Journal of Information Engineering and Applications
ISSN 2224-5782 (print) ISSN 2225-0506 (online)
Vol.4, No.4, 2014

New Algorithm for Drawings of 3-Planar Graphs

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Abstract

Graphs arise in a natural way in many applications, together with the need to be drawn. Except for very small instances, drawing a graph by hand becomes a very complex task, which must be performed by automatic tools. The field of graph drawing is concerned with finding algorithms to draw graph in an aesthetically pleasant way, based upon a certain number of aesthetic criteria that define what a good drawing, (synonyms: diagrams, pictures, layouts), of a graph should be. This problem can be found in many such as in the computer networks, data networks, class inter-relationship diagrams in object oriented databases and object oriented programs, visual programming interfaces, database design systems, software engineering…etc.

Given a plane graph *G*, we wish to find a drawing of *G* in the plane such that the vertices of *G* are represented as grid points, and the edges are represented as straight-line segments between their endpoints without any edge-intersection. Such drawings are called planar straight-line drawings of *G*. An additional objective is to minimize the area of the rectangular grid in which *G* is drawn. In this paper we introduce a new algorithms that finds an embedding of 3-planar graph.

Keywords: 3- Planar Graph; Graph Drawing; drawing on grid.

1. Introduction

Visualization is the presentation of information graphically, rather than textually. To comprehend a large body of information presented in textual form, users must read each entry in turn, and recall related earlier entries. This task quickly exceeds the user's ability to remember the earlier entries. On the other hand, if the same information is presented graphically by an appropriate visualization, then the user simply "sees" the relationship. The drawing of graphs is widely recognized as a very important task in diverse fields of research and development. Examples include VLSI design, plant layout, software engineering and bioinformatics [1,2]. Large and complex graphs are natural ways of describing real world systems that involve interactions between objects: persons and/or organizations in social networks, articles incitation networks, web sites on the World Wide Web, proteins in regulatory networks, etc [3,4].

Graphs that can be drawn without edge crossings (i.e. planar graphs) have a natural advantage for visualization [5]. When we want to draw a graph to make the information contained in its structure easily accessible, it is highly desirable to have a drawing with as few edge crossings as possible.

A straight-line embedding of a plane graph *G* is a plane embedding of *G* in which edges are represented by straight-line segments joining their vertices, these straight line segments intersect only at a common vertex.

A straight-line drawing is called a convex drawing if every facial cycle is drawn as a convex polygon. Note that not all planar graphs admit a convex drawing. A straight-line drawing is called an inner-convex drawing if every inner facial cycle is drawn as a convex polygon [6].

A strictly convex drawing of a planar graph is a drawing with straight edges in which all faces, including the outer face, are strictly convex polygons, i. e., polygons whose interior angles are less than 180 [7,8]. However, a problem with graph layout methods which are capable of producing satisfactory results for a wide range of graphs is that they often put an extremely high demand on computational resources [9]. Visualizing graphs using virtual physical models is probably the most heavily used technique for drawing graphs in practice. There are many techniques to produce length-sensitive drawings for large graphs by reformulating the energy function [10,11,12].

One of the most popular drawing conventions is the straight-line drawing, where all the edges of a graph are drawn as straight-line segments. Every planar graph is known to have a planar straight-line drawing [13]. A straight-line drawing is called a convex drawing if every facial cycle is drawn as a convex polygon. Note that not all planar graphs admit a convex drawing. Tutte [14] gave a necessary and sufficient condition for a triconnected plane graph to admit a convex drawing. Thomassen [15] also gave a necessary and sufficient condition for a biconnected plane graph to admit a convex drawing. Based on Thomassen's result, Chiba et al. [16] presented a linear time algorithm for finding a convex drawing (if any) for a biconnected plane graph with a specified

convex boundary. Tutte [14] also showed that every triconnected plane graph with a given boundary drawn as a convex polygon admits a convex drawing using the polygonal boundary. That is, when the vertices on the boundary are placed on a convex polygon, inner vertices can be placed on suitable positions so that each inner facial cycle forms a convex polygon.
In paper [17], it was proved that every triconnected plane graph admits an inner-convex drawing if its boundary is fixed with a star-shaped polygon *P*, i.e., a polygon P whose kernel (the set of all points from which all points in P are visible) is not empty. Note that this is an extension of the classical result by Tutte [14] since any convex polygon is a star-shaped polygon. We also presented a linear time algorithm for computing an inner-convex drawing of a triconnected plane graph with a star-shaped boundary [13].
Rosenstiehl and Tarjan [18] posed the question of whether it is always possible to find such an embedding into a polynomial-size grid. Later, de Fraysseix, Pach and Pollack [19] indeed gave a method that embeds an n-vertex planar graph into the $(2n\text{-}4)\times(n\text{-}2)$ grid in an $O(n \log n)$ time. Kant [20] developed a method for constructing convex grid drawing of 3-connected plane graphs in linear-time. His algorithm, related to those of Refs. [21] and [4a], uses a $(2n\text{-}4)\times(n\text{-}2)$ grid, and the grid size was improved to $(n\text{-}2)\times(n\text{-}2)$ by Schnyder and Totter [22] and Chrobak and Kant [20], independently. All these algorithms can be implemented in linear time.

