Numerical Solutions for Linear Fredholm Integro-Differential Difference Equations with Variable Coefficients by Collocation Methods

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Abstract
We employed an efficient numerical collocation approximation methods to obtain an approximate solution of linear Fredholm integro-differential difference equation with variable coefficients. An assumed approximate solutions for both collocation approximation methods are substituted into the problem considered. After simplifications and collocations, resulted into system of linear algebraic equations which are then solved using MAPLE 18 modules to obtain the unknown constants involved in the assumed solution. The known constants are then substituted back into the assumed approximate solution. Numerical examples were solved to illustrate the reliability, accuracy and efficiency of these methods on problems considered by comparing the numerical solutions obtained with the exact solution and also with some other existing methods. We observed from the results obtained that the methods are reliable, accurate, fast, simple to apply and less computational which makes the valid for the classes of problems considered.

Keywords: Approximate solution, Collocation, Fredholm, Integro-differential difference and linear algebraic equations

Introduction
The theory of integral equation is one of the most important branches of Mathematics. Basically, its importance is in terms of boundary value problem in equation theories with partial derivatives. Integral equations have many applications in Mathematics, chemistry and engineering e.t.c. In recent years, the studies of integro-differential difference equations i.e equations containing shifts of unknown functions and its derivatives, are developed very rapidly and intensively [see Gulsu and Sezer (2006), Cao and Wang (2004), Bhrawy et al., (2012)]. These equations are classified into two types; Fredholm integro-differential-difference equations and Volterra integro-differential-difference equations, the upper bound of the integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type which are often difficult to solve analytically, or to obtain closed form solution, therefore, a numerical method is needed.

The study of integro-differential difference equations have great interest in contemporary research work in which several numerical methods have been devoloped and applied to obtain their approximate solutions such as Taylor and Bernoulli matrix methods [Gulsu and Sezer, 2006, Bhrawy et al., 2012], Chebyshev finite difference method [Dehghan and Saadatmandi, 2008], Legendre Tau method [Dehghan and Saadatmandi, 2010], Bessel matrix method [Yuzbas et al., 2011], and Variational Iteration Method (VIM) [Biazar and Gholami Porshokouhi, 2010]. Homotopy analysis method (HAM) was first introduced by Liao (2004) to obtain series solutions of various linear and nonlinear problems of this type of equation.

In this study, the basic ideas of the above studies motivated the work to apply a numerical collocation approximation methods that is reliable, fast, accurate and less computational to obtain an approximate solutions to the mth order linear Fredholm integro-differential difference equation with variable coefficients of the form:

$$\sum_{k=0}^{m} P_k y^{(k)}(x) + \sum_{r=0}^{n} P_r^*(x) y^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t) y(t-\tau) dt \qquad (1)$$

with the mixed conditions

$$\sum_{k=0}^{m-1} \left(a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b) + c_{ik} y^{(k)}(c) \right) = \mu_i,$$
$$i = 0,1,\cdots,m-1, \; a \le c \le b \qquad (2)$$

where $P_k(x), P_r^*(x), K(x,t)$ and $f(x)$ are given continuous smooth functions defined on $a \le x \le b$. The real coefficients a_{ik}, b_{ik}, c_{ik} and μ_i are appropriate constants , τ is refer to as the delay or difference constant (Gulsu and Sezer, 2006).

Basic Definitions

Integro-Differential Equations (IDEs)
An integro-differential equation is an equation which involves both integral and derivatives of an unknown function. A standard integro-differential equation is of the form:

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) y(t) dt \qquad (3)$$

where $g(x), h(x), f(x)$ and λ and the kernel $K(x,t)$ are as prescribed in definition (2.2) and n is the order of the IDE.

Equation (3) is called Fredholm Integro-Differential Equation if both the lower and upper bounds of the region of the integration are fixed numbers while it is called Volterra Integro-Differential Equation if the lower bound of the region of integration is a fixed number and the upper bound is not.

