators of all other eleven columns of the Matrix H_{16}.Columns (rows) of Matrix H_{16} can however be generated by dot products of other 4 independent generator rows(columns).

Generalizing results of matrix H_{16} to matrix H_{2^k} which is the successive k times Kronecker product of the Hadamardmatrix H_2 by itself

$$H_2 \otimes H_2 \otimes H_2 \ldots \otimes H_2 = H_2 \otimes H_{2^{k-1}} = H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}$$

yields the Normalized Sylvester-Hadamard matrixof order 2^k (i.e. power of 2) which is symmetric and of size 2^kx2^k and where the number of sign changes in its rows (columns) range from [0,1,2,…,(2^k-1)] but are not in increasing order. In addition, matrix H_{2^k} has the property that its k rows numbered {2,3, 2^2+1 , 2^3+1, …, 2^{k-1}+1 } are independent and can generate the remaining (2^k-k-1) rows by dot products (twice, thrice, …, k times). Row 2 alternates between 1 and -1 2^{k-1} times , row 3 alternates between the double {1,1} and the double{ -1,-1} 2^{k-2} times, row 5 alternates between the quadruple{ 1,1,1,1} and the quadruple {-1,-1,-1,-1 } 2^{k-3} times . Row (2^{k-1}+1) consists of 2^{k-1} +1's followed by 2^{k-1} -1's. The k generator columns of the matrix H_{2^k} having the same properties as the k generator rows are columns with sign changes {2, 3, 2^2+1 , 2^3+1, …, 2^{k-1}+1 }.

Now rearranging the 15 columns of matrix H_{16} (except the first of +1's) in increasing order of sign changes (from 1 up to 15) yields Table (3.2),where they are renamed A_i (i=1,2,…,15).

Table (3.2): The Normalized Sylvester-Hadamard Matrix H_{16} (in increasing columns level changes) along with its columns' time trend resistance

Run Order	Columnsof the matrix H_{16}in increasing level changes														
	A_1	**A_2**	**A_3**	**A_4**	**A_5**	**A_6**	**A_7**	**A_8**	**A_9**	**A_{10}**	**A_{11}**	**A_{12}**	**A_{13}**	**A_{14}**	**A_{15}**
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
3	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
4	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
5	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
6	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
7	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
8	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
9	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
10	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
11	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
12	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
13	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
14	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
15	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
16	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
Number of column sign changes	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Time Count(Linear)	-64	0	-32	0	0	0	-16	0	0	0	0	0	0	0	-8
Time Count (Quadratic)	-1088	256	-544	64	0	128	-272	16	0	0	0	32	0	64	-136

Information in the bottom two rows of Table (3.2) summarize the level of orthogonality of the columns of the Hadamard Matrix H_{16}(except the first column) with the first column of the time order (from 1 up to 2^4). The linear Time Count Statisticassesses the level of orthogonality between columns of H_{16} and the column of time run order ,whereas the quadratic Time Count Statisticassesses the level of orthogonality between these 15 columns and the squares of the entriesin the first column (from 1^2 up to 16^2).These two statistics are defined as dot products as follows:

Linear Time Count for columnA_i= $\Sigma t_j * A_{ij}$, (3.1)

QuadraticTimeCount for columnA_i= $\Sigma t_j * t_j * A_{ij}$(3.2)

WherecolumnA$_i$ (i=1,2,…,15)is any column of Table (3.2)and wheret$_j$ is the jthentry of the run order column(j= 1,2,…,16). Zero value for the statistic in (3.1) ensures linear trend resistance of columnA$_i$ (i=1,2,…,15), while zero value for the statistic in (3.2) ensures quadratic trend resistance of column A$_i$ (i=1,2,…,15). Hence, a glance at the values of these two Time Count statistics in the bottom of Table (3.2) reveals the following: none of the 4 columns(A_1,A_3 ,A_7 ,A_{15}) of the matrix H_{16} is linear trend free nor quadratic trend free,while the remaining 11 columns (A_2 ,A_4,A_5 ,A_6 , A_8 ,A_9 ,A_{10} ,A_{11} , A_{12} ,A_{13},A_{14}) are linear trend free but only a subset of them are both linear as well as quadratic trend free,namely the 5 columns (A_5 ,A_9 ,A_{10} ,A_{11} ,A_{13}).
Similar results as those in Tables(3.1)and (3.2) were generated inductively with the help of a statistical package for the larger size Normalized Sylvester-Hadamard matrices H_{32}, H_{64}, H_{128}, H_{256}, H_{512}andH_{1024},where we state the following conclusions forNormalized Sylvester –Hadamard matrices $H_{2^k}$of size 2^kx$2^k$and about their columns' time trend resistance:
(i) MatricesH_{2^k}have their (2^k -1) columns (except the first) pairwise orthogonal and each of these columns is balanced containing2^{k-1} +1's and 2^{k-1} -1's.This fact is true whether columns are arranged in increasing sign changes or not.
(ii) the number of sign changes in these (2^k -1) columnsrange from 1 up to(2^k-1) , where these columns can be rearranged in increasing order,as illustrated in Table (3.2) for k =4. Hence, the (2^k -1) columns of the matrixH_{2^k} (except the first)can be identified in two equivalent ways : either by the number of column sign changes or by their column number,whereeach identification ranges from 1 up to(2^k -1).
(iii)the(2^k -1) columns of the matrixH_{2^k}(except the first) have the following properties about linear/quadratic time trend resistance ,whether these columns are arranged in increasing order of sign changesor not:
(a)only the n columns having the respective number of level changes{(2^k-1),(2^{k-1}1), (2^{k-2}-1) , (2^{k-3}-1) ,… , ($2^{k-(k-1)}$-1) }are not orthogonal to the linear time trend, where their Linear Time Countsare not zeros.
(b)all remaining (2^k-k- 1) columns of thematrixH_{2^k}(except the first)are orthogonal to the linear time trend, where their linear Time Counts are zeros.A subset of these(2^k-k- 1) columns are at least quadratic trend free besides being linear trend free. The exactsize of this subset is [2^k-k- 1 – k(k-1)/2] columns.
Inductive results in (i) , (ii) and (iii) aboutthe Normalized Sylvester –Hadamardmatrices of size 2^kx$2^k$and their columns' time trend resistance will be utilized in Sections 4 and 5 for the construction of thethree proposed systematic minimum cost/trend resolution III and IV $2^{n-(n-k)}$ designs.

4. Normalized Sylvester-Hadamard matrices of size 2^k x$2^k$and their relationship with 2^{n-k} fractional designs through factor projections

This section discusses the relationship between the Normalized Sylvester-Hadamard matrices H_{2^k} ofsize 2^kx2^k(introduced in Section 3) and the full 2^k and fractional 2^{n-k} factorial experiments, where it is documented in [20]that deleting the first column of+1's in this matrix results in a saturated orthogonal array OA(2^k, 2^k -1,2,2) in maximum number of two-level factors N=(2^k-1) having level changes from 1 up to N, but not arranged in increasing order. That is, these orthogonal arrays are saturated regular$2^{N-(N-k)}$ designs of resolution III in N= (2^k-1) factors and in only 2^k=(N+1)experimental runs. However, these saturated $2^{N-(N-k)}$ designs are not time trend resistant, where k of their columns (i.e. factors) are not orthogonal to the time effect, as shown in the conclusion at the end of Section 3 and as can be seen from the bottom two rows of Table (3.2), for k=4. Therefore, removing these k non-trend free columns {1,3, 7, 15,31,…,(2^k-1)} from all (2^k -1) columns of theOA(2^k, 2^k -1,2,2)result in a minimum cost trend free resolution III $2^{M-(M-k)}$ designof 2^kexperimental runsin maximum number of trend free factors,namely M=(2^k-1- k).

Applying factor deletion by deleting columnsof the saturated OA(2^k, 2^k -1,2,2) with large level changes to economize experimentation cost result in a sequence (or catalog) of unsaturated minimum cost resolution III $2^{n-(n-k)}$ fractional factorial designs(2^{k-1}≤n≤2^k-1), where factor bounds ensure that runs are not duplicated. Also reducing the number of factors by deleting columnsfrom the minimum cost trend free resolution III $2^{M-(M-k)}$ designin maximum number of trend free factorsM=(2^k-1- k) result in another sequence of unsaturated minimum cost trend free resolution III $2^{n-(n-k)}$ fractional factorial experiments (2^{k-1}≤n≤2^k-1- k) .

On the other hand applyingnow factor addition on the minimum cost $2^{n-(n-k)}$design [from the OA(2^k, 2^k -1,2,2)] with smallest number of factors (n=k+1) and with the smallest number of factor level changesby adding factors sequentially in increasing number of factor level changes produces a sequence of minimum cost resolution III $2^{n-(n-k)}$ designs(k+1≤n≤2^{k-1}- 1+k) without getting into run duplication.Similarly, applying factor addition on the smallest minimum cost trend free $2^{m-(m-k)}$design[from the OA(2^k, 2^k -1,2,2) where (m=k+1)] by adding trend free factors sequentially in increasing number of factor level changes produces a sequence of minimum cost trend free resolution III $2^{m-(m-k)}$ designs(k+1≤m≤2^{k-1}- 2+k). These two backward and forward factor projectionsof the saturated OA(2^k, 2^k -1,2,2) will be illustrated in the followingsubsections.

In addition, restricting the number of factors to exactly n=k projects the saturated OA(2^k, 2^k -1,2,2) into full 2^k factorial design, where the remaining(2^k-1- k) columns become factor interactions of all orders (from 2 up to k).When n=k , factor projection of the OA(2^k, 2^k -1,2,2) may however reduce thisOAinto $2^{n-(n-k)}$ fractional factorial designsin duplicated runs if the n=k columns chosen are not linearly independent.Fractional factorial $2^{n-(n-k)}$ designs in duplicated runs can also be generated if projections involve n<k factors.There are many choices for the (n=k) generator columns as factor main effects, where some selections (i.e. projections) lead to full 2^k factorial designs in minimum number of factor level changes (i.e. minimum experimentation cost), while other column choices produce full 2^k factorial designs in maximum number of factor level changes (i.e. maximum experimentation cost).Projecting the saturated OA (2^k, 2^k -1, 2, 2) onto its (n=k) non-trend free columns{(2^k-1),(2^{k-1}1), (2^{k-2}-1) , (2^{k-3}-1) ,… , ($2^{k-(k-1)}$-1) } result in the standard order (1.1) of the full 2^k factorial experiment. None of these three projected full 2^kfactorial designsare however time trend resistant,since they involve column factors having non-zero Time Counts. Of course, there are other projections of the saturated OA (2^k, 2^k -1, 2, 2) into (n=k) factors producing time trend free full 2^k factorial designs,where this is achieved by avoiding assigning any of the (n=k) non-trend free columns{(2^k-1),(2^{k-1}1), (2^{k-2}-1) , (2^{k-3}-1) ,… , ($2^{k-(k-1)}$-1) } as factor main effects and also by avoiding selecting any of the dependent columns of the (2^k-k -1) trend- free columns of this OA.These full 2^k factorial projections will also be illustrated in the following subsections.