In this paper, we will describe a new technique for graph layout that attempts to satisfy edge length constraints. This technique uses a modified Kant approach of convex drawing. In this paper we will show how to construct convex drawings of 3-connected plane graphs into a smaller, $(n\text{-}3)\times(n\text{-}3)$, grid in linear time. In addition, The paper present a different techniques for orthogonal drawing of 3- planar graph aiming to improve them to get the optimal upper and lower area bounds.
The remainder of the paper is organized as follows. In section 2, we give some definitions in graph drawing, specially, the canonical decomposition of plane graph . In sections 3, we introduce an algorithm that finds an embedding of *G* into a grid, $(n\text{-}2)\times(n\text{-}2)$. In sections 4, We will show how to modify the previous algorithm in order to reduce the grid size to $(n\text{-}3)\times(n\text{-}3)$. Section 5 present a new algorithm of 3-planar graph in orthogonal drawing. In section 6,we improve the grid size of orthogonal drawing into a smaller grid in linear time.

2. The Canonical Decomposition of Plane Graph

In this section we introduce the concept of canonical decomposition for triconnected planar graphs, The canonical decomposition is a generalization of the canonical ordering of De Fraysseix et al. [23]. Define a plane graph *G* to be *internally 3-connected* if (a) *G* is 2-connected, and (b) if removing two vertices *u,v* disconnects *G* then *u, v* belong to the outer face and each connected component of $G\text{-}\{u, v\}$ has a vertex of the outer face. In other words, *G* is internally 3-connected iff it can be extended to a 3-connected graph by adding a vertex and connecting it to all vertices on the outer face. Let *G* be an *n*-vertex 3-connected plane graph with an edge $e(v_1,v_2)$ on the outer face.

Let $\pi=(V_1,...,V_m)$ be an ordered partition of *V*, that is , $V_1\cup...\cup V_m = V$ and for $V_i \cap V_j \neq \Phi$ for $i\neq j$. Define G_k to be the subgraph of *G* induced by $V_1\cup...\cup V_k$, and denote by C_k the external face of G_k. We say that π is a *canonical decomposition* of *G* with bottom edge $e(v_1,v_2)$ if:

(CD1) V_m is a singleton, $\{z_0\}$, where z_0 lies on the outer face and $z_0 \notin \{v_1,v_2\}$.

(CD2) C_1 is a face of *G*, and each C_k is a cycle containing $e(v_1,v_2)$.

(CD3) Each G_k is 2-connected and internally 3-connected.

(CD4) For each $2\leq k \leq m\text{-}1$, one of the two following conditions holds:

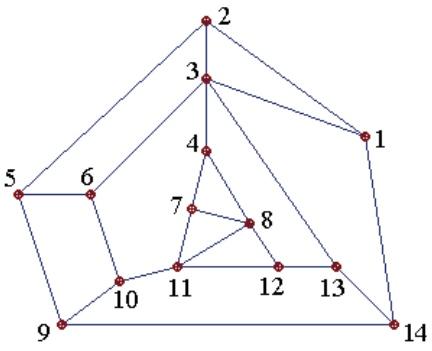
(i) V_k is a singleton, $\{z\}$, where *z* belongs to C_k and has at least one neighbor in $G\text{-}G_k$.

(ii) V_k is a chain, $(z_1, z_2,..., z_t)$, where each z_i has at least one neighbor in $G\text{-}G_k$, and where z_1 and z_t each have one neighbor on $G_{k\text{-}1}$ and these are the only two neighbors of V_k in $G_{k\text{-}1}$.

By an ordered plane graph (G,π) we will understand a plane graph *G* with a given canonical decomposition $\pi=V_1,...,V_m$. By the *contour* of G_k we will mean its outer face written as C_k.We will commonly view C_k as a path $(w_1,w_2,...,w_j)$ starting with $w_1=v_1$ and ending with $w_j=v_2$, ignoring the edge $e(v_1,v_2)$. We will also view C_k as being ordered from "left" to "right", where w_1 is the leftmost and w_j is the rightmost vertex on C_k. Let w_p be the leftmost and w_q be the rightmost neighbors of *v* in C_k, we will say that the vertex *v covers* the vertices $w_{p+1},..., w_{q\text{-}1}$. Throughout the rest of the paper we will call a plane graph *internally convex* if all its internal faces are convex.

