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Discussion of a Problem in Analytic Geometry

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Abstract

Analytic geometry is a branch of mathematics that studies geometric figures through algebraic methods and coordinate systems to solve geometric problems. Concepts such as the relationship between lines and planes, projections, and solid rotations have wide applications in fields such as engineering, physics, and computer graphics. To help students better understand and master these concepts, this paper will explore the basic methods and teaching strategies of analytic geometry through a specific problem.

Keywords: Plane, Projected line, Pencil of planes, Rotational surface

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1. Introduction

Find the line l_0 :

$$\frac{x-1}{1}=\frac{y}{1}=\frac{z-1}{-1}$$

Determine the equation of the projection of the line l_0 in the plane $\pi: x-y+2z-1=0$, and find the equation of the surface formed by rotating l_0 around the y-axis for one full revolution.

This problem examines the fundamental concepts of spatial geometry and calculus, requiring students to comprehensively apply these concepts. By solving the problem, students can deepen their understanding of projection and the rotational surface equation, while also improving their ability to solve complex spatial geometry problems.

The problem has a certain level of difficulty and challenge, making it suitable as a practice problem for intermediate to advanced spatial analytic geometry or calculus courses. Below, we break down the problem into smaller subproblems using a systematic approach and solve them accordingly based on standard methods.

Let us assume the vector parametric equation of the line l is given as:

$$\overrightarrow{r(t)}=\vec{r_0}+t\vec{v},$$

where $\vec{r_0}=\{x_0, y_0, z_0\}$ is a point on the line, $\vec{v}=\{X, Y, Z\}$ is the direction vector of the line, and t is the parameter. From the vector parametric equation, we can obtain

$\begin{cases} x=x_0+Xt \\ y=y_0+Yt \\ z=z_0+Zt \end{cases}$, this is also referred to as the coordinate parametric equation of the line.

2. Methods to find the projection of a line onto a plane

To find the projection of a line onto a plane, several different geometric and algebraic methods can be used. Below are some commonly used approaches:

2.1 Method 1: Using vectors and parametric equations

(1) Direction vector of the projection line

The direction vector of the projection line can be obtained by projecting the direction vector of the line onto the plane. First, calculate the component of the line's direction vector in the direction of the plane's normal vector:

$$\overrightarrow{n_1} = \left(\frac{\vec{v} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \right) \vec{n} .$$

Then, subtract this component to get the direction vector of the projected line in the plane: $\vec{s} = \vec{v} - \overrightarrow{n_1}$.

(2) Point of the projection line
The projected line l_p intersects the plane π. To find the intersection point, the method is to substitute the parametric equation of the original line l into the equation of the plane π:

$$A(x_0 + tX) + B(y_0 + tY) + C(z_0 + tZ) + D = 0$$

Solving for $t = t_0$, substitute t_0 into the parametric equation of the line to obtain the coordinates of the intersection point:

$$(x_0 + t_0 X,\, y_0 + t_0 Y,\, z_0 + t_0 Z).$$

(3) The equation of the projected line

The parametric equation of the projected line l_p is

$$\overrightarrow{r(t)} = (x_0 + t_0 X,\, y_0 + t_0 Y,\, z_0 + t_0 Z) + t\vec{s} .$$

Detailed Solution:

Rewrite the line l as: $\begin{cases} x = 1 + t \\ y = t \\ z = 1 - t \end{cases}$ .The line passes through the point $(1, 0, 1)$, and the direction vector of the line is $\vec{v} = \{1, 1, -1\}$. For the plane $\pi : x - y + 2z - 1 = 0$, its normal vector is $\vec{n} = \{1, -1, 2\}$. Based on the solution steps above, the direction vector of the projection of the line onto the plane is:

$$\overrightarrow{n_1} = \left(\vec{v} \cdot \vec{n} / \vec{n} \cdot \vec{n}\right) \vec{n} = \{-1/3, 1/3, 2/3\}.$$

The direction vector of the projection of the line onto the plane: $\vec{s} = \vec{v} - \overrightarrow{n_1} = \{4/3, 2/3, -1/3\}$. Substitute the parametric equations of the line l into the plane equation $\pi : (1 + t) - t + 2(1 - t) - 1 = 0$， Solve to get $t = 1$.