It should be noted here that preceding factor projectionsof the saturated OA (2^k, 2^k -1, 2, 2)into unsaturated resolution III $2^{n-(n-k)}$ fractional factorial designsor into full 2^kfactorial designshave been found through an inductive analysis of the Normalized Sylvester-Hadamard matrices $H_2{}^k$ and their associated OA (2^k, 2^k -1, 2, 2) for k=4,5,6,7,8,9,10.The following three subsections will illustrate these factor projections(Backward/Forward) utilizing the Normalized Sylvester-Hadamard matrix H_{16} in Table (3.1) and its associated OA (2^4, 2^4 -1, 2, 2) in Table (3.2) when (i) time trend is negligible /non- negligibleand when (ii) the projected $2^{n-(n-k)}$ design is of resolution III or IV, where Subsection 4.3 will discuss the problem of raising the design's resolution from III into IV while securing minimum factor level and/or factors' time trend resistance.

4.1. Minimum cost /trend free resolution III $2^{n-(n-4)}$ designs ($2^{4-1}\leq n\leq 2^4$-1- 4)

This subsection will illustrate projectionof the saturated OA (2^4, 2^4 -1, 2, 2) by factor deletion(i.e. backwardly), where we start when time trend is negligible.So,referring to reference [20],the OA(2^4, 2^4 -1, 2,2) in Table (3.2) is a saturated highly fractionated $2^{15-(15-4)}$design of resolution III in 15 two-level factors A_i(i = 1,2,…,15),where the total number of factor level changes is 120= (1+2+3+…+15).Not allof these 15 factors arehowever time trend resistant if this trend is non-negligible,where this can clearly be seen from the bottom two rows of Table(3.2) since Time Counts for some columns (i.e. factors) are not zeros, namely columns (A_1,A_3,A_7 and A_{15}) . If time trend is negligible then, the foldover of the 16 runs of this saturated resolution III $2^{15-(15-4)}$design can be sequenced by the GFS approach using the following 4 independent run generators:

$$\mathbf{g_1}=a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15},\mathbf{g_2}=a_4a_5a_6a_7a_8a_9a_{10}a_{11},\mathbf{g_3}=a_2a_3a_4a_5a_{10}a_{11}a_{12}a_{13}\text{and}\quad \mathbf{g_4}=a_1a_2a_5a_6a_9a_{10}a_{13}a_{14} \tag{4.1}$$

where starting with the null treatment (1)=000…0,thissaturated foldover$2^{15-(15-4)}$design is sequenced as follows:

$$(1),\ g_1,\ g_2,\ g_1g_2,\ g_3,\ g_1g_3,\ g_2g_3,\ g_1g_2g_3,\ g_4,\ g_1g_4,\ g_2g_4,\ g_1g_2g_4,\ g_3g_4,\ g_1g_3g_4,\ g_2g_3g_4,\ g_1g_2g_3g_4 \quad (4.2)$$

where, for instance, the fourth run in (4.2) is $g_1g_2=a_4a_5a_6a_7a_{12}a_{13}a_{14}a_{15}$ computed modulo 2.

That is, only 4 independent run generators suffice to sequence all 16 runs of this saturated $2^{15-(15-4)}$design. These 4 generator runs are locatedatthe 2nd,3rd,5th and 9th run of the sequence (4.2) , which they are exactly the four independent runs in (4.1).

This run order in (4.2) is only one of 16!=2092278989000 possible run ordersfor this $2^{15-(15-4)}$design, where a small subset of these run orders can be generated by the GFS approachby employing only 4 independent runs such as the 4 generator runsin (4.1). The alias structure of this foldover$2^{15-(15-4)}$design in (4.2) when three- factor and higher order interactions are negligible is given in Table (4.1),showing that the resolution is really III.

Table (4.1): The alias structure of the saturated resolution III $2^{15-(15-4)}$ design in (4.2)

$A_{-1}A_2A_3 - A_4A_5 - A_6A_7 - A_8A_9 - A_{10}A_{11} - A_{12}A_{13} - A_{14}A_{15}$
$A_2 - A_1A_3 - A_4A_6 - A_5A_7 - A_8A_{10} - A_9A_{11} - A_{12}A_{14} - A_{13}A_{15}$
$A_3 - A_1A_2 - A_5A_6 - A_4A_7 - A_9A_{10} - A_8A_{11} - A_{13}A_{14} - A_{12}A_{15}$
$A_4 - A_1A_5 - A_2A_6 - A_3A_7 - A_8A_{12} - A_9A_{13} - A_{10}A_{14} - A_{11}A_{15}$
$A_5 - A_1A_4 - A_3A_6 - A_2A_7 - A_9A_{12} - A_8A_{13} - A_{11}A_{14} - A_{10}A_{15}$
$A_6 - A_2A_4 - A_3A_5 - A_1A_7 - A_{10}A_{12} - A_{11}A_{13} - A_8A_{14} - A_9A_{15}$
$A_7 - A_3A_4 - A_2A_5 - A_1A_6 - A_{11}A_{12} - A_{10}A_{13} - A_9A_{14} - A_8A_{15}$
$A_8 - A_1A_9 - A_2A_{10} - A_3A_{11} - A_4A_{12} - A_5A_{13} - A_6A_{14} - A_7A_{15}$
$A_9 - A_1A_8 - A_3A_{10} - A_2A_{11} - A_5A_{12} - A_4A_{13} - A_7A_{14} - A_6A_{15}$
$A_{10} - A_2A_8 - A_3A_9 - A_1A_{11} - A_6A_{12} - A_7A_{13} - A_4A_{14} - A_5A_{15}$
$A_{1-1}\ A_3A_8 - A_2A_9 - A_1A_{10} - A_7A_{12} - A_6A_{13} - A_5A_{14} - A_4A_{15}$
$A_{12} - A_4A_8 - A_5A_9 - A_6A_{10} - A_7A_{1-1}\ A_1A_{13} - A_2A_{14} - A_3A_{15}$
$A_{13} - A_5A_8 - A_4A_9 - A_7A_{10} - A_6A_{1-1}\ A_1A_{12} - A_3A_{14} - A_2A_{15}$
$A_{14} - A_6A_8 - A_7A_9 - A_4A_{10} - A_5A_{1-1}\ A_2A_{12} - A_3A_{13} - A_1A_{15}$
$A_{15} - A_7A_8 - A_6A_9 - A_5A_{10} - A_4A_{1-1}\ A_3A_{12} - A_2A_{13} - A_1A_{14}$

Therefore , to estimate the 15 main effects A_i (i = 1,2,…,15) we assume all two-factor interactions are negligible.Also to make tests of significance on these 15 factor main effects, we need to remove saturation by reducing the number of factors (i.e. by factor projection),since the experimental error cannot be estimated and it has zero degrees of freedom.

To remove saturation and to reduce the fractionation level by deleting factors fromthis saturated $2^{15-(15-4)}$ designin (4.2) without getting intorun duplication while maintaining factor level changes minimum, seven unsaturated minimum cost resolution III $2^{n-(n-4)}$designs(n=8,9,…,14) can be constructed by successively dropping the column factors (A_{15}, A_{14}, A_{13}, A_{12},A_{11} , A_{10}, A_9) ,where the $2^{14-(14-4)}$ design drops column A_{15} while the $2^{13-(13-4)}$ design drops the two columns (A_{15} and A_{14}), etc. This factor deletion scheme ensures that the total cost of factor level changes is kept minimum, where factors deleted are those with the highest level changes.

The total cost of factor level changes in each of these seven unsaturated minimum cost $2^{n-(n-4)}$ designs(n=8,9,…,14) is 120 reduced successively by 15,(15+14), (15+14+13), (15+14+13+12), (15+14+13+12+11), (15+14+13+12+11+10), (15+14+13+12+11+10+9), where the pattern of factor level changes in each $2^{n-(n-4)}$designis increasing, indicating that factors having hard-to-vary levels are assigned to factor columns with minimal level changes and indicating that such factors are introduced later into these$2^{n-(n-4)}$ fractionated experiments. The alias structure for each of these seven $2^{n-(n-4)}$ designs(n=8,9,…,14) can be read from alias structure of the saturated $2^{15-(15-4)}$ design in Table (4.1) by just dropping the deleted factor(s).In addition, the 4 GFS run generators for the foldover of each of these seven $2^{n-(n-4)}$ designs (n=8,9,…,14) can be obtained from the 4 GFS run generators in (4.1) by dropping factor levels of the deleted factor(s).

These seven minimum cost resolution III $2^{n-(n-4)}$designs (n=8,9,…,14) include theminimum cost resolution III $2^{8-(8-4)}$ designin the smallest number of factors,factors A_i (i = 1,2,…,8) and with the lowest fractionation level . The 4 independent defining contrast interactions of this $2^{8-(8-4)}$ designare I=$A_1A_2A_3$=$A_1A_4A_5$=$A_2A_4A_6$=$A_3A_4A_7$, where the 4 GFS generatorsare : $\mathbf{g_1}=a_8$, $\mathbf{g_2}=a_4a_5a_6a_7a_8$, $\mathbf{g_3}=a_2a_3a_4a_5$ and $\mathbf{g_4}=a_1a_2a_5a_6$, and where the full layout of thefoldoverby these 4 GFS generators using (4.2) is:

(1) , a_8 ,$a_4a_5a_6a_7a_8$,$a_4a_5a_6a_7$,$a_2a_3a_4a_5$, $a_2a_3a_4a_5a_8$, $a_2a_3a_6a_7a_8$, $a_2a_3a_6a_7$,
$a_1a_2a_5a_6$,$a_1a_2a_5a_6a_8$,$a_1a_2a_4a_7a_8$,$a_1a_2a_7$, $a_1a_3a_4a_6$, $a_1a_3a_4a_6a_8$, $a_1a_3a_5a_7a_8$, $a_1a_3a_5a_7$ (4.3)

where the total cost of factor level changes for its8 factors A_i (i = 1,2,…,8) is minimal equaling 36=(1+2+3+4+5+6+7+8).