We will use the following lemma proved by Kant in [21], and our explanation is similar to the one given by Chrobak and Kant [20]:

Lemma 1: *Each 3-connected plane graph has a canonical decomposition*.
Proof: We present only a sketch. Pick an edge $e(v_1,v_2)$ and a vertex $z_0 \notin \{v_1,v_2\}$ on the outer face of G. Let $V_m=\{z_0\}$. Inductively, suppose that $V_m,...,V_{k-1}$ have been defined, and $G_m,...,G_{k-1}$ satisfy the lemma. If G_k is 3-connected, let V_k be $\{z\}$, where z is an arbitrary vertex from $C_k - \{v_1,v_2\}$ that has neighbor in $G-G_k$.
Otherwise, if C_k contains a chain whose removal does not destroy 2-connectivity, let V_k be a maximal such chain –its members will have degree 2 in G_k (and will have a neighbor in $G-G_k$ by 3-connectivity of G), and its two neighbors will have greater degree. If, however, no such chain exists, pick two vertices in C_k whose removal disconnects G_k that are as close to each other as possible in the ordering of C_k. The 3-connected component in between, by 3-connectivity of G, contains a vertex z having a neighbor in $G-G_k$. Let V_k be $\{z\}$. □



k	V_k	C_k
1	9-14	9-14
2	8	9,10,11,8,12,13,14
3	7	9,10,11,7,8,12,13,14
4	5,6	9,5,6,10,11,7,8,12,13,14
5	4	9,5,6,10,11,7,4,8,12,13,14
6	3	9,5,6,3,13,14
7	2	9,5,2,3,13,14
8	1	9,5,2,1,14

Figure 1: The canonical decomposition with bottom edge $e(9,14)$

As it was shown by Kant [21] (Theorem 2.3) a canonical decomposition can be constructed in linear time. In Figure 1 an example (which is given in [20]) of a canonical decomposition of a triconnected planar graph given, with bottom edge $e(9,14)$.
By $P(v)$ we will denote the current position of vertex v in the grid, i.e., $P(v):=(x(v),y(v))$. By $P(u,v)$ we denote the embedding of edge $e(u,v)$, that is, the line segment that connects $P(u)$ with $P(v)$. To each vertex w we assign a set of vertices, $U(w)$, that will contain certain vertices that are located below w and have to be shifted right whenever w is shifted right.
We will describe first an algorithm that uses the $(n-2)\times(n-2)$ grid, $n\geq 3$, and then show how to improve it to $(n-3)\times(n-3)$, $n>3$.

3. ConvexDraw Algorithm

The algorithm will be to add sets V_k, one by one, in forward order $V_1, ..., V_m$, adjusting the embedding at every step. For z_i, $i=1,2,\ldots,t$, $P(z_i) :=(x(z_i), y(z_i))$, since $x(z_i)$ and $y(z_i)$ are integers so $P(z_i)$ is always a grid point.
Let (G,π) be a given ordered plane graph with n vertices, where $\pi= V_1,..., V_m$ and $n\geq 3$. Suppose that $2\leq k\leq m$ and that we are about to add V_k to G_{k-1}.

Algorithm ConvexDraw
Input: A convex graph G with β vertices and m contours.
Output: Outline graph embedded in $(\beta-2)\times(\beta-2)$ grid.
Begin
We initialize the embedding by drawing $C_1 =(v_1=z_1, z_2,..., z_t= v_2)$, as follows :

- $P(z_1):=(0,0)$;
- $P(z_t):=(t-2,0)$;
- $P(z_i):=(i-2,1)$, for all $i=2,\ldots,t-1$;
- $U(z_i) = \{z_i\}$, $i = 1,2,3,...,t$.

Then, for each $k= 2,\ldots,m$, we do the following:

- Let $C_{k-1}= (v_1=w_1,w_2,...,w_j= v_2)$ be the contour of G_{k-1} .
- Let $V_k =(z_1, z_2,..., z_t)$, and V_k may be a singleton or a chain.

- Let w_p be the leftmost and w_q be the rightmost neighbors of V_k in C_{k-1}.

