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Research on the Problem Chain of Linear Algebra Under the Main Thread of Systems of Linear Equations

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Abstract

Addressing the cognitive challenges of "many abstract concepts and fragmented knowledge" in linear algebra teaching, this paper proposes to use "solving systems of linear equations" as the central main thread running through the entire course, and reconstructs the mathematics classroom using the "problem chain" teaching method based on APOS theory and deep learning concepts. The paper analyzes the epistemological dominant position of this main thread and three core design principles of the problem chain. Organized around five progressive stages—"introduction, operation, structure, sublimation, and summary"—the framework constructs an interlocking spiral problem network: beginning from system modeling and data encapsulation, progressing through the excavation of invariants in elimination and the geometric characterization of solution structures, and culminating in the search for optimal observation perspectives and the integration of the knowledge system. Through carefully designed questioning to induce cognitive conflict, obscure concepts such as matrices, rank, and eigenvalues are restored as inevitable products of overcoming bottlenecks in solving systems of equations. Teaching empirical research shows that this model significantly activates students' internal motivation and effectively drives the educational transition from mechanical problem-solving to cultivating holistic computational thinking.

Keywords: Linear Algebra; System of Linear Equations Main Thread; Problem Chain; APOS Theory; Deep Learning; Teaching Reconstruction

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1. Introduction

Linear algebra is an indispensable mathematical cornerstone in modern science, big data analysis, and artificial intelligence (Strang, 2016). However, current linear algebra teaching universally faces the pain points of "two excessives and two deficiencies": excessive abstract concepts and mechanical calculations; meanwhile lacking descriptions of the natural generative process of knowledge, and lacking support driven by practical engineering problems (Jiang et al., 2021). The traditional closed deductive loop of "Definition—Theorem—Proof", although ensuring logical rigor, often conceals the original process of guessing and exploring during the early stages of mathematical knowledge creation. Many engineering students fail to grasp the algebraic structural characteristics and geometric significance behind matrix operations after class, making it difficult for them to achieve flexible knowledge transfer and application in subsequent professional courses (Qiu, 2020).

Based on this, the present paper aims to break the traditional didactic paradigm and explore a classroom reconstruction model driven by the "problem chain" with "systems of linear equations" as the central disciplinary thread. It attempts to guide students to autonomously deduce high-order abstract concepts such as matrix rank and maximal linearly independent systems through carefully arranged series of situational problems during the exploration of multi-dimensional constraint systems, seeking a feasible pedagogical path to shift from "mechanical memorization" to "active construction" through empirical research.

2. Theoretical Basis and Literature Review

In order to scientifically and completely reconstruct the teaching system of linear algebra, this section will first analyze the essence of the "problem chain" teaching concept based on APOS and deep learning theories, and then summarize the current research status of the teaching main thread and problem chains from the perspective of literature review.

2.1 Theoretical Basis of "Problem Chain" Teaching

A "problem chain" refers to translating teaching content into a logical problem system with interlocking and progressively advancing steps according to educational goals and students' existing cognitive levels. To break through the limitations of empiricism, the instructional design of this study is deeply rooted in APOS cognitive theory and modern deep learning concepts.

In terms of micro-mathematical cognitive mechanisms, the APOS theory proposed by E. Dubinsky emphasizes that the understanding of mathematical concepts must go through the stages of "Action—Process—Object—Schema" (Dubinsky, 1991). Due to the high abstraction of linear algebra, students often encounter a cognitive break when transforming from "Action" to "Object." The "problem chain" serves as cognitive scaffolding here, guiding students to internalize and encapsulate the "dynamic behavior of solving equations" into abstract algebraic objects through gradient questioning, ultimately weaving it into a global cognitive schema.

Meanwhile, at the classroom occurrence level, the problem chain is the core triggering mechanism for inducing "deep learning." Deep learning, which focuses on critical thinking and concept transfer, requires breaking the cognitive inertia brought about by the single-direction transmission of the traditional "Definition—Theorem—Proof" framework. Conflict-oriented problem chains continuously create cognitive imbalance, forcing students to engage in autonomous reasoning and reflection when solving complex systems, thereby cultivating higher-order computational thinking and modeling abilities (Wang, 2017).