All precedingseven unsaturated minimum cost resolution III $2^{n-(n-4)}$ designs (n=8,9,10,…,15) are however not time trend free, since some of their column factors have non-zero Time Counts. Therefore, to construct minimum cost trend free resolution III $2^{n-(n-4)}$ designs from the saturated OA (2^k, 2^k -1, 2, 2) by factor projection , we first delete the 4 non-trend free columns (A_1, A_3, A_7, A_{15}) of Table (3.2) then the remaining eleven columns (A_2, A_4, A_5, A_6,A_8, A_9, A_{10},A_{11}, A_{12}, A_{13}, A_{14})constitute an unsaturated minimum cost trend free resolution III $2^{11-(11-4)}$ designin the largest number of trend free factors (i.e. n=11) with total cost of factor level changes 94=(2+4+5+6+8+9+10+11+12+13+14) and 4 foldover GFS generators:

$\mathbf{g_1}=a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}$,$\mathbf{g_2}=a_4a_5a_6a_8a_9a_{10}a_{11}$,$\mathbf{g_3}=a_2a_4a_5a_{10}a_{11}a_{12}a_{13}$and $\mathbf{g_4}=a_2a_5a_6a_9a_{10}a_{13}a_{14}$ (4.4)

These4GFS generators in (4.4) are those 4 generators in (4.1) after deleting levels of the dropped non-trend free factors (A_1, A_3, A_7, A_{15}) from each generator run.

To reduce both the number of factors and the fractionation level from this minimum cost trend free resolution III trend free$2^{11-(11-4)}$ design in (4.4), three minimum cost resolution III trend free $2^{n-(n-4)}$ designs(n=8,9,10)can be produced. The minimum cost trend free resolution III $2^{8-(8-4)}$ design with the smallest number of trend free factors (A_2, A_4, A_5, A_6, A_8, A_9, A_{10} ,A_{11}) renamed as (C_i : i=1,2,…8) has the 4 independent defining contrast interactions I = $C_1C_2C_4$= $C_2C_3C_5C_6$= $C_1C_5C_7$= $C_1C_2C_3C_5C_8$. The 4 GFS generators of the foldover of this $2^{8-(8-4)}$ designare :

g_1 $=c_5c_6c_7c_8$, **g_2**$=c_2c_3c_4c_5c_6c_7c_8$, **g_3**$= c_1c_2c_3c_7c_8$ and **g_4**$=c_1c_3c_4c_6c_7$, where the full layout of this foldover$2^{8-(8-4)}$ designby these 4 GFS generators using (4.2) is:

(1) , $c_5c_6c_7c_8$, $c_2c_3c_4c_5c_6\,c_7c_8$, $c_2c_3c_4$, $c_1c_2c_3c_7c_8$, $c_1c_2c_3c_5c_6$, $c_1c_4\,c_5c_6$, $c_1c_4\,c_7c_8$,$c_1c_3c_4c_6c_7$, $c_1c_3c_4c_5c_8$,$c_1c_2c_5c_8$, $c_1c_2c_6\,c_7$, $c_2c_4c_6\,c_8$, $c_2c_4c_5\,c_7$,$c_3c_5\,c_7$, $c_3c_6\,c_8$ (4.5)

and where the total cost of factor level changes is45=(2+4+5+6+8+9+10+11),which is higher than the total cost (i.e. 36) of the minimum cost resolution III $2^{8-(8-4)}$ design in (4.3).That is , factors' time trend resistance has raised the total cost of factor level changes by 9.

All 8 main effects of this minimum cost trend free resolution IV $2^{8-(8-4)}$ designin (4.5) are free from (or resistant to) the linear time trend, where all these columns have zero Time Counts. Time trend resistance of this $2^{8-(8-4)}$ design can also be confirmed if we consider linear modeling its output (Y_{ij}) in terms of these8 main effects(C_i : i=1,2,…8) including the linear time trend as follows:

$Y_{ij}= \mu+C_1x_1 + C_2x_2 +C_3x_3 +C_4x_4 +C_5x_5 +C_6x_6 + C_7x_7+C_8x_8+\alpha t +\mathcal{E}$,t and j= 1,2,3,…,16 , x_i = -1,=1 , i=1,2,…8(4.6)

where the design's matrix will have all eight main effect columns orthogonal to the linear time effect column (i.e. zero Linear Time Counts).

Above factor deletionsand projections of the saturated OA (2^4, 2^4 -1, 2, 2) under resolution III have reduced the number of factorsfrom 15 to 8 and the fractionation level from (15-4)=11 into (8-4)=4when time trend is negligible, whereas when time trend is non-negligible factor reduction was from 11 into 8 and fractionation reduction was from (1-14)=7 into (8-4)=4.Therefore, to reduce the fractionation level further below (8-4)=4 while maintaining resolution to be III and avoiding runs duplication, we consider in subsection 4.2 the second approach of factor projection of the saturated OA (2^k, 2^k -1, 2, 2) , which involves factor addition (not factor deletion)to a minimum cost highly unsaturated $2^{n-(n-k)}$ design derived from this OA (2^k, 2^k -1, 2, 2) inthe smallest number of factors .

4.2. Minimum cost /trend free resolution III $2^{n-(n-4)}$ designs ($4+1\leq n\leq 2^{4-1}-2+4$)

This subsection will work in the reverse direction of subsection 4.1 (i.e. forwardly) by adding factorssequentially to a minimal cost highly unsaturated $\mathbf{2^{n-(n-4)}}$design in the smallest number of factors [derivedfrom the OA (2^4, 2^4 -1, 2, 2)]to generate a sequence of minimum cost resolution III $2^{n-(n-k)}$ designs ($4+1\leq n\leq 2^{4-1}$ -1+4)in increasing number of factors and in increasing fractionation level. That is,factor addition approach starts with half and quarter fractionsor it may start with full 2^4 designs then moves upward in the fractionation level.This subsection will illustrate this factor addition process under the two situations when the time trend effect is negligible and when non-negligible.

We start with the minimal cost full 2^4 factorial design derivable from the saturated OA (2^4, 2^4 -1, 2, 2) in Table (3.2) having the four main effect columns (A_1 ,A_2, A_4, A_8) renamed as (F_1,F_2,F_3,F_4)with level changes {1,2,4,8}. The 4 GFS generators for the foldover of this full 2^4design are : [$g_1=f_4$, $g_2= f_3f_4$, $g_3= f_2f_3$, $g_4= f_1f_2$],where the layout of itsfoldover by GFS is:

(1), f_4 , f_3f_4 , $f_1f_2f_3f_4$, f_3 ,f_2f_3 ,$f_2f_3f_4$,f_2f_4 , f_2 ,f_1f_2, $f_1f_2f_4$, $f_1f_2f_3f_4$, $f_1f_2f_3f_4$, $f_1f_2f_3$, $f_1f_2f_3f_4$,f_1f_3 ,$f_1f_3f_4$,f_1f_4 , f_1
(4.7)

With minimal total cost of factor level changes 15=(1+2+4+8)=(2^4-1).This column selection(A_1 ,A_2, A_4, A_8) in minimal cost is unique , otherwise runs will be duplicated or the total cost of factor level changes will be above minimal.

Next weapply factor addition to this minimal cost full 2^4 factorial design in (4.7)by addingthe seven columns (A_9,A_{10}, A_{11}, A_{12} ,A_{13} ,A_{14}, A_{15}) of Table (3.2) sequentially to generate seven new minimum cost resolution III $2^{n-(n-4)}$ designs n={5,6,7,8,9,10,11} in 16=2^4 runs each. Hence, adding column $A_9$to the 4 columns of the minimal cost full 2^4factorial design in (4.7) leads to the minimum cost resolution III $2^{5-(5-4)}$half fraction with defining contrast I= $F_1F_4F_5$, where (F_i :i=1,2,…,5) are its 5 factors. The 4 GFS generators for its foldover are:[$g_1=f_4f_5$, $g_2= f_3f_4f_5$, $g_3= f_2f_3$, $g_4= f_1f_2f_5$] and total cost 24=(1+2+4+8+9). This minimum cost resolution III $2^{5-(5-4)}$half fraction is however not trend freeif time trend is non-negligible, since factor F_1 is not trend free. Error degrees of freedom is 3 provided the three two-factor interactions (F_1F_4, F_1F_5 ,$F_4 F_5$) are negligible,hence effects estimation and tests of hypothesis can be conducted if time trend effect is negligible.

We now add the two columns (A_9 and A_{10}) of Table (3.2) to the minimal cost full 2^4 factorial design in (4.7) or equivalently we select the six columns (A_1 ,A_2, A_4, A_8 ,A_9 ,A_{10}) together as the 6 factors (F_i :i=1,2,…,6)of the minimum cost resolution III $2^{6-(6-4)}$ quarter fraction with defining contrast $I=F_1F_4F_5=F_2F_4F_6=F_1F_2F_5F_6$and 4 foldover GFS generators:[$g_1=f_4f_5f_6$, $g_2= f_3f_4f_5f_6$, $g_3= f_2f_3f_6$, $g_4= f_1f_2f_5f_6$] and with total cost 34 =(1+2+4+8+9+10).This minimum cost resolution III $2^{6-(6-4)}$quarterfraction is also not trend free, since factor F_1 is not trend free.

Proceeding further adding the three columns (A_9 and A_{10} , A_{11}) of Table (3.2) to the minimal cost full 2^4 factorial design in (4.7) yields the minimum cost resolution III $2^{7-(7-4)}$ design in the 7 factors (F_i : i=1,2,…,7)with defining contrast independent interactions $I=F_1F_4F_5=F_2F_4F_6=F_1F_6F_7$and with 4 foldover GFS generators : [$g_1=f_4f_5f_6f_7$, $g_2= f_3f_4f_5f_6f_7$, $g_3= f_2f_3f_6f_7$, $g_4= f_1f_2f_5f_6$] yielding a total cost of factor level changes 45=(1+2+4+8+9+10+11).This

minimum cost resolution III $2^{7-(7-4)}$design is also not trend free containing the non-trend free factor F_1.
Finally , adding all seven columns (A_9 ,A_{10},A_{11} , A_{12} ,A_{13} ,A_{14} ,A_{15}) of Table (3.2) to the minimal cost full 2^4 factorial design in (4.7) yields a minimum cost resolution III $2^{11-(11-4)}$ design with the eleven factors (F_i : i=1,2,…,11)and alias structure given in Table 4.2 :
TABLE 4.2: The alias structure of the minimum cost resolution III $2^{11-(11-4)}$ design derived from the full 2^4 factorial design in (4.7)by factor addition

Main effects alias chains	Two-Factor interactions alias chains
Intercept	
$F_1 + F_4F_5+ F_6F_7 +F_8F_9+ F_{14}F_{15}F_1F_2+F_4F_7+F_5F_6+F_8F_{15}+F_9F_{14}$	
$F_2+F_4F_6+F_5F_7+F_8F_{14}+F_9F_{15}F_1F_3+F_4F_9+F_5F_8+F_7F_{14}+F_6F_{15}$	
$F_3+F_4F_8+F_5F_9+F_6F_{14}+F_7F_{15}F_2F_3+F_4F_{14}+F_5F_{15}+F_6F_8+F_6F_{15}+F_7F_9$	
$F_4+F_1F_5+F_2F_6+F_3F_8F_5F_{14}+F_6F_9+F_7F_8+F_4F_{15}$	
$F_5+F_1F_4+F_2F_7+F_3F_9$	
$F_6+F_1F_7+F_2F_4+F_3F_{14}$	
$F_7+F_1F_6+F_2F_5+F_3F_{15}$	
$F_8+F_1F_9+F_2F_{14}+F_3F_4$	
$F_9+F_1F_8+F_2F_{15}+F_3F_5$	
$F_{10}+F_1F_{15}+F_2F_8+F_3F_6$	
$F_{11}+F_1F_{14}+F_2F_9+F_3F_7$	