The equation of the projection line is solved as

$$\overrightarrow{r(t)} = (4t + 2, 1 + 2t, -t) \tag{1}$$

2.2 Method 2: Using the plane pencil equation

Definition[Lv2019]

The collection of all planes passing through the same line in space is called a plane pencil, and the line is called the axis of the plane pencil.

Theorem[Lv2019]

If two planes $\pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0$ and $\pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0$ intersect in a line l, then the equation of the plane pencil with this line l as its axis is

$$\lambda(A_1 x + B_1 y + C_1 z + D_1) + \mu(A_2 x + B_2 y + C_2 z + D_2) = 0 ,$$

where λ and μ are arbitrary non-zero real numbers.

(1) Converting the symmetric equation of a line to the general equation of a line

From the parametric equation of the line's direction vector, the symmetric equation of the line can be written as

$$\frac{x - x_0}{X} = \frac{y - y_0}{Y} = \frac{z - z_0}{Z} .$$

This can be transformed into the following general form

$$\begin{cases} \pi_1 : \dfrac{x-x_0}{X} = \dfrac{y-y_0}{Y} \\ \pi_2 : \dfrac{y-y_0}{Y} = \dfrac{z-z_0}{Z} \end{cases}.$$

(2) Determining the plane pencil equation passing through line l

The plane pencil equation is obtained by forming a linear combination of the two plane equations, resulting in the plane pencil equation passing through line l :

$$\pi_1 : \lambda\left(x - x_0 - \frac{X}{Y}y + \frac{X}{Y}y_0\right) + \mu\left(y - y_0 - \frac{Y}{Z}z + \frac{Y}{Z}z_0\right) = 0,$$

where λ and μ are parameters. The rewritten form of plane π_1 is

$$\lambda x - \left(\lambda\frac{X}{Y} - \mu\right)y - \mu\frac{Y}{Z}z - \left(\lambda x_0 - \lambda\frac{X}{Y}y_0 + \mu y_0 - \mu\frac{Y}{Z}z_0\right) = 0.$$

Therefore, the normal vector of plane π_1 is $\overrightarrow{n_1} = \left(\lambda, \mu - \lambda\dfrac{X}{Y}, -\mu\dfrac{Y}{Z}\right)$.

(3) Determining the equation of the plane passing through line l and perpendicular to plane π

For a plane that passes through line l and is perpendicular to plane π , their normal vectors are perpendicular to each other, so

$$\vec{n} \cdot \overrightarrow{n_1} = (A, B, C) \cdot \left(\lambda, \mu - \lambda\frac{X}{Y}, -\mu\frac{Y}{Z}\right) = A\lambda + B\left(\mu - \lambda\frac{X}{Y}\right) - C\mu\frac{Y}{Z} = 0.$$

From this, we obtain

$$\frac{\lambda}{\mu} = \frac{\left(C\dfrac{Y}{Z} - B\right)}{\left(A - B\dfrac{X}{Y}\right)}.$$

Therefore, we can let $\lambda = \lambda_0 = CY/Z - B$ and $\mu = \mu_0 = A - BX/Y$. Thus, the equation of the plane π_1 that passes through line l and is perpendicular to plane π is

$$\lambda_0 x - \left(\lambda_0\frac{X}{Y} - \mu_0\right)y - \mu_0\frac{Y}{Z}z - \left(\lambda_0 x_0 - \lambda_0\frac{X}{Y}y_0 + \mu_0 y_0 - \mu_0\frac{Y}{Z}z_0\right) = 0.$$

(4) The general equation of the projection line

The projection line can be considered as the intersection of planes π and π_1 . Therefore, the general equation of the projection line is

$$\begin{cases} \pi_1 : \lambda_0 x - \left(\lambda_0\dfrac{X}{Y} - \mu_0\right)y - \mu_0\dfrac{Y}{Z}z - \left(\lambda_0 x_0 - \lambda_0\dfrac{X}{Y}y_0 + \mu_0 y_0 - \mu_0\dfrac{Y}{Z}z_0\right) = 0 \\ \pi : Ax + By + Cz + D = 0 \end{cases}.$$

Detailed Solution:

Rewrite the symmetric equation of the line l

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{-1},$$

and convert it to the general equation of the line

$$\begin{cases} \pi_1 : \dfrac{x-1}{1} = \dfrac{y}{1} \\ \pi_2 : \dfrac{y}{1} = \dfrac{z-1}{-1} \end{cases},$$

which gives $l : \begin{cases} \pi_1 : x - y - 1 = 0 \\ \pi_2 : y + z - 1 = 0 \end{cases}$. The plane pencil equation passing through line l is

$$\pi_1 : \lambda(x - y - 1) + \mu(y + z - 1) = 0,$$

which simplifies to $\pi_1 : \lambda x - (\lambda - \mu)y + \mu z - (\lambda + \mu) = 0$. Thus, the normal vector of plane π_1 is $\overrightarrow{n_1} = (\lambda, \mu - \lambda, \mu)$.

Since the plane passing through line l is perpendicular to plane π, their normal vectors are perpendicular, i.e., $\vec{n} \cdot \overrightarrow{n_1} = 0$. Thus, we obtain $\lambda/\mu = -1/2$. Therefore, we can take $\lambda = 1$ and $\mu = -2$. Hence, the equation of the plane passing through line l and perpendicular to π is

$$x - 3y - 2z + 1 = 0.$$

The projection line can be regarded as the intersection of π and π_1, so the general equation of the projection line is

$$\begin{cases} x - y + 2z - 1 = 0 \\ x - 3y - 2z + 1 = 0 \end{cases}.$$

The general equation of the projection line can be written in the following parametric form

$$\begin{cases} x = 2y \\ y = y \\ z = -\dfrac{1}{2}(y - 1) \end{cases}. \tag{2}$$

3. Equation of surface from line rotation around an axis

The parametric equation of the line l is $\begin{cases} x = x_0 + Xt \\ y = y_0 + Yt \\ z = z_0 + Zt \end{cases}$.

Without loss of generality, assume that the line l rotates around the y-axis. Let A be any point on line l. The coordinates of point A can be expressed as $(x_0 + Xt, y_0 + Yt, z_0 + Zt)$. Next, let $B(x, y, z)$ be an arbitrary point on the trajectory of A when it rotates around the y-axis. The trajectory of point A after rotating around the y-axis forms a circle. The center of the circle is denoted as Q, and the radius is the distance from point A to the y-axis. Let the origin of the coordinate system be O. Then the triangles ΔAQO and ΔBQO are right triangles.

From the properties of a right triangle, we know

$$|AQ|^2 + |OQ|^2 = |OA|^2 \text{ and } |BQ|^2 + |OQ|^2 = |OB|^2.$$

Since $|AQ| = |BQ|$, it follows that $|OA| = |OB|$. Therefore, we have

$$(x_0 + Xt)^2 + (y_0 + Yt)^2 + (z_0 + Zt)^2 = x^2 + y^2 + z^2.$$

Because the rotation is around the y-axis, the y-coordinate remains unchanged before and after rotation, i.e., $y = y_0 + Yt$. We can write

$$t = \frac{y - y_0}{Y}.$$

Substitute it into the equation

$$(x_0 + Xt)^2 + (z_0 + Zt)^2 = x^2 + z^2,$$

which gives the equation of the surface of rotation

$$(x_0 + X\frac{y - y_0}{Y})^2 + (z_0 + Z\frac{y - y_0}{Y})^2 = x^2 + z^2. \quad (3)$$

Detailed Solution:

From (1) or (2), we know $X = 4, Y = 2, Z = -1$, $x_0 = 2,\ y_0 = 1, z_0 = 0$. Substituting these values into (3), the equation of the surface of rotation is obtained as

$$4x^2 - 17y^2 + 4z^2 + 2y - 1 = 0.$$

4. Conclusion

The problems of lines and planes, projections, and solids of revolution in analytic geometry not only hold significant theoretical importance but also have broad practical applications. Through the analysis of specific problems and the exploration of teaching methods, students' understanding and application skills in analytic geometry can be effectively enhanced. Teachers should focus on integrating theory with practice, fostering students' thinking abilities, providing personalized instruction, and delivering timely feedback and evaluation to improve teaching outcomes.

References

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