2.2 Domestic and International Research Status

In response to the teaching difficulties caused by the "two excessives and two deficiencies" in linear algebra, multi-dimensional educational reform attempts have been carried out in domestic and foreign educational circles. In reshaping the teaching thread, domestic scholars such as Shangzhi Li strongly advocate for "returning to the origin," emphasizing the reduction of course cognition to the basic problem of solving systems of equations (Li, 2007). International frontier literature, such as David C. Lay's classic textbook *Linear Algebra and Its Applications*, has also thoroughly implemented the macro-architecture with linear systems as the primary narrative (Lay, 2018). Regarding overcoming algebraic cognitive obstacles, international scholars like Carlson emphasized the critical role of meaningful problem sequences in the psychological construction of concepts (Carlson & Bloom, 2005); Sfard's "Process-Object" theory proved that operational cognition must occur before object understanding (Sfard, 1991); Harel pointed out through the DNR principles that students desperately need to generate the "intellectual need" to explore the unknown (Harel, 2008).

In specific instructional intervention means, the spiraling "problem chain" has attracted much attention for its effectiveness in building cognitive scaffolding. Some early empirical explorations (Jiang & Dai, 2018; Qiu & Zhang, 2015) have proven that problem chains can significantly maintain classroom logical tension and thinking depth. In recent years, domestic scholars have further deepened their research in this field, focusing on theoretical and practical explorations covering construction strategies of problem chains (Wang, 2017), targeted cultivation of computational thinking (Cao & Sun, 2021), and mechanisms for achieving high-level thinking goals (Yu & Bao, 2022).

In summary, although reflections on reconstructing the teaching thread and independent research on the "problem chain" concept have begun to form a system, how to achieve global and deep integration of the highly dominant disciplinary root thread, "systems of linear equations," and the micro-advanced design of "problem chains" remains relatively lacking.

3. Analysis of the Main Thread Connotation and Design Principles of Problem Chains

Revealing the necessity of "systems of linear equations" as the core thread of the entire book from the perspective of mathematical epistemology, and clarifying the specific design principles of the problem chain based on this, are the key to ensuring the true implementation of the teaching.

3.1 The Natural "Guiding" Role of "Systems of Linear Equations" in the Knowledge System

"Systems of linear equations" can be said to be the absolute "origin" of the development of algebra. Historically, both the determinant theory in Leibniz's era and matrix theory in Cayley's era originated from the exploration and solving of multivariate linear equation systems. From the modern logical architecture of the course, various seemingly isolated abstract modules in linear algebra can all be viewed as multi-faceted characterizations of the general system $A\vec{x}=\vec{b}$. For example: determinants are scalar measurement tools for macroscopically determining whether square matrix systems have unique solutions; matrices and their elementary transformations are compact expressions and dynamic operating operators of equation coefficient information stripped of constants; vector spaces and maximal linearly independent systems are high-dimensional theoretical spaces built to structurally map the equation solution sets (such as null spaces) from a geometric perspective; eigenvalues and eigenvectors are optimal observational perspectives and pure dimensionality reduction tools used to explore how to decouple and simplify equations when the system continuously iterates. It can be seen that "systems of linear equations" is by no means an independent chapter but the core carrier that runs through the whole book and aggregates various algebraic tools.

3.2 Advanced Design Principles of Problem Chains

After clarifying the core thread, to ensure that the "problem chain" truly triggers deep learning in class, rather than degenerating into rote "Q&A" interactions due to overly fragmented questioning, its overall system design must strictly abide by the following three core principles:

First is the principle of thread orientation (convergence). No matter what abstract concepts are taught (such as matrix rank, inner products, orthogonalization, etc.), the end of the problem chain must loop back to "how this new concept will help us more deeply analyze or simplify systems of linear equations." This targeted positioning ensures that students remain cognitively focused.

Second is the principle of cognitive progression (gradient). The design of problems must fit Vygotsky's "Zone of Proximal Development." By pre-setting the knowledge ladder of "scenario introduction—local analysis—pattern conflict—conceptual encapsulation," we ensure that every core problem is an inevitable logical dead knot faced by the previous scenario, thereby maximizing cognitive imbalance to stimulate students' intrinsic drive to fill in new knowledge.