The 4 foldover GFS generators of this $2^{11-(11-4)}$ design are: [g_1=$f_4f_5f_6f_7f_8f_9f_{10}f_{11}$, g_2= $f_3f_4f_5f_6f_7$, g_3= $f_2f_3f_6f_7f_8f_9$, g_4= $f_1f_2f_5f_6f_9f_{10}$] and the total cost of factor level changes is 99 = (1+2+4+8+9+10+11+12+13+14+15), which is higher than the total cost 66=(1+2+3+4+5+6+7+8+9+10+11) of the unsaturated minimum cost resolution III $2^{11-(11-4)}$ designof subsection 4.1 and also higher than the total cost94 =(2+4+5+6+8+9+10+11+12+13+14) of the unsaturated minimum cost trend free resolution III $2^{11-(11-4)}$ designof that subsection.
The seven independent interactions of the defining contrast of this minimum cost resolution III $2^{11-(11-4)}$ design of total cost 99 are : I= $F_1F_4F_5$= $F_2F_4F_6$= $F_1F_2F_4F_7$= $F_3F_4F_8$= $F_1F_3F_4F_9$= $F_2F_3F_4F_{10}$= $F_1F_2F_3F_4F_{11}$
(4.8)
Which yield the same alias structure as that of Table (4.2) after deleting all three-factor and higher order interactions.This minimum cost resolution III $2^{11-(11-4)}$ design is however not time trend free.
Soif time trend is non-negligible , we consider for factor addition as the initial minimum cost trend free full 2^4 factorial design the 2^4 design having the 4 time trend free columns (A_2 ,A_4, A_5, A_8) of the OA (2^4, 2^4 -1, 2, 2) in Table (3.2) with factor level changes {2,4,5,8}) totaling 19=(2+4+5+8) , being above the minimal 2^4design of (4.7) by only 4. Denoting these four factors by (F_1,F_2,F_3,F_4),the 4 foldover GFS generators are:
[g_1=f_4 , g_2= $f_2f_3f_4$, g_3= $f_1f_2f_3$, g_4= f_1f_3],where the layout of itsfoldover is:
(1), f_4 , $f_2f_3f_4$, f_2f_3 ,$f_1f_2f_3$, $f_1f_2f_3f_4$, f_1f_4 ,f_1 , f_1f_3, $f_1f_3f_4$, $f_1f_2f_4$, f_1f_2 , f_2 , f_2f_4 , f_3f_4 , f_3 (4 .9)
Applying factor addition to this minimum cost trend free full 2^4 factorial design in (4.9) by adding the six trend free columns (A_9,A_{10}, A_{11}, A_{12} ,A_{13} ,A_{14}) of Table (3.2) sequentially to its columns (A_2 ,A_4, A_5, A_8) to generate six new minimum cost trend free resolution III $2^{n-(n-4)}$ designs n={5,6,7,8,9,10} in 16=2^4 runs each . Hence, adding the trend free column A_9 to the 4 columns of the 2^4 design in (4.9) leads to the minimum cost trend free resolution IV half fraction $2^{5-(5-4)}$ with defining contrast I= $F_2F_3F_4F_5$, where (F_i i=1,2,…,5) are its 5 factors. The 4 GFS generators for its foldover are : [g_1=f_4f_5, g_2= $f_2f_3f_4f_5$, g_3= $f_1f_2f_3$, g_4= $f_1f_3f_5$] with total cost 28=(2+4+5+8+9).
Adding the two trend free columns (A_9 and A_{10}) of Table 3.2 to the minimum cost trend free full 2^4 factorial design in (4.9) or equivalently selecting the six columns (A_2,A_4, A_5, A_8 ,A_9 ,A_{10}) from Table (3.2) together as the 6 factors (F_i i=1,2,…,6) lead to the minimum cost trend free resolution III $2^{6-(6-4)}$ quarter fraction with defining contrast I = $F_1F_4F_6$= = $F_2F_3F_4F_5$= $F_1F_2F_3F_5F_6$ and the 4 foldover GFS generators : [g_1=$f_4f_5f_6$, g_2= $f_2f_3f_4f_5f_6$, g_3= $f_1f_2f_3f_6$, g_4= $f_1f_3f_5f_6$] having total cost of factor level changes 38 =(2+4+5+8+9+10).
Proceeding further adding the three trend free columns (A_9 , A_{10} , A_{11}) of Table (3.2) to the minimal cost trend free full 2^4 factorial design in (4.9) yields the minimum cost trend free resolution III $2^{7-(7-4)}$ design in the 7 factors (F_i i=1,2,…,7) with defining contrast independent interactions I = $F_1F_4F_6$= $F_2F_4F_8$= $F_1F_5F_7$ and with the 4 foldover GFS generators : [g_1=$f_4f_5f_6f_7$, g_2= $f_2f_3f_4f_5f_6f_7$, g_3= $f_1f_2f_3f_6f_7$, g_4= $f_1f_3f_5f_6$] having total cost of factor level changes 49=(2+4+5+8+9+10+11).
Finally, adding all sixtrend free columns (A_9 ,A_{10},A_{11} , A_{12} ,A_{13} ,A_{14}) of Table (3.2) to the minimal cost trend free full 2^4 factorial design in (4.9) yields a minimum cost trend free resolution III $2^{10-(10-4)}$ design in the ten factors (F_i i=1,2,…,10) and with the 4 foldover GFS generators :
[g_1= $f_4f_5f_6f_7f_8f_9f_{10}$, g_2=$f_2f_3f_4f_5f_6f_7$, g_3= $f_1f_2f_3f_6f_7f_8f_9$, g_4= $f_1f_3f_5f_6f_9f_{10}$] with total cost of factor level changes88 =(2+4+5+8+9+10+11+12+13+14).
Comparingtotal cost of factor level changes(24)of the minimum cost resolution III $2^{5-(5-4)}$ half fraction with the total cost of the minimum cost trend free resolution III $2^{5-(5-4)}$ half fraction(28)shows that time trend resistance requires more factor level changes. A similar conclusion canalso be reached if we compare the two $2^{6-(6-4)}$ quarter

fractions of the two types of time trend resistance(i.e. negligible or not).
Finally,it is worth to note that candidate columnsof this subsectionfrom the OA (2^4, 2^4 -1, 2, 2)in Table (3.2) for minimum cost trend free resolution III $2^{n-(n-4)}$ designs($2^{k-1} \leq n \leq 2^k - 1 - k$) are different from candidate columns of subsection 4.1 for minimum cost trend free resolution III $2^{n-(n-4)}$ designs ($k+1 \leq n \leq 2^{k-1} - 2+k$).There is however some overlap in the number of candidate columns between these two design categories, but $2^{n-(n-4)}$ designs of subsection 4.1 have smaller total number of factor level changes than $2^{n-(n-4)}$ designs of this subsection for the same number of factors. Whereas designs of this subsection produce half and quarter fractions while designs of subsection 4.1 do not.

4.3. Minimum cost/trend free resolution IV $2^{n-(n-4)}$ designs[$2^{4-2} \leq n \leq 2^{4-1} - 2$]

This subsection raises the resolution in the factor projection of the saturatedOA (2^4, 2^4-1,2,2)in Table (3.2) from III to IV,where the 15 columns of this OA reduce under resolution IVto only the 8=2^3candidate column factors (A_2, A_3, A_4, A_5, A_8, A_9, A_{14} ,A_{15}) ,where two of these 8 columns, namely (A_3 ,A_{15}) are not time trend free.Hence, the largest minimum cost resolution IV by factor projection of the saturatedOA (2^4, 2^4-1,2,2) is theminimum cost resolutionIV $2^{8-(8-4)}$ design having allthese8 column factors, with total cost of factor level changes 60 = (2 + 3 +4 + 5 + 8 + 9 + 14 + 15). This cost is certainly higher than the total cost 36=(1+2+3+4+5+6+7+8)of the unsaturated minimum cost resolution III $2^{8-(8-4)}$ fractionin (4.3) of Subsection 4.1.Thus indicating that minimum cost resolution IV $2^{n-(n-k)}$ designs require generally larger factor level changes (i.e. more costly)than minimum cost resolution III $2^{n-(n-k)}$ designs.This is mainly due to the fact that resolution IV has excluded half the columns of the OA (2^4, 2^4-1,2,2) before the factor projection process.
The 8 factors of this minimum cost resolution IV $2^{8-(8-4)}$ designare now renamed as(B_i i=1,2,…,8), where the defining contrast is

$$I = B_1B_2B_3B_4 = B_1B_2B_5B_6 = B_3B_4B_5B_6 = B_1B_3B_5B_7 = B_2B_4B_5B_7 = B_2B_3B_6B_7 = B_1B_4B_6B_7 = B_2B_3B_5B_8 = B_1B_4B_5B_8 = B_1B_3B_6B_8 = B_2B_4B_6B_8 = B_1B_2B_7B_8 = B_3B_4B_7B_8 = B_5B_6B_7B_8 = B_1B_2B_3B_4B_5B_6B_7B_8 \quad (4.10)$$

which confirms that the resolutionis really IV.
The 4 independent GFS run generators are :

$$\mathbf{g_1}=b_5b_6b_7b_8 ,\mathbf{g_2}=b_3b_4b_5b_6 , \mathbf{g_3}=b_1b_2b_3 b_4 \quad \text{and} \quad \mathbf{g_4}=b_1b_4b_6b_7 \quad (4.11)$$

where applying the GFS technique using these GFS 4 run generators in (4.11) yields itsfoldover as:
(1), $b_5b_6b_7b_8$,$b_3b_4b_5b_6$,$b_3b_4b_7b_8$, $b_1b_2b_3b_4$, $b_1b_2b_3b_4b_5b_6b_7b_8$,$b_1b_2b_5b_6$, $b_1b_2b_7b_8$, $b_1b_4b_6b_7$,$b_1b_4b_5b_8$,$b_1b_3b_5b_7$, $b_1b_3b_6b_8$, $b_2b_3b_6b_7$, $b_2b_3b_5b_8$,$b_2b_4b_5b_7$,$b_2b_4b_6b_8$ (4.12)
The detailed alias structure of this minimum cost resolution IV $2^{8-(8-4)}$ design can be found from the defining contrast in (4.10) ,where this alias structure is given explicitly in Table (4.3), assuming 3-factor and higher order interactions are negligible.