Collocation Method
This is a method of evaluating a given differential equation at some points in order to nullify the values of a differential equation or intgro-differential equation at those points.

Approximate Solution
This is the expression obtained after the unknown constants have been found and substituted back into the assumed solution. It is referred to as an approximate solution since it is a reasonable approximation to the exact solution. It is denoted by $y_{N(x)}$, and taken as an inexact representation of the exact solution, where N is the degree of the approximant used in the calculation. Methods of approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be given. In this work, approximate solution used are given as

$$y_{N(x)} = \sum_{i=0}^{N} a_n \phi_n(x)$$

where x represents the independent variables in the problem, $a_n (n \ge 0)$ are the unknown constants to be determined and $\phi_n(x), (n \ge 0)$ is the basis function which is either Chebyshev or Legendre Polynomials.

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Chebyshev Polynomials

The Chebyshev polynomials of degree n of first kind which is valid in the interval $-1 \le x \le 1$ and is given by

$$T_n(x) = cos(ncos^{-1}x) \tag{4}$$

$$T_\circ(x) = 1, T_1(x) = x$$

and the recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \ge 1$$

$$T_n(x) = \cos\left[n\cos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right], \ a \le x \le b$$

and this satisfies the recurrence relation

$$T_{n+1}(x) = 2\left(\frac{2x-a-b}{b-a}\right)T_n(x) - T_{n-1}(x), \ n \ge 0. a \le x \le b \tag{5}$$

Equation (5) is the recurrence relation of the Chebyshev polynomials in the interval $[-1,1]$, thus we have

$$\begin{aligned} &T_\circ^*(x) = 1 \\ &T_1^*(x) = x \\ &T_2^*(x) = 2x^2 - 1 \\ &T_3^*(x) = 4x^3 - 3x \\ &T_4^*(x) = 8x^4 - 8x^2 + 1 \text{ andsoon.} \end{aligned} \tag{6}$$

Legendre's Polynomial

The Legendre's polynomial is defined and denoted by

$$P_{n+1}(x) = \frac{1}{n+1}\left\{(2n+1)xP_n(x) -^n P_{n-1}(x)\right\}$$

and

$$P_n(x) = \frac{1}{2^n n! dx^n}(x^2-1)^n; \ n = 0,1,\cdots$$

with the first few polynomial as

$$\begin{aligned} &P_\circ(x) = 1 \\ &P_1(x) = x \\ &P_2(x) = \frac{1}{2}(3x^2-1) \\ &P_3(x) = \frac{1}{2}(5x^3-3x) \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \quad -1 \le x \le 1 \end{aligned} \tag{7}$$

Discussion of Methods

Problem Considered

We consider the m^{th} order linear Fredholm integro-differential difference equation with variable coefficients of the forms:

$$(a) \qquad \sum_{k=0}^{m} P_k y^{(k)}(x) + \sum_{r=0}^{n} P_r^*(x) y^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t) y(t-\tau) dt \tag{8}$$

with the mixed conditions

$$\sum_{k=0}^{m-1} \left(a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b) + c_{ik} y^{(k)}(c) \right) = \mu_i,$$
$$i = 0,1,\cdots,m-1, \; a \le c \le b \tag{9}$$

Equation (8) is referred to as Linear Fredholm Integro-differential difference equation with variable coefficients, where $P_k(x), P_r^*(x), K(x,t)$ and $f(x)$ are given continuous smooth functions defined on $a \le x \le b$. The real coefficients a_{ik}, b_{ik}, c_{ik} and μ_i are appropriate constants , τ is refer to as the delay term or difference constant (Gulsu and Sezer, 2006).