Last is the principle of exploratory openness (heuristic). In explicit language, decidedly discard closed confirmatory questions like "is it right?" or "yes or no?" Instead, extensively use heavily loaded questions such as "what happens to the result if a condition is changed?" or "how many patterns do you think exist?". Additionally, in selecting materials, incorporate backgrounds of engineering constraints, network flow, or probabilistic graph algorithms to imbue them with rich colors of applied exploration.

4. Teaching Practice Design Based on the Problem Chain Under the Main Thread

Based on the aforementioned teaching principles and the core thread, we have modularly reconstructed the entire linear algebra course. To more clearly demonstrate this spiral upward teaching vein of "problem-driven concept construction," we constructed a core concept construction problem chain evolution chart (Figure 1). This figure illustrates how to induce students' cognitive conflicts through a four-stage progressive problem chain starting from practical engineering scenarios, and then independently deduce the complete closed loop of core higher-algebra knowledge.

The following specifically demonstrates a series of representative extended problem chains crafted under the guidance of Figure 1, traversing the five dimensions of introduction, operation, structure, sublimation, and summary.

4.1 Introductory Phase: Natural Occurrence of Concepts—From System Modeling to Data Encapsulation

In the early stages of learning, students are most confused about "Why learn matrices?". We no longer directly throw out definition structures but start from real cases of Kirchhoff's circuit networks or chemical production material balances to build grand high-dimensional simultaneous equation systems.

Core Problem Chain 1:

(1) With the increasing number of system constraints, is it inefficient and error-prone for us to continue writing lengthy equations comprising variables x_1, x_2, …?

(2) On mathematical laws, are the only two things determining the nature of the equation system genuinely the "arrangement of the coefficients of the unknowns" and "the constant terms on the right side"?
(3) Can we strip off the variables and create an entirely new mathematical carrier, to globally pack and represent this equation system merely using a "number table" of constant arrangements? ——This naturally derives the concept of a "matrix."

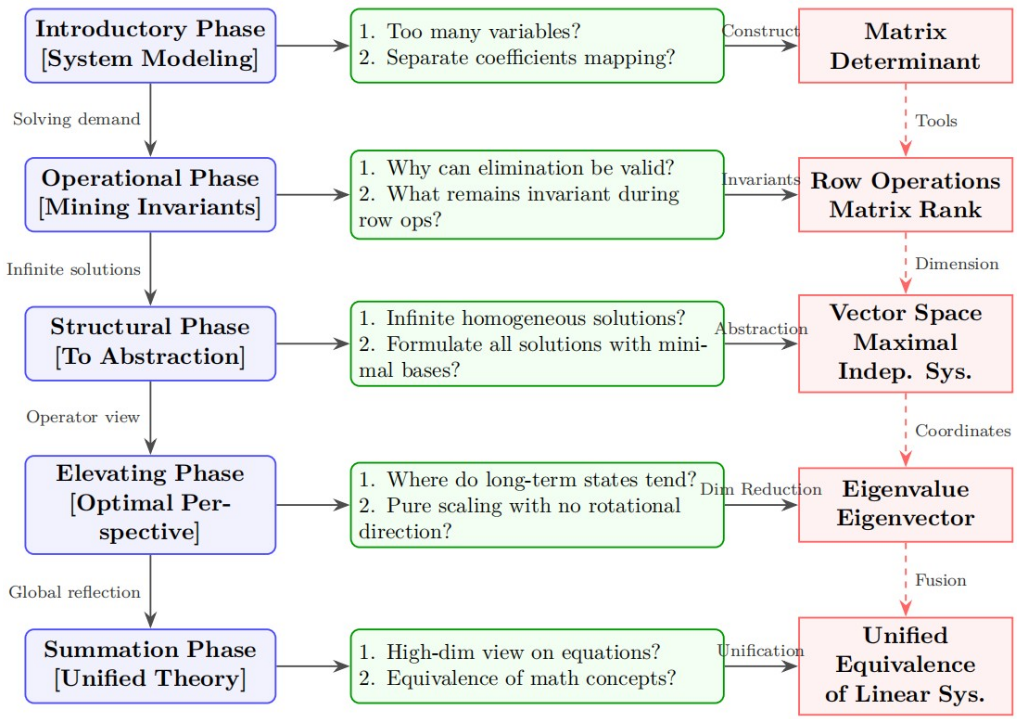

Figure 1. Evolution chart of core concept construction problem chain based on the "systems of linear equations" thread.