Table (4. 3) : The alias structure for the minimum cost resolution IV $2^{8-(8-4)}$ design in (4.12)

Main effects (free from aliasing)	Two-Factor interactions Alias chains
Intercept	$B_1B_2 \pm B_3B_4 \pm B_5B_6 \pm B_7B_8$
B_1	$B_1B_3 \pm B_2B_4 \pm B_5B_7 \pm B_6B_8$
B_2	$B_1B_4 \pm B_2B_3 \pm B_5B_8 \pm B_6B_7$
B_3	$B_1B_5 \pm B_2B_6 \pm B_3B_7 \pm B_4B_8$
$B_4B_1B_6 \pm B_2B_5 \pm B_3B_8 \pm B_4B_7$	
$B_5B_1B_7 \pm B_2B_8 \pm B_3B_5 \pm B_4B_6$	
B_6	$B_1B_8 \pm B_2B_7 \pm B_3B_6 \pm B_4B_5$
B_7	
B_8	

Therefore, factor effects can be estimated unbiased by any interaction effect, while to make tests of significance onthese factor main effects we may assume all two-factor interactions negligible,yielding an experimental error with 7 degrees of freedom.
On the other hand,thesmallest minimum cost resolution IV $2^{n-(n-4)}$ design from the candidate column factors (A_2, A_3, A_4, A_5, A_8, A_9, A_{14} ,A_{15})under resolution IV is the minimum cost resolution IV $2^{5-(5-4)}$ half fraction having the first 5 column factors (A_2, A_3, A_4, A_5, A_8) as its 5 factors renamed as (F_i i=1,2,…,5), with total cost of factor level changes 22 = (2 + 3 + 4 + 5 + 8) and defining contrast $I=F_1F_2F_3F_4$. This total cost (i.e.22) turns out to be smaller than the total cost (i.e. 24) of the minimum cost resolution III $2^{5-(5-4)}$ half fraction of subsection 4.2. The 4 GFS generators[$g_1=f_5$, $g_2=f_3f_4f_5$, $g_3=f_1f_2f_3f_4$, $g_4= f_1f_4$] where applying the GFS technique using these GFS 4 run generators yields the foldover as:
(1) ,f_5, $f_3f_4f_5$, f_3f_4, $f_1f_2f_3f_4$, $f_1f_2f_3f_4f_5$, $f_1f_2f_5$, f_1f_2, f_1f_4, $f_1f_4f_5$, $f_1f_3f_5$, f_1f_3, f_2f_3, $f_2f_3f_5$, $f_2f_4f_5$, f_2f_4(4.13)
Hence, only four unsaturated minimum cost resolution IV $2^{n-(n-4)}$designs(n=5,6,7,8)can be constructed by either the forward or the backward factor projection approacheson the 8 candidate column factors (A_2, A_3, A_4, A_5, A_8, A_9, A_{14} ,A_{15})of the saturatedOA (2^4, 2^4-1,2,2) under resolution IV.The backward approach starts with the minimum cost resolution IV $2^{8-(8-4)}$ design in (4.12) deleting factors successively while the forward approach starts with the

minimum cost resolution IV $2^{5-(5-4)}$ design in (4.13)adding factors successively .This situation is unlike the resolution III case,where candidate columns fromthe saturatedOA (2^4, 2^4-1,2,2) under the backward approach [i.e. subsection 4.1]weredifferent from those of subsection 4.2 under factor addition .

Allabove four minimum cost resolution IV fractional $2^{n-(n-4)}$ designs (n=5,6,7,8) are not time trend free , since referring to Table (3.2) some of their columns are not orthogonal to the linear time trend. Therefore, to achieve factors' trend freeness under resolution IV the 8 candidate columns (A_2, A_3, A_4, A_5, A_8, A_9, A_{14} ,A_{15}) reduce to only to the 6 trend free columns (A_2, A_4, A_5, A_8, A_9, A_{14}) with factor level changes (2, 4, 5, 8, 9, 14). Hence, only two unsaturated minimum cost trend free resolution IV $2^{n-(n-4)}$designs(n=5 ,6) can be constructed without run duplication , the largest is the $2^{6-(6-4)}$ quarter fraction having the 6 column factors (A_2,A_4, A_5, A_8, A_9, A_{14}) renamed as factors (C_i i=1,2,…,6), with total cost of factor level changes 42 = (2 + 4+ 5 + 8 + 9 + 14) and defining contrast I =$C_2C_3C_4C_5$=$C_1C_2C_4C_6$=$C_1C_3C_5C_6$.

The 4 independent run generators to sequence the foldover by the GFS approach are [$\mathbf{g_1}$=$c_4c_5c_6$,,$\mathbf{g_2}$=$c_2c_3c_4c_5$,, $\mathbf{g_3}$=$c_1c_2c_3$,$\mathbf{g_4}$=$c_1c_3c_5c_6$] where applying the GFS , the foldover is:

(1) ,$c_4c_5c_6$, $c_2c_3c_4c_5$, $c_2c_3c_6$, $c_1c_2c_3$, $c_1c_2c_3c_4c_5c_6$, $c_1c_4c_5$, c_1c_6,$c_1c_3c_5c_6$, $c_1c_3c_4$, (4.14)

On the other hand , the smallestminimum cost trend free resolution IV fraction is the $2^{5-(5-4)}$ half fraction having the5 column factors (A_2, A_4, A_5, A_8, A_9) renamed as (F_i i=1,2,…,5) , with total cost of factor level changes 28 = (2 + 4 + 5 + 8 + 9) and defining contrast I= $F_2F_3F_4F_5$. This total cost (i.e. 28) turns out to be larger than the total cost (i.e. 22) of the minimum cost resolution IV $2^{5-(5-4)}$ half fraction in (4.13) due to securing factors' time trend resistance. The 4 GFS generators are[g_1= f_4f_5, g_2= $f_2f_3f_4f_5$, g_3=$f_1f_2f_3$, g_4= f_1] ,where applying the GFS technique yields the foldover $2^{5-(5-4)}$ half fraction as:

(1) , f_4f_5 ,$f_2f_3f_4f_5$, f_2f_3, $f_1f_2f_3$, $f_1f_2f_3f_4f_5$, $f_1f_4f_5$, f_1, $f_1f_3f_5$, $f_1f_3f_4$, $f_1f_2f_4$, $f_1f_2f_5$, f_2f_5, f_2f_4, f_3f_4, f_3f_5(4.15)

Finally, the trend free $2^{4-(4-4)}$ fractionunder the 4 column factors (A_2, A_4, A_5 and A_8)of the saturated OA(2^4 ,2^4 ,2,2) in Table 3.2 is a minimum cost trend free full 2^4 factorial designwith total cost of factor level changes 19=(2+4+5+8), where the remaining eleven columns (A_1, A_3, A_6, A_7, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}) are the interactions of all orders.Renaming these 4 factors (A_2, A_4, A_5 and A_8) as (D_i i=1,2..,4), the correspondence between columns of this minimum cost trend free full 2^4 factorial design and the columns of the saturated OA(2^4 ,2^4 ,2,2) is as follows:

D_1= A_2 , D_2= A_4 , D_3 =A_5 , D_4=A_8 ,D_1D_2=A_6 ,D_1D_3 =A_7 , D_1D_4=A_{10} , D_2D_3=A_1 , D_2D_4 =A_{12} , D_3D_4 =A_{13} ,$D_1D_2D_3$=A_3, $D_1D_2D_4$ =A_{14} , $D_1D_3D_4$ =A_{15} , $D_2D_3D_4$ =A_9 ,$D_1D_2D_3D_4$=A_{11} (4.16)

The foldover of this minimum cost trend free full 2^4 factorial design under the 4 column factors (A_2,A_4,A_5,A_8) or equivalently under columns (D_i i=1,2..,4) is

(1),d_4 , $d_2d_3d_4$,d_2d_3 , $d_1d_2d_3$,$d_1d_2d_3d_4$,d_1d_4 ,d_1 , d_1d_3,$d_1d_3d_4$, $d_1d_2d_4$,d_1d_2,d_2, d_2d_4 ,d_3d_4,d_3 .(4.17)

where the 4 GFS generator runs are:$\mathbf{g_1}$=d_4, g_2 = $d_2d_3d_4$,$\mathbf{g_3}$ =$d_1d_2d_3$ and $\mathbf{g_4}$=d_1d_3, located at the 2nd,3rd , 5th and 9th runs of the foldover sequence (4.17) .

Of course,there are other column selections (i.e. projections) of the saturated OA(2^4, 2^4 -1,2,2)into 4 factors which lead to full 2^4 factorial designs .For instance , the three selections (A_{15},A_7 ,A_3 ,A_1), (A_1,A_2 ,A_4 ,A_8) and (A_{15},A_{14} ,A_{13} ,A_{11}) each leads to full 2^4 factorial design with the following properties : in standard order, in minimum cost and in maximum cost, respectively . But unlike the full 2^4 factorial design in (4.17) none of these three full 2^4 factorial designs is time trend free.

Having finally completed all aspects of the factor projection of the OA(2^4, 2^4 -1,2,2) , generalizationstoOA(2^k, 2^k -1,2,2) will be considered in Section 5 leading to the construction of three proposed categories of minimum cost /trend free resolution III/IV $2^{n-(n-k)}$ designs.

5. Catalog of Minimum Cost /trend free $2^{n-(n-k)}$ Designs of resolution III and IV

Section 4has illustrated various factor projectionsof the saturated OA(2^k, 2^k-1, 2,2) when k=4 (backward and forward)under resolutions III and IV.Similar projectionshave been worked out using a statistical package for k=5,6,7,8,9,10 .This extensive computer work has lead to the following generalizationswhich involves the proposition of the following threecategories of minimum cost / trend free $2^{n-(n-k)}$ designsof resolutions III and IV.

(i) Minimum cost / trend free resolution III $2^{n-(n-k)}$ designs [2^{k-1}≤n≤(2^k-1)-k](Category One Designs): Candidate columns for projection of the OA(2^k, 2^k-1, 2,2) by factor deletion under resolution III are all its (2^k-1) columns {1,2,3,…., (2^k-1)} representing a saturated resolution III $2^{n-(n-k)}$ designin N=(2^k-1) factors with total cost of factor level changes =[1+2+3+…. + (2^k-1)]=2^{k-1}(2^k-1) ,where its k GFS run generators are located at the k runs numbered {1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,…, 2^{k-1}+1}. This saturated OA(2^k, 2^k-1,2, 2)of (2^k-1) columns is then reduced successively by factor deletion deleting columns of large level changes (to minimize experimentation cost) until reaching the first 2^{k-1} columns {1,2, 3 ,…., 2^{k-1}}which represent together the smallestminimum cost resolution III $2^{n-(n-k)}$ design in N=2^{k-1} factors with total cost of factor level changes =[1+2+3 + ….+ 2^{k-1}]=2^{k-2}(2^{k-1} +1),where its k GFS run generators are also located at the k runs numbered {1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,…, 2^{k-1}+1}. More factor deletion involving less than the first 2^{k-1}columnswill however result in runs duplication. Therefore , successive factor deletionsof the OA(2^k, 2^k-1, 2,2) starting deletion with factors of highest level changes produce a sequence of $(2^k-1)-2^{k-1}+1=(2^k-2^{k-1})=2^{k-1}$ minimum cost resolution III $2^{n-(n-k)}$ designs$(2^{k-1}\le n\le(2^k-1))$.On the other hand, tThisdesign sequence can equivalently be generated by factor addition starting forwardly with the first 2^{k-1} columnsrepresenting the smallest minimum cost resolution III $2^{n-(n-k)}$ design in $N=2^{k-1}$factorsthen adding factor columns successively until exhausting all (2^k-1) columns of the OA$(2^k, 2^k-1, 2,2)$ .Experimental runs, the k GFS generators ,total cost of factor level changes and the alias structure of these minimum cost resolution III $2^{n-(n-k)}$ designs$(2^{k-1}\le n\le(2^k-1))$ can be found from the saturated resolution III $2^{n-(n-k)}$ designin $N=(2^k-1)$ factors by dropping deleted factors.