In this section, standard collocation methods is applied to solve equation of the form (a) using the following bases functions:

(i) Chebyshev Polynomials
(ii) Legendre Polynomials

Method I: Standard Collocation Method by Chebyshev Polynomial Basis
In order to solve equations (8)-(9) using the collocation approximation method, we used an approximate solution of the form

$$y_N(x) = \sum_{i=0}^{N} a_i T_i(x) \tag{10}$$

where N is the degree of our approximant, $a_i (i \ge 0)$ are constants to be determined and $T_i (i \ge 0)$ are the Chebyshev Polynomials defined in equation (5). Thus, differentiating equation (10) with respect to x m-times (m is the order of the given problem), we obtain

$$\left. \begin{aligned} y' &= \sum_{i=0}^{n} a_i T_i'(x) \\ y'' &= \sum_{i=0}^{n} a_i T_i''(x) \\ &\vdots \\ y^m &= \sum_{i=0}^{n} a_i T_i^m(x) \end{aligned} \right\} \tag{11}$$

and then substituting equation (10) and its derivatives in equation (11) into equation (8), we obtain

$$\sum_{k=0}^{m} P_k y_N^{(k)}(x) + \lambda \sum_{r=0}^{n} P_r^*(x) y_N^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t) y_N(t-\tau) dt \tag{12}$$

Evaluating the integral part of equation (12) and after simplifications, we collocate the resulting equation at the point $x = x_k$ to get

$$\sum_{k=0}^{m} P_k y_N^{(k)}(x_k) + \lambda \sum_{r=0}^{n} P_r^*(x_k) y_N^{(r)}(x_k - \tau) = f(x_k) + G(x_k) \tag{13}$$

where $G(x)$ is the evaluated integral part and

$$x_k = a + \frac{(b-a)k}{N+1}; k = 1(1)N-1 \quad (14)$$

Thus, equation (13) gives rise to $(N-1)$ system of linear algebraic equations in $(N+1)$ unknown constants and m extra equations are obtained using the conditions given in equation (9). Altogether, we now have $(N+m+1)$ system of linear algebraic equations. These equations are then solved using MAPLE software to obtain (N+1) unknown constants $a_i (i \geq 0)$ which are then substituted back into the approximate solution given by equation (10).

Method II: Standard Collocation Method by Legendre Polynomial Basis

We consider here also the problem of the form (a) using the collocation approximation method, we used an approximate solution of the form

$$y_N(x) = \sum_{i=0}^{N} a_i L_i(x) \quad (15)$$

where N is the degree of our approximant, $a_i (i \geq 0)$ are constants to be determined and $L_i (i \geq 0)$ are the Legendre Polynomials defined in equation (7). Thus, differentiating equation (15) with respect to x m-times (m is the order of the given problem), we obtain

$$\left.\begin{aligned} y' &= \sum_{i=0}^{n} a_i L_i'(x) \\ y'' &= \sum_{i=0}^{n} a_i L_i''(x) \\ &\vdots \\ y^m &= \sum_{i=0}^{n} a_i L_i^m(x) \end{aligned}\right\} \quad (16)$$

and then substituting equation (15) and its derivatives in equation (16) into equation (8), we obtain

$$\sum_{k=0}^{m} P_k y_N^{(k)}(x) + \lambda \sum_{r=0}^{n} P_r^*(x) y_N^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t) y_N(t-\tau) dt \quad (17)$$

Hence, evaluating the integral part of equation (20) and after simplification, we collocate the resulting equation at the point $x = x_k$ to get

$$\sum_{k=0}^{m} P_k y_N^{(k)}(x_k) + \lambda \sum_{r=0}^{n} P_r^*(x_k) y_N^{(r)}(x_k - \tau) = f(x_k) + G(x_k) \quad (18)$$

where $G(x)$ is the evaluated integral part and

$$x_k = a + \frac{(b-a)k}{N+1}; k = 1(1)N-1 \quad (19)$$

Thus, equation (18) gives rise to $(N-1)$ system of linear algebraic equations in $(N+1)$ unknown constants and m extra equations are obtained using the conditions given in equation (9). Altogether, we now have $(N+m+1)$ system of linear algebraic equations. These equations are then solved using MAPLE software to obtain (N+1) unknown constants $a_i (i \geq 0)$ which are then substituted back into the approximate solution given by equation (10).