(4) Furthermore, when analyzing whether a simplistic binary network can undergo stable solving (has a unique solution), how can we quickly judge relying on a fixed scalar mapping? ——This prompts the geometric and algebraic significance of the "determinant."

4.2 Operational Phase: The Algebraic Essence of Elimination—Mining the Invariant "Rank"

For the algorithms that solve the system of equations, we do not require students to blindly memorize dry operational rules, but link them with Gaussian elimination inherited from middle school, delving into the rule of the topological invariability of the algorithm.

Core Problem Chain 2:

(1) What specific operations on the newly invented system's "augmented matrix" do the familiar addition and subtraction elimination from junior high schools correspond to? ——Concluding the "elementary row operations" of matrices.

(2) In the complete cancellation process of to-be-solved systems (sequentially transforming to Row Echelon Form and Reduced Row Echelon Form), what informational quantity remains perennially unchanged though the extrinsic form of matrices changes? (The solution of the system remains unchanged, as does the essential dimensionality of the constraints).

(3) In the row echelon form matrix, what rigorous algebraic mapping associates the volume of non-zero rows with deducing whether the equation system comprises a solution or counting the free independent variables? ——Through this conflict-based exploration, the most obscure concept: "Rank" is deeply analyzed.

4.3 Structural Phase: Isomorphism Between Solution Sets and Spaces—From Concrete to Abstract

When there's a need to grasp and characterize an infinity of multiple solution systems, analytical algorithmic procedures lose efficacy, thus mandating guidance for students mapping out the "great leap" from elementary algebraic arithmetic to spatially-geometric considerations.

Core Problem Chain 3:

(1) When homogeneous equation models yield infinitely many solutions, conceptually integrating via spatial layouts how do you perceive these solutions? Is its identity maintained as a solution upon scaling or aggregating alternative solutions under original systemic bounds? ——Leads to "vector subspaces" globally encasing closure

propositions under scaling mapping and aggregations.
(2) Given that enumerating boundless solution spaces is physically infeasible in practical deployment environments, might we discover relatively scarce "core elemental representative solutions", whose "linear combinations" can synthesize practically every mathematical solution? ——This inevitably prompts establishing the skeletons underpinning vectors encompassing dimensions like maximal linearly independent spanning sets, fundamental solution systems, alongside bases encompassing multidimensional volumes.

4.4 Elevating Phase: Searching for the Optimal Observation Perspective

In the application expansion part of linear algebra, the teaching shifts thoughts away from statically deriving answers to dynamically evaluating iterating matrices, bringing the dimensional reductions characterizing subsequent analytics contexts.

Core Problem Chain 4:

(1) When a certain apparatus (Markov chains) procedurally iterates spanning evolving timespans encapsulating matrix scaling parameters mapped under $x_{k+1} = Ax_k$, how does it fundamentally traverse asymptotic infinity endpoints?

(2) To minimize excessive powers, might discrete optimal perspectives embedded spatially facilitate transforming mapping operators A such that orientations perpetually persist without rotations exclusively accommodating purely fractional "expansions" or "shrinkages"? ——Unearthing eigenvalues (scaling rates), corresponding distinct invariant vectors, highlighting diagonal mapping effectively uncoupling intrinsic constraints.

4.5 Summation Phase: Returning to a Simplified Unified Theory

Having completed the preliminary introductory, operational mapping, dimensional constructs, plus dimensional reductions phases, sequentially unifying these domains structurally constitutes the definitive summary stage.

Core Problem Chain 5:

(1) Structurally revisiting the inaugural equations arrays denoted via $A\vec{x}=\vec{b}$, can spatial mappings, projections coupled intimately alongside dimensional paradigms project entirely revamped geometry perspectives?
(2) Conceptually interconnecting the intrinsic determinants mapping, ranks, uncoupled sub-space independent bases mapped synchronously adjacent eigenvalues; alongside establishing precisely amidst which resolving junctions these definitions emerged collectively establishing cross-boundary equivalences? ——Via culminating overarching synthesis validating conceptually bridged interconnectedness achieving thoroughly overarching comprehension.