To achieve both minimum cost and factors' time trend resistancewhen projecting the OA$(2^k, 2^k-1, 2,2)$ by factor deletion under resolution III, candidate columns are now all (2^k-1) columns of this OA excluding the k non-trend free columns $\{1,3,7,15,31,\ldots,(2^k-1)\}$, where the remaining $[(2^k-1)-k]$ trend free columns represent an unsaturated minimum cost trend free resolution III $2^{n-(n-k)}$ designin the largest $N=[(2^k-1)-k]$ trend free factorswith total cost of factor level changes $=[1+2+3+\ldots + (2^k-1)] -[1+3+7+15+31+\ldots+ (2^k-1)]= (2^{k-1}-2)(2^k-1)-k$, where the k GFS run generators are located at the k runs numbered $\{1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,\ldots, 2^{k-1}+1\}$. Projectionsof theminimum cost trend free resolution III $2^{n-(n-k)}$ designinN$=[(2^k-1)-k]$ trend free factors successively by factor deletion deleting trend free columns of large level changes (to minimize experimentation cost) until reaching the first 2^{k-1}trend-free columns with the least factor level changes represent the smallest minimum cost trend free resolution III $2^{n-(n-k)}$ design in $N=2^{k-1}$factors.Therefore, factor deletion of the OA$(2^k, 2^k-1, 2,2)$ produces a sequence of $(2^k-2^{k-1}-k)$minimum cost trend free resolution III $2^{n-(n-k)}$ designs $[2^{k-1}\le n\le(2^k-1)-k]$ without getting into run duplication.This design sequence can equivalently begenerated by factor addition starting factor addition with the first trend free 2^{k-1} columns of the OA$(2^k, 2^k-1, 2,2)$representing a minimum cost trend free resolution III $2^{n-(n-k)}$ design in the smallest number of trend free factors $N=2^{k-1}$then adding factor columns successively until exhausting all $(2^k-2^{k-1}-k)$candidate trend free columns.Experimental runs, the k GFS generators , total cost of factor level changes and the alias structure of these minimum cost trend free resolution III $2^{n-(n-k)}$ designs ($2^{k-1}\le n\le(2^k-1)$) can be found from the unsaturated minimum cost trend free resolution III $2^{n-(n-k)}$ designin $N=(2^k--1k)$ factors by dropping deleted factors.Putting k =4 reducethese general conclusions into the illustrative minimum cost /trend free resolution III $2^{n-(n-4)}$ designs$[2^{4-1}\le n\le(2^4-1)-4]$ of subsection 4.1.

(ii) Minimum cost / trend free resolution III $2^{n-(n-k)}$ designs$[k+1\le n\le(2^{k-1}- 1)+k]$(category Two Designs): Candidate columns for projectionof the OA$(2^k, 2^k-1, 2,2)$ by factor addition under resolution III are the $(k+ 2^{k-1} -1)$ columns numbered $\{1,2,4,8,\ldots, 2^{k-1} , (2^{k-1} +1),(2^{k-1} +2) , (2^{k-1} +3), \ldots, (2^k-1)\}$ .These $(k+ 2^{k-1} -1)$ columns can be grouped in two groups :

Group 1: contains the k columnsin the firs 2^{k-1} columns of the OA$(2^k, 2^k-1, 2,2)$ numbered $\{2^0,2^1 , 2^2 ,\ldots, 2^{k-1}\}$, where the first column is the only non-linear trend free column having nonzero Time Count.These k columns$\{2^0,2^1 , 2^2 ,\ldots, 2^{k-1}\}$represent together the minimal cost full 2^k factorial design in minimal total factor level changes $(1+2+4+8+\ldots+2^{k-1})=(2^k-1)$, which will be utilized to start the factor addition process.

Group 2:contains the last$(2^{k-1}1)$ columnsof the OA$(2^k, 2^k-1, 2,2)$,namely columns $\{ (2^{k-1}+1)+ (2^{k-1}+2)+\ldots+ (2^k-1)\}$,where the last column [i.e. (2^k-1)] is the only non-linear trend free column.These$(2^{k-1} - 1)$ column factors $\{(2^{k-1} +1),(2^{k-1} +2) , (2^{k-1} +3), \ldots, (2^k-1)\}$ have increasing number of level changes starting with $(2^{k-1}+1)$ increasing one by one until (2^k-1).

The $(k+ 2^{k-1} -1)$ columns of groups(1 and 2)produce together by factor addition a minimum cost resolution III $2^{n-(n-k)}$ design in the largest number of factors$M=(2^{k-1} -1+k)$, where its k GFS run generators are located at the k runs numbered $\{1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,\ldots, 2^{k-1}+1\}$.This largest minimum cost resolution III $2^{M-(M-k)}$ designcan also be used to start the factor deletion process.On the other hand, the first $(k+1)$ candidate columns of the OA$(2^k, 2^k-1, 2,2)$,namely columns $\{1,2,4,8,\ldots, 2^{k-1} , (2^{k-1} +1)\}$ containing all group1 and the first column of group 2 produce together the smallestminimum cost resolution III $2^{n-(n-k)}$ design,namely the $2^{(k+1)-(k+-1k)}$ half fraction with total cost of factor level changes $=[1+2+4+8\ldots+2^{k-1} + (2^{k-1}+1)]= 2^{k-1} + 2^{k-2}$, where its k GFS run generators are located at the k runs numbered $\{1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,\ldots, 2^{k-1}+1\}$.This smallestminimum cost resolution III $2^{n-(n-k)}$ designcan be used to start the factor addition process.Both factor addition and factor deletion applied on the $M=(2^{k-1} -1+k)$ candidate columnsof the OA$(2^k, 2^k-1, 2,2)$ in this design category lead to the same catalog of minimum cost resolution III $2^{n-(n-k)}$ designs $(k+1\le n\le k+ 2^{k-1} -1)$, one process works backwardly and the other forwardlyon these candidate $(k+ 2^{k-1} -1)$ columns .

The k GFS run generators for the foldover of the largest minimum cost resolution III $2^{n-(n-k)}$design of category two in the $n=(2^{k-1} + k - 1)$two-level factors$(A_1 ,A_2 ,A_3, \ldots,A_n)$ are :

$g_1=\prod_{i=k}^{2^{k-1}+k-1} a_i$

$g_2=\prod_{i=k-1}^{2^{k-2}+k-1} a_i$

$g_{k-2}= a_3a_4\left(\prod_{i=(k+2^2)+0(2^{k-(k-3)})}^{(k+2^2)+1(2^{k-(k-3)})-1} a_i\right)\left(\prod_{j=(k+2^2)+2(2^{k-(k-3)})}^{(k+2^2)+3(2^{k-(k-3)})-1} a_j\right)\ldots$

$\ldots\left(\prod_{l=(k+2^2)+(2^{k-4}-2)(2^{k-(k-3)})}^{(k+2^2)+(2^{k-4}-1)(2^{k-(k-3)})-1} a_l\right)$

$g_{k-1}= a_2a_3(\prod_{i=(k+2^1)+0(2^{k-(k-2)})}^{(k+2^1)+1(2^{k-(k-2)})-1} a_i)(\prod_{j=(k+2^1)+2(2^{k-(k-2)})}^{(k+2^1)+3(2^{k-(k-2)})-1} a_j)\ldots$
$\ldots(\prod_{l=(k+2^1)+(2^{k-3}-2)(2^{k-(k-2)})}^{(k+2^1)+(2^{k-3}-1)(2^{k-(k-2)})-1} a_l)$
$g_k= a_1a_2(\prod_{i=(k+2^0)+0(2^{k-(k-1)})}^{(k+2^0)+1(2^{k-(k-1)})-1} a_i)(\prod_{j=(k+2^0)+2(2^{k-(k-1)})}^{(k+2^0)+3(2^{k-(k-1)})-1} a_j)\ldots(\prod_{l=(k+2^0)+(2^{k-2}-2)(2^{k-(k-1)})}^{(k+2^0)+(2^{k-2}-1)(2^{k-(k-1)})-1} a_l)$ (5.1)

where the total cost of level changes for thesen=(2^{k-1} + k – 1)factors is the sum of their level changes , which is:
$C=\{2^0+2^1+2^2+\ldots+2^{k-1}\}+\{(2^{k-1}+1)+(2^{k-1}+2)+\ldots+(2^k-1)\}$ (5.2)

Experimental runs, the k GFS generators , total cost of factor level changes and the alias structure of the minimum cost resolution III $2^{n-(n-k)}$ designs (k+1≤n≤(2^{k-1}1+k)) of category two can be found from the unsaturated minimum cost trend free resolution III $2^{n-(n-k)}$ designin M=(2^{k-1}-1+k) factors by dropping deleted factors.

It is worth to note that minimum cost resolution III $2^{n-(n-k)}$ designs (2^{k-1}≤n≤(2^k-1)) of category one and minimum cost resolution III $2^{n-(n-k)}$ designs (k+1≤n≤k+ 2^{k-1} -1) of category two are based on different candidate sets of columns from the saturated OA(2^k, 2^k-1, 2,2), where category one $2^{n-(n-k)}$ designs do not produce half and quarter $2^{n-(n-k)}$ fractions while category two $2^{n-(n-k)}$ designs do. There is also an overlap in the number of factors between these two$2^{n-(n-k)}$ design categories but$2^{n-(n-k)}$ designs of category one (in the overlap region) have smaller total factor level changes.