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Numerical Examples

Numerical Example 1

Consider the Second order linear Fredholm integro-differential difference equation with variable coefficients

$$y''(x) - xy(x) + xy(x-1) + y'(x-1) + y(x-1) = e^{-x} + e + \int_{-1}^{0} ty(t-1)dt \quad (20)$$

with the initial conditions

$$y(0) = 1,\ y'(0) = -1 \quad (21)$$

The exact solution is given as $y(x) = e^{-x}$ [Gulsu and Sezer, 2006].

Numerical Example 2

Consider third order linear Fredholm integro-differential-difference equation with variable coefficients

$$y'''(x) - xy'(x) + y''(x-1) - xy(x-1) = -(x+1)(\sin(x-1) + \cos(x))$$
$$-\cos 2 + 1 + \int_{-1}^{1} y(t-1)dt \quad (22)$$

with the initial conditions

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0 \quad (23)$$

The exact solution is given as $y(x) = \sin x$ [Gulsu and Sezer, 2006].

Numerical Example 3

Consider first order linear Fredholm integro-differential-difference equation with variable coefficients

$$y'(x) - y(x) + xy'(x-1) - y(x-1) = x - 2 + \int_{-1}^{1} (x+t)y(t-1)dt \quad (24)$$

with the mixed condition

$$y(-1) - 2y(0) + y(1) = 0 \quad (25)$$

The exact solution is given as $y(x) = 3x + 4$ [Gulsu and Sezer, 2006].

Remark: We defined absolute error as:

$$Error = \left| y(x) - y_N(x) \right|,\quad a \le x \le b,\ N = 1,2,3,\cdots$$

Here, $y(x)$ is the given exact solution and $y_N(x)$ is the approximate solution respectively.

Numerical Results and Eerror for Examples

Table 1: Results obtained for example 1: Case N=6

x	EXACT	APPROXIMATE SOLUTIONS		
		CHEBYSHEV	LEGENDRE	TAYLOR
.0	1.0000000000	1.0000000000	1.0000000000	0.00000000000
-0.1	1.1051709181	1.1052508587	1.1052508587	1.10530000000
-0.2	1.2214027582	1.2216333968	1.2216333968	1.22160000000
-0.3	1.3498588076	1.3501660185	1.3501660185	1.35000000000
-0.4	1.4918246976	1.4919754149	1.4919754149	1.49190000000
-0.5				

	1.6487212707	1.6483097916	1.6483097916	1.64850000000
-0.6	1.8221188004	1.8205524611	1.8205524611	1.82110000000
-0.7	2.0137527075	2.0102358013	2.0102358013	2.01140000000
-0.8	2.2255409285	2.2190555779	2.2190555779	2.22100000000
-0.9	2.4596031112	2.4488856333	2.4488856334	2.45220000000
-1.0	2.7182818285	2.7017929401	2.7017929400	2.70690000000

Table 2: Results obtained for example 1: Case N=7

x	EXACT	APPROXIMATE SOLUTIONS		
		CHEBYSHEV	LEGENDRE	TAYLOR
.0	1.0000000000	1.0000000000	1.0000000000	0.00000000000
-0.1	1.1051709181	1.1052038643	1.1052038642	1.10530000000
-0.2	1.2214027582	1.2214978470	1.2214978469	1.22160000000
-0.3	1.3498588076	1.3499847405	1.3499847404	1.35000000000
-0.4	1.4918246976	1.4918832368	1.4918832367	1.49190000000
-0.5	1.6487212707	1.6485413049	1.6485413048	1.64850000000
-0.6	1.8221188004	1.8214503126	1.8214503126	1.82110000000
-0.7	2.0137527075	2.0122500401	2.0122600401	2.01140000000
-0.8	2.2255409285	2.2227947307	2.2227947307	2.22100000000
-0.9	2.4596031112	2.4550703277	2.4550703277	2.45220000000
-1.0	2.7182818285	2.7113130423	2.7113130423	2.70690000000