5. Teaching Effectiveness and Reflection

During a two-semester teaching practice in engineering programs at a university, the reform class adopting the linear equation main thread problem chain mode was compared with a parallel class using the conventional teaching model. At the end of the semester, both quantitative examination results and qualitative feedback were collected to evaluate instructional outcomes.

In terms of academic performance, the experimental class showed a notable improvement over the control class, particularly on examination items involving abstract concepts such as linear dependence and matrix rank. Students in the experimental group demonstrated stronger ability to connect these concepts to the underlying structure of systems of equations, rather than treating them as isolated definitions to memorize.

A "Linear Algebra Learning Status Questionnaire" was administered to gather student feedback across four dimensions: (1) "Learning Motivation and Interest"; (2) "Core Concept Comprehension"; (3) "Computational Thinking and Modeling Ability"; (4) "Overall Knowledge System Construction." The results consistently indicated that students in the problem chain class reported higher engagement and a stronger sense of conceptual coherence. Many noted that the progressive problem chains helped them understand why abstract tools such as rank and eigenvalues were necessary, rather than simply how to compute them. The fragmented feeling commonly associated with traditional linear algebra instruction was markedly reduced.

In reflecting on the teaching practice, we recognize that the rapid development of Artificial Intelligence (AI) not only transforms traditional mathematical tools but also opens new avenues for pedagogical innovation. On one hand, generative AI can assist instructors in efficiently drafting auxiliary problem chains before class, generating tailored examples for students from different professional backgrounds. On the other hand, students are encouraged to use AI as an auxiliary learning tool outside class—interacting with AI to test their understanding of concepts (such as maximal linearly independent systems) and using programming tools to solve complex large-scale matrix problems. This shifts the instructional emphasis toward computational thinking and logical verification.

Future improvements will focus on further developing application-oriented problem chains grounded in authentic engineering cases, cultivating innovative talent equipped for the intelligent era.

6. Conclusion

Drawing on both the theoretical discussions and empirical findings of this teaching reform research, constructing a "problem chain" around the main thread of systems of linear equations represents a meaningful return to the roots of linear algebra instruction. It clarifies the intricate logical threads running through the concepts, enabling students—driven by the intrinsic motivation of problem-solving—to achieve a cognitive leap from local operations to a holistic understanding of algebraic structure.

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Appendix: Questionnaire on Linear Algebra Learning Status

Assessment Dimension	Specific Questionnaire Items
Dimension 1: Motivation (Assess enthusiasm for exploration)	1. When encountering unbalanced data issues in engineering, I am willing to build equation systems to describe it. 2. Instead of merely memorizing concepts, I prefer autonomous thinking spanning "scenario-conflict-construction" chains. 3. Solving practical equation issues with professional contexts (e.g. network flow) amplifies my success sentiments. 4. The progressive evolution of problem chains effectively stimulates my thirst to probe successive parameters.
Dimension 2: Concept Grasping (Assess mastery of abstract concepts)	5. I accurately interpret the "matrix rank" logic and its affiliation alongside architectural parameter bounds. 6. Regarding "linearly dependent sets", I logically link mappings alongside redundant structural boundaries. 7. I structurally map matrices beyond numerical tabulations into high-dimensional geometries scaling resolving computations. 8. I thoroughly articulate abstract derivations bridging structural baselines encompassing "maximal independent systems".
Dimension 3: Computations (Assess modeling problem-solving)	9. Tasked with multivariate system bounds, I readily convert operational layouts mapping algebraic matrices parameters. 10. Navigating insoluble boundaries spanning overdetermined equations, I actively target optimal least square evaluations. 11. Immersed within intricate matrix analytics spanning multiple matrices, I organically integrate software bridging computations. 12. Facilitating protracted systematic algorithms, I fundamentally validate overarching calculations bridging rational inferences structurally.
Dimension 4: Architectures (Assess knowledge connectedness)	13. I functionally grasp how "matrices", "determinants", and "vectors" synchronize mapping structural equations boundaries conceptually. 14. Assimilating eigenvalues alongside related parameters, I recognize optimal simplified coordinates mapping intrinsic abstractions comprehensively. 15. Applying high-dimensional algorithms comprehensively, I architect updated foundational interpretations mapping structural transformations seamlessly. 16. Globally framing frameworks spanning applied mathematics, this architecture actively eradicates fragmented pedagogical abstractions entirely.