To achieve both minimum cost and factors' time trend resistance under resolution III while projecting the saturated OA(2^k, 2^k-1, 2,2) utilizing the M=(2^{k-1} + k – 1) candidate columns of groups 1 and 2 of category two, we need to delete the two non-trend free columns : column 1 of group1 and column (2^k-1) of group 2. Group 1 is compensated by adding the trend free column numbered 5 to keep factor level changes small, hencegroup1 contains nowthe k trend free columns {2,4,5,8,16,32,..,2^{k-1}}.On the other hand, group 2 is reduced by 1 for deleting its last column.Therefore, candidate trend free columns for factor projection is now reduced into [2^{k-1} + k – 2] columns , which all together produce by either forward factor addition or backward factor deletion a sequence of minimum cost trend free resolution III $2^{n-(n-k)}$ designs (k+1≤ n≤(2^{k-1} + k – 2)),where their k GFS run generators are located at the k runs numbered {1, 2 ,3, 2^2+1 , 2^3+1 ,2^4+1 ,…, 2^{k-1} +1}.The k GFS generators for the foldover of the largestminimum cost linear trend free resolution III $2^{n-(n-k)}$designin the n=(2^{k-1} + k – 2) two-level factors(A_1 ,A_2 ,A_3, …,A_n)for k ≥ 5 are :

$g_1=\prod_{i=k}^{2^{k-1}+k-2} a_i$
$g_2=\prod_{i=k-1}^{2^{k-2}+k-1} a_i$
$g_{k-4}= a_5a_6(\prod_{i=(k+2^4)+0(2^{k-(k-5)})}^{(k+2^4)+1(2^{k-(k-5)})-1} a_i)(\prod_{j=(k+2^4)+2(2^{k-(k-5)})}^{(k+2^4)+3(2^{k-(k-5)})-1} a_j)\ldots$
$\ldots(\prod_{l=(k+2^4)+(2^{k-6}-2)(2^{k-(k-5)})}^{(k+2^4)+(2^{k-6}-1)(2^{k-(k-5)})-1} a_l)$
$g_{k-3}= a_4a_5(\prod_{i=(k+2^3)+0(2^{k-(k-4)})}^{(k+2^3)+1(2^{k-(k-4)})-1} a_i)(\prod_{j=(k+2^3)+2(2^{k-(k-4)})}^{(k+2^3)+3(2^{k-(k-4)})-1} a_j)\ldots$
$\ldots(\prod_{l=(k+2^3)+(2^{k-5}-2)(2^{k-(k-4)})}^{(k+2^3)+(2^{k-5}-1)(2^{k-(k-4)})-1} a_l)$
$g_{k-2}= a_2a_3a_4(\prod_{i=(k+2^2)+0(2^{k-(k-3)})}^{(k+2^2)+1(2^{k-(k-3)})-1} a_i)(\prod_{j=(k+2^2)+2(2^{k-(k-3)})}^{(k+2^2)+3(2^{k-(k-3)})-1} a_j)\ldots$
$\ldots(\prod_{l=(k+2^2)+(2^{k-4}-2)(2^{k-(k-3)})}^{(k+2^2)+(2^{k-4}-1)(2^{k-(k-3)})-1} a_l)$
$g_{k-1}= a_1a_2a_3(\prod_{i=(k+2^1)+0(2^{k-(k-2)})}^{(k+2^1)+1(2^{k-(k-2)})-1} a_i)(\prod_{j=(k+2^1)+2(2^{k-(k-2)})}^{(k+2^1)+3(2^{k-(k-2)})-1} a_j)\ldots$
$\ldots(\prod_{l=(k+2^1)+(2^{k-3}-2)(2^{k-(k-2)})}^{(k+2^1)+(2^{k-3}-1)(2^{k-(k-2)})-1} a_l)$
$g_k= a_1a_3(\prod_{i=(k+2^0)+0(2^{k-(k-1)})}^{(k+2^0)+1(2^{k-(k-1)})-1} a_i)(\prod_{j=(k+2^0)+2(2^{k-(k-1)})}^{(k+2^0)+3(2^{k-(k-1)})-1} a_j)\ldots(\prod_{l=(k+2^0)+(2^{k-2}-2)(2^{k-(k-1)})}^{(k+2^0)+(2^{k-2}-1)(2^{k-(k-1)})-1} a_l)$ (5.3)

with the total cost of level changes for these n=(2^{k-1} + k – 2)factors is the sum of their level changes in each of the two groups,which is:
$C=\{2^1+2^2+(2^2+1)+2^3+2^4+\ldots+2^{k-1}\}+\{(2^{k-1}+1)+(2^{k-1}+2)+\ldots+(2^k-2)\}$ (5.4)

Experimental runs, the k GFS generators , total cost of factor level changes and the alias structure of these minimum cost trend free resolution III $2^{n-(n-k)}$ designs (k+1≤n≤(2^{k-1} - 1 +k)) of category two can be found from the unsaturated minimum cost trend free resolution III $2^{n-(n-k)}$ designin M=(2^{k-1}- 1 +k) factors by dropping deleted factors.Putting k= 4 into these general resultsin (5.1), (5.2), (5.3) and (5.4) reduce to the illustrative minimum cost/ trend free resolution IV $2^{n-(n-4)}$ designs(4+1≤n≤(2^{4-1}-1+k)) of subsection 4.2.

(iii) Minimum cost / trend free resolution IV $2^{n-(n-k)}$ designs (2^{k-2} ≤ n≤(2^{k-1}2) (Category Three Designs):

It is documented in the literature that the maximum number of column factors for projection of the saturated OA(2^k, 2^k-1, 2,2)into$2^{n-(n-k)}$ fractional factorial designs under resolution IV is 2^{k-1},where this number is half the number of runs,namely = $2^{-1}.2^k$. These 2^{k-1}candidate column factors under resolution IV were found by extensive computer work on Sylvester Hadamard Matrices of size 2^kx2^k and their associated OA(2^k, 2^k-1, 2,2) for

$k=4,5,6,7,8,9,10$.These 2^{k-1} candidate column factorscan however be grouped into three groups of consecutive columns each, where:

Group 1:contains 2^{k-2}columns , which are columns $\{2^{k-3}, (2^{k-3}+1), (2^{k-3}+2),\ldots, (2^{k-3}+2^{k-2}-1)\}$, where the single column $(2^{k-2}-1)$ is non-linear trend free having nonzero Time Count.These 2^{k-2} column factors have increasing number of level changes starting with 2^{k-3} and increase one by one until $(2^{k-3}+2^{k-2}-1)$.

Group 2: contains 2^{k-3}columns , which are the columns $\{2^{k-1}, (2^{k-1}+1), (2^{k-1}+2),\ldots,(2^{k-1}+2^{k-3}-1)\}$,where all are linear trend free each having zero Time Count. These 2^{k-3} column factors have increasing number of level changes starting with $2^{k-1}$and increase one by one until $(2^{k-1}+2^{k-3}-1)$.

Group 3: contains $(2^{k-1}- 2^{k-2} -2^{k-3})$ columns, which are the columns $\{(2^{k}- 2^{k-3}), (2^{k}-2^{k-3}+1), (2^{k}- 2^{k-3} +2),\ldots, (2^{k}-1)\}$, where the last column $(2^{k}-1)$ is the only non-linear trend free column.These $(2^{k-1} - 2^{k-2}-2^{k-3})$column factors have increasing number of level changes starting with $(2^{k} – 2^{k-3})$ and increase one by one until $(2^{k}-1)$.

Therefore, the largestminimum cost resolution IV $2^{n-(n-k)}$ design in $N=2^{k-1}$factors that can be constructed from the saturated $OA(2^{k}, 2^{k}-1, 2,2)$ is the$2^{N-(N-k)}$ designhaving the $N=2^{k-1}$ columns of groups 1,2 and 3,where these $n=2^{k-1}$ columns will be denoted by factors $(A_1 ,A_2 ,A_3, \ldots,A_n)$.Level changes of these $n=2^{k-1}$ two-level factors are in increasing order, the smallest level change is 2^{k-3} while the largest factor level change is $(2^{k}-1)$.The k GFS generators for the foldover of this minimum cost resolution IV $2^{n-(n-k)}$designin $n=2^{k-1}$factors are located at the k runs numbered $\{1, 2 ,3, 2^{2}+1 , 2^{3}+1 ,2^{4}+1 ,\ldots, 2^{k-1}+1\}$ and they are :

$g_1=\prod_{i=2^{k-2}+1}^{2^{k-1}} a_i$

$g_2=\prod_{i=2^{k-3}+1}^{2^{k-3}+2^{k-2}} a_i$

$g_3=\prod_{i=1}^{2^{k-2}} a_i$

$g_4=(\prod_{i=1}^{2^{k-4}} a_i)(\prod_{j=2^{k-4}+2^{k-3}+1}^{2(2^{k-4})+2^{k-3}} a_j)(\prod_{l=3(2^{k-4})+2^{k-3}+1}^{3(2^{k-4})+2(2^{k-3})} a_l)$

$g_{k-2}=(\prod_{i=(2^{k-(k-2)}+1)+0(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+1(2^{k-(k-3)})-1} a_i)(\prod_{j=(2^{k-(k-2)}+1)+2(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+3(2^{k-(k-3)})-1} a_j)\ldots$

$\ldots(\prod_{l=(2^{k-(k-2)}+1)+(2^{k-4}-2)(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+(2^{k-4}-1)(2^{k-(k-3)})-1} a_l)$

$g_{k-1}=(\prod_{i=(2^{k-(k-1)}+1)+0(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+1(2^{k-(k-2)})-1} a_i)(\prod_{j=(2^{k-(k-1)}+1)+2(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+3(2^{k-(k-2)})-1} a_j)\ldots$

$\ldots(\prod_{l=(2^{k-(k-1)}+1)+(2^{k-3}-2)(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+(2^{k-3}-1)(2^{k-(k-2)})-1} a_l)$

$g_k=(\prod_{i=(2^{k-k}+1)+0(2^{k-(k-1)})}^{(2^{k-k}+1)+1(2^{k-(k-1)})-1} a_i)(\prod_{j=(2^{k-k}+1)+2(2^{k-(k-1)})}^{(2^{k-k}+1)+3(2^{k-(k-1)})-1} a_j)\ldots(\prod_{l=(2^{k-k}+1)+(2^{k-2}-2)(2^{k-(k-1)})}^{(2^{k-k}+1)+(2^{k-2}-1)(2^{k-(k-1)})-1} a_l)$

(5.5)

The total cost of level changes for these $n=2^{k-1}$factorsis the sum of their level changes in each of the three groups ,which equals:

$C=\{2^{k-3}+ (2^{k-3}+1)+ (2^{k-3}+2)+\ldots+ (2^{k-3}+2^{k-2}-1)\}+ \{2^{k-1}+ (2^{k-1}+1)+ (2^{k-1}+2)+\ldots+(2^{k-1}+2^{k-3}-1)\} +\{(2^{k} – 2^{k-3})+ (2^{k} – 2^{k-3}+1)+ (2^{k} – 2^{k-3} +2)+\ldots+ (2^{k}-1)\}$ or more compactly

$C=\sum_{i=2^{k-3}}^{2^{k-3}+2^{k-2}-1} i +\sum_{i=2^{k-1}}^{2^{k-1}+2^{k-3}-1} i + \sum_{l=2^{k}-2^{k-3}}^{2^{k-1}+n-1} l$ (5.6)

Therefore, a total of $(2^{k-1} -2^{k-2})$ unsaturated minimum cost resolution IV $2^{n-(n-k)}$ designs$(2^{k-2} \leq n\leq 2^{k-1})$ can be constructed from the minimum cost resolution IV $2^{n-(n-k)}$ design with the largest number of factors $N = 2^{k-1}$ by either factor addition or deletion . Experimental runs, the k GFS generators , total cost of factor level changes and the alias structure of these minimum cost resolution IV $2^{n-(n-k)}$ designs $(2^{k-2} \leq n\leq 2^{k-1})$ of category three can be found from the unsaturated minimum cost resolution IV $2^{n-(n-k)}$ designin $N=2^{k-1}$ factors by dropping deleted factors.