Table 3: Absolute Errors for Example 1: Case N=6 and 7

x	CHEBYSHEV	LEGENDRE	TAYLOR	CHEBYSHEV	LEGENDRE	TAYLOR
	N=6			N=7		
.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

-0.1	7.9941E-05	7.9941E-05	0.1000E-03	3.2946E-05	7.9941E-05	0.00000000
-0.2	2.3064E-04	2.3064E-04	0.2000E-03	9.5089E-05	2.3064E-04	0.1000E-03
-0.3	3.0721E-04	3.0721E-04	1.0000E-03	1.2593E-04	1.2593E-04	0.1000E-03
-0.4	1.5072E-04	1.5072E-04	1.5072E-03	5.8539E-05	5.8539E-05	0.00000000
-0.5	4.1148E-04	4.1148E-04	0.2000E-03	1.7996E-04	1.7996E-04	0.1000E-03
-0.6	1.5663E-03	1.5663E-03	0.1000E-02	6.6849E-04	6.6849E-04	0.8000E-03
-0.7	3.1569E-03	3.1569E-03	0.2400E-02	1.4927E-03	1.4927E-03	0.1600E-02
-0.8	6.4853E-03	6.4853E-03	0.4500E-02	2.7462E-03	2.7462E-03	0.3100E-02
-0.9	1.0717E-02	1.0717E-02	0.7400E-02	4.5328E-03	4.5328E-03	0.5100E-02
-1.0	1.6489E-02	1.6489E-02	0.1140E-01	6.9688E-03	6.9688E-03	0.7800E-02

Table 4: Results and Errors obtained for example 1: Case $N = 15$

x	Exact	CHEBYSHEV	LEGENDRE	E_{CP}	E_{LP}
.0	1.0000000000	1.0000000000	1.0000000001	0.0000000	1.0000E-10
-0.1	1.1051709181	1.1051709225	1.1051630547	4.400E-09	6.7863E-06
-0.2	1.2214027582	1.2214027712	1.2213800716	1.300E-08	2.2687E-05
-0.3	1.3498588076	1.3498588248	1.3498286988	1.720E-08	3.0109E-05
-0.4	1.4918246976	1.4918247058	1.4918104108	8.200E-09	1.4287E-05
-0.5	1.6487212707	1.6487212466	1.6487632901	2.410E-08	4.2019E-05
-0.6	1.8221188004	1.8221187100	1.8222762751	9.040E-08	1.5747E-04
-0.7	2.0137527075	2.0137525054	2.0141050354	2.021E-07	3.5233E-04
-0.8	2.2255409285	2.2255405565	2.2261896364	3.720E-07	6.4871E-04
-0.9	2.4596031112	2.4596024968	2.4606741794	6.144E-07	1.0711E-03
-1.0	2.7182818285	2.7182808839	2.7199286075	9.446E-07	1.6468E-03

Here, we denoted E_{CP} as the Error of results obtained using Chebyshev polynomials and E_{LP} as the Error of results obtained using Legendre polynomials.

Table 5: Results obtained for example 2: Case $N = 6$

x	EXACT	APPROXIMATE SOLUTIONS		
		CHEBYSHEV	LEGENDRE	TAYLOR
-1.0	-0.8414709848	-0.8412178340	-0.8183268604	-0.9273450000
-0.8	-0.7173560909	-0.7171927056	-0.7064218929	-0.7567230000
-0.6	-0.5646424734	-0.5645686751	-0.5604519509	-0.5797120000
-0.4	-0.3894183423	-0.3883086444	-0.3883086444	-0.3935390000
-0.2	-0.1986693308	-0.1985475309	-0.1985475310	-0.1991540000
.0	0.00000000000	0.0000000000	0.0000000000	0.0000000000
.2	0.19866933080	0.1985811922	0.1985811920	0.1991280000
.4	0.3894183423	0.3888427347	0.3888427345	0.3931170000
.6	0.5646424734	0.5631179243	0.5631179242	0.5774680000
.8	0.7173560909	0.7146801750	0.7146801750	0.7491370000
.0	0.8414709848	0.8379628764	0.8379628762	0.9072650000

Table 6: Results obtained for example 2: Case $N = 7$

x	EXACT	APPROXIMATE SOLUTIONS		
		CHEBYSHEV	LEGENDRE	TAYLOR
-1.0	-0.8414709848	-0.8459516158	-0.8459531673	-0.9018320000
-0.8	-0.7173560909	-0.7194740946	-0.7194748278	-0.7401870000
-0.6	-0.5646424734	-0.5654548269	-0.5654551080	-0.5712780000
-0.4	-0.3894183423	-0.3896336970	-0.3896337714	-0.3906190000
-0.2	-0.1986693308	-0.1986930013	-0.1986930094	-0.1987380000
.0	0.00000000000	0.0000000000	0.0000000000	0.0000000000
.2	0.19866933080	0.1986865239	0.1986865299	0.19861600000
.4	0.3894183423	0.3895309176	0.3895309564	0.3886090000
.6	0.5646424734	0.5649417357	0.5649418384	0.5608220000
.8	0.7173560909	0.7178844600	0.7178846405	0.7058770000
.0	0.8414709848	0.8421710782	0.8421713155	0.8140980000

Table 7: Errors obtained for example 2: Case N=6 and 7

x	CHEBYSHEV	LEGENDRE	TAYLOR	CHEBYSHEV	LEGENDRE	TAYLOR
	N=6			N=7		
-1.0	2.3144E-02	2.3144E-02	8.5870E-02	4.4806E-03	4.8218E-03	6.0360E-02
-0.8	1.0934E-02	1.0934E-02	3.9360E-02	2.1180E-03	2.1187E-03	2.2830E-02
-0.6	4.1905E-03	4.1905E-03	1.5070E-02	8.1235E-04	8.1263E-04	6.6360E-03
-0.4	1.1097E-03	1.1097E-03	4.1210E-03	2.1535E-04	2.1543E-04	1.2010E-03
-0.2	1.2180E-04	1.2180E-04	4.8500E-04	2.3671E-05	2.3679E-05	6.9000E-05
.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
.2	8.8139E-05	8.8139E-05	4.5900E-04	1.7193E-05	1.7199E-05	5.3000E-05
.4	5.7561E-04	5.7560E-04	3.6990E-03	1.1258E-04	1.1261E-04	8.0900E-04
.6	1.5245E-03	1.5245E-03	1.2820E-02	2.9926E-04	2.9927E-04	3.8200E-03
.8	2.6759E-03	2.6759E-09	3.1780E-02	5.2837E-04	5.2855E-04	1.1470E-02
.0	3.5081E-03	3.5081E-03	6.5790E-02	7.0009E-04	7.0033E-04	2.7370E-02

NOTE:
On solving this numerical example (3) using the two methods, the same exact solution is obtained.

Presentation of Results in Graphical Forms

Conclusion

We have presented and illustrated the collocation approximation methods using two different bases functions namely; Chebyshev and Legendre polynomials to solve linear Fredholm integro-differential difference equations with variable coefficients which are very difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. One of the advantages of these methods is that the numerical solutions of the problems considered is converted into system of linear algebraic equations which are very easy to solve for the constants involved. Another considerable advantage of these methods is that if the exact solution is a polynomial function, with the methods used, the analytical solution is obtained.

Moreover, satisfactory results of illustrative examples were obtained when the value of N increases for both methods, the approximate solutions obtained are closer to the exact solution (where the exact solution are known in closed form) which are compared with some other existing methods and makes these methods valid for solving linear Fredholm Integro-differential difference and Fredholm Integral equations.

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