These minimum cost resolution IV $2^{n-(n-k)}$ designs($2^{k-2} \leq n\leq 2^{k-1}$) are economic in minimum total number of factor level changes but not all their factors are resistant to the time trend ,since some of these factors have nonzero Time Counts.Therefore,to construct minimum cost trend free resolution IV$2^{n-(n-k)}$ designs from the $N=2^{k-1}$candidate columns under resolution IV in groups (1,2 and 3) we need to drop the two non-trend free columns : column $(2^{k-2}-1)$ in the first group and column($2^{k}-1$) in the third group ,leaving a total of ($2^{k-1}-2$) candidate trend free columns for factor projection under resolution IV.These ($2^{k-1}-2$) candidate column factors produce the largest minimum cost trend free resolution IV $2^{n-(n-k)}$design in $n=(2^{k-1}-2)$two-level factors,where factor level changes are in increasing order, the smallest factor level change is 2^{k-3} and the largest is $(2^{k}-2)$.The k GFS generators for the foldover of the largest minimum cost trend free resolution IV $2^{n-(n-k)}$ designs in $n=(2^{k-1}-2)$two-level factors $(A_1 ,A_2 ,A_3, \ldots,A_n)$ are:

$g_1=\prod_{i=2^{k-2}}^{2^{k-1}-2} a_i$

$g_2=\prod_{i=2^{k-3}}^{2^{k-3}+2^{k-2}-1} a_i$

$g_3=\prod_{i=1}^{2^{k-2}-1} a_i$

$g_4=(\prod_{i=1}^{2^{k-4}} a_i)(\prod_{j=2^{k-4}+2^{k-3}}^{2(2^{k-4})+2^{k-3}-1} a_j)(\prod_{l=3(2^{k-4})+2^{k-3}}^{3(2^{k-4})+2(2^{k-3})-1} a_l)$

.

$g_{k-2}=(\prod_{i=(2^{k-(k-2)}+1)+0(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+1(2^{k-(k-3)})-1} a_i)(\prod_{j=(2^{k-(k-2)}+1)+2(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+3(2^{k-(k-3)})-1} a_j)\ldots$

…

$(\prod_{l=(2^{k-(k-2)}+1)+(2^{k-6}-2)(2^{k-(k-3)})}^{(2^{k-(k-2)}+1)+(2^{k-6}-1)(2^{k-(k-3)})-1} a_l)(\prod_{m=(2^{k-(k-2)}+1)+(2^{k-6}-1)(2^{k-(k-3)})+(2^{k-(k-3)}-1)}^{(2^{k-(k-2)}+1)+(2^{k-6})(2^{k-(k-3)})+(2^{k-(k-3)}-2)} a_m)(\prod_{p=(2^{k-(k-2)}+1)+(2^{k-6}+1)(2^{k-(k-3)})+(2^{k-(k-3)}-1)}^{(2^{k-(k-2)}+1)+(2^{k-6}+2)(2^{k-(k-3)})+(2^{k-(k-3)}-2)} a_p)\ldots$

$\ldots(\prod_{q=(2^{k-(k-2)}+1)+(2^{k-4}-3)(2^{k-(k-3)})+(2^{k-(k-3)}-1)}^{(2^{k-(k-2)}+1)+(2^{k-4}-2)(2^{k-(k-3)})+(2^{k-(k-3)}-2)} a_q)$

$g_{k-1}=(\prod_{i=(2^{k-(k-1)}+1)+0(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+1(2^{k-(k-2)})-1} a_i)(\prod_{j=(2^{k-(k-1)}+1)+2(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+3(2^{k-(k-2)})-1} a_j)\ldots$

…

$(\prod_{l=(2^{k-(k-1)}+1)+(2^{k-5}-2)(2^{k-(k-2)})}^{(2^{k-(k-1)}+1)+(2^{k-5}-1)(2^{k-(k-2)})-1} a_l)(\prod_{m=(2^{k-(k-1)}+1)+(2^{k-5}-1)(2^{k-(k-2)})+(2^{k-(k-2)}-1)}^{(2^{k-(k-1)}+1)+(2^{k-5})(2^{k-(k-2)})+(2^{k-(k-2)}-2)} a_m)(\prod_{p=(2^{k-(k-1)}+1)+(2^{k-5}+1)(2^{k-(k-2)})+(2^{k-(k-2)}-1)}^{(2^{k-(k-1)}+1)+(2^{k-5}+2)(2^{k-(k-2)})+(2^{k-(k-2)}-2)} a_p)\ldots$

$\ldots(\prod_{q=(2^{k-(k-1)}+1)+(2^{k-3}-3)(2^{k-(k-2)})+(2^{k-(k-2)}-1)}^{(2^{k-(k-1)}+1)+(2^{k-3}-2)(2^{k-(k-2)})+(2^{k-(k-2)}-2)} a_q)$

$g_k=(\prod_{i=(2^{k-k}+1)+0(2^{k-(k-1)})}^{(2^{k-k}+1)+1(2^{k-(k-1)})-1} a_i)(\prod_{j=(2^{k-k}+1)+2(2^{k-(k-1)})}^{(2^{k-k}+1)+3(2^{k-(k-1)})-1} a_j)\ldots$

…

$(\prod_{l=(2^{k-k}+1)+(2^{k-4}-2)(2^{k-(k-1)})}^{(2^{k-k}+1)+(2^{k-4}-1)(2^{k-(k-1)})-1} a_l)(\prod_{m=(2^{k-k}+1)+(2^{k-4}-1)(2^{k-(k-1)})+(2^{k-(k-1)}-1)}^{(2^{k-k}+1)+(2^{k-4})(2^{k-(k-1)})+(2^{k-(k-1)}-2)} a_m)(\prod_{p=(2^{k-k}+1)+(2^{k-4}+1)(2^{k-(k-1)})+(2^{k-(k-1)}-1)}^{(2^{k-k}+1)+(2^{k-4}+2)(2^{k-(k-1)})+(2^{k-(k-1)}-2)} a_p)\ldots$

$\ldots(\prod_{q=(2^{k-k}+1)+(2^{k-2}-3)(2^{k-(k-1)})+(2^{k-(k-1)}-1)}^{(2^{k-k}+1)+(2^{k-2}-2)(2^{k-(k-1)})+(2^{k-(k-1)}-2)} a_q)$

(5.7)

The total cost of level changes for these $n=(2^{k-1}-2)$factors is the sum of their level changes in each of the three groups ,which equals:

$C=\{2^{k-3}+(2^{k-3}+1)+(2^{k-3}+2)+\ldots+(2^{k-2}-2)+2^{k-2}+(2^{k-2}+1)+\ldots+(2^{k-3}+2^{k-2}-1)\}+\{2^{k-1}+(2^{k-1}+1)+(2^{k-1}+2)+\ldots+(2^{k-1}+2^{k-3}-1)\}+\{(2^k-2^{k-3})+(2^k-2^{k-3}+1)+(2^k-2^{k-3}+2)+\ldots+(2^k-2)\}$, or more compactly

$C=\sum_{i=2^{k-3}}^{2(2^{k-3})-2} i+\sum_{j=2(2^{k-3})}^{2^{k-3}+2^{k-2}-1} j+\sum_{i=2^{k-1}}^{2^{k-1}+2^{k-3}-1} l+\sum_{i=2^k-2^{k-3}}^{2^k-2} m$ (5.8)

Experimental runs, the k GFS generators , total cost of factor level changes and the alias structure of the minimum cost trend free resolution IV $2^{n-(n-k)}$ designs ($2^{k-2}\le n\le 2^{k-1}-2$) of category three can be found from the unsaturated minimum cost trend free resolution IV $2^{n-(n-k)}$ designin $N=(2^{k-1}-2)$ factors by dropping deleted factors.Putting k =4 into these general resultsin (5.5),(5.6),(5.7) and (5.8) reduce to the illustrative minimum cost/ trend free resolution IV $2^{n-(n-4)}$ designs($2^{4-2}\le n\le 2^{4-1}-2$) of subsection 4.3.

6. Discussion and Conclusion

Fractional 2^{n-k}factorial experiments with factors having levels hard- to- vary should be carried out sequentially (i.e.not randomly) either run after run or block of runs after block in order to economize the cost of varying factor levels between successive runs. However,systematic fractional 2^{n-k}factorial experiments suffer from the problem that factor effects may be adversely affected by a time trend which might be present among responses of the successive runs. Therefore,2^{n-k} fractional factorial experiments should be sequenced but overcome this time trend problem and also economize the experimental cost. There are a total of $2^{n-k}!$ run orders (i.e. permutations) to carry out fractional 2^{n-k}factorial experiments run after run but not all these run orders are resistant to the time trend nor economic. Also not all these $2^{n-k}!$ run orders can be sequenced by the GFS technique, yet economic run orders resistant to the time trend can be generated by the GFS approach.

This research has utilized the Normal Sylvester-Hadamard matrices of size 2^k x $2^k$and their associated saturated orthogonal arrays OA(2^k ,2^k -1,2,2) to construct (by factor projection) three systematic $2^{n-(n-k)}$ fractional factorial designsof resolutions III and IV that are economic regarding the cost of factor level changesand/or resistant to the non-negligible time trend.Proposed $2^{n-(n-k)}$ fractional factorial designs have the merit that all their 2^k experimental runs can be sequenced run- after- run by the GFS technique using only k independent run generators,where these k independent generator runs aregiven.The other merit is that all factor effects can be estimated unbiased by the non-negligible time trend. Comparison with existing counterpart systematic 2^{n-k}designs shows that the proposed$2^{n-(n-k)}$designs compete well with and sometimes are better,since they have: (i) smaller cost of factor level changes between successive runs and (ii) securedall factor effects to be orthogonal and unbiased by the non-negligible time trend. All this is done without fixingan upper limit for either the number of factors or the fractionation level, while maintaining resolution III or IV without duplicatingany experimental run. It is however worth to conduct a comparison among existing runs sequencing algorithms for the 2^{n-k} fractional factorial experiment (run after run or block after block) in terms of thethe following parameters: the total cost of factor level changes,pattern of factor level changes, factors' time trend resistance, the resolution and the GFS generators.

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