We now execute the following steps:

Step 1: (Shift operation) for each vertex v is belong to $\{U(w_i), i = p+1, ...,j\}$ do

$$x(v) = x(v) + t; \quad (1)$$

Step 2: (Install operation) For each i=1,…,t, let $P(z_i)$ be defied by :

$$x(z_i) = x(w_p)+i-1, \quad (2)$$

$$y(z_i) = y(w_q) + x(w_q) - x(w_p) - t + 1; \quad (3)$$

Step 3: (Update operation) $U(z_1) = \{z_1\} \cup \{U(w_i), i = p+1, ...,q-1\}$ and $U(z_i) = \{z_i\}, i = 2,3,...,t.$

End

In the other words, in step 2, we draw the V_k horizontally, in such a way that the slope of the segment $P(z_t,w_q)$ is -45°. Vertex z_1 is placed above w_p, that the slope of the segment $P(w_p,z_1)$ is 90°. Note that by moving some of the points $P(w_i)$ in step 1, we ensure that all neighbors of V_k will be visible from $P(z_i)$ for i=1,2,…,t.

Lemma 2: *Let* $1 \le k \le m$, *and* $C_k=(w_1=v_1,w_2,...,w_j=v_2)$ *and* β *is the number of vertices of* G_k . *Then* $P(v_1)=(0,0)$, $P(v_2)=(\beta-2,0)$, *and all contour segments* $e(w_i,w_{i+1})$, i=1,2,...,j-1, *have slopes in* {-45°,0°,90°}.

Proof: the proof is by induction on k. For G_1 the lemma is obvious, the segment $e(w_1,w_2)$ has slope of 90°, the segments $e(w_i,w_{i+1})$, i=2,3,…,j-2 have slope of 0°, and the segment $e(w_{j-1},w_j)$ has slope of -45°, and $P(v_2)=(j-2,0)$.

So suppose that it holds for G_{k-1}. As in the algorithm, before installing V_k, the contour $C_{k-1}=(w_1=v_1,w_2,...,w_j=v_2)$, $P(v_1)=(0,0)$.and $P(v_2)=(\alpha-2,0)$ where α is the number of vertices in G_{k-1} . Let $w_p,w_{p+1},..., w_q$ be the neighbors of V_k in C_{k-1}.

When we are going to install V_k, we always have $w_j=v_2$, $x(w_j)= \alpha-2$ and from (1), by moving all the vertices $w_{p+1},...,w_j$ by t to the right, $x(w_j)= \alpha-2+t$, but $\alpha+t$ equal to the number of vertices in G_k , hence $P(v_2)=(\beta-2,0)$. Let $w_p,w_{p+1},..., w_q$ be the neighbors of V_k in C_{k-1}.After installing V_k we can divide the segments of the contour C_k into three intervals, the first interval is $\{e(w_i,w_{i+1}), i:=1,2,..,p-1\}$, the second interval is $\{e(w_p,z_1),e(z_1,z_2), ….,e(z_{t-1},z_t), e(z_t,w_q)\}$ and the third interval is $\{e(w_i,w_{i+1}), i:=q,..,j-1\}$.

In the first interval, if it contains any line-segment, the slope will be the same as its slope at the contour C_{k-1}. But for the second interval, the line-segment $e(w_p,z_1)$, has slope $[y(z_1)-y(w_p)]/[x(z_1)-x(w_p)]$, from (2) the denominator $x(z_1)-x(w_p)=0$, from (3) the numerator $y(z_1)-y(w_p)$ greater than zero and less than infinity , so the line-segment $e(w_p,z_1)$ has the slope equal to 90°. The line-segments $e(z_i,z_{i+1})$, i=1,2,…,t-1, have the slope equal to 0°, because $y(z_{i+1})-y(z_i)$ equal to zero from (3). But for the line-segments $e(z_t,w_q)$, $y(w_q)-y(z_t)= -\{x(w_q)-x(z_t)\}$, i.e., the line-segments $e(z_t,w_q)$ has the slope equal to -45°. For the third interval, the line-segments has the same slopes as C_{k-1} because the only change that we have shifted vertices $w_q,...,w_j$ to the right by t and this will not effect the slopes of the line-segments from C_{k-1} to C_k. Hence, the contour segments $e(w_i,w_{i+1})$, i=1,2,...,j-1 of G_k have slopes in {-45°,0°,90°}. □

The lemma above implies immediately that adding V_k does not destroy the embedding, as stated in the corollary below.

Corollary 1: *For each* $1 \le k \le m$, *when we add* V_k, *then after applying the shift operation*, *all neighbors of* V_k *are visible*, *that the edges between* V_k *and* C_{k-1} *do not intersect themselves or edges in* C_{k-1}.

What remains to show is that do destroy the planarity property and convexity when we apply the shift operation. This is proven in the next lemma.

Lemma 3: *Let* G_k *be straight-line embedded and internally convex*. *Additionally*, *it has the following property: Suppose* $C_k=(v_1=w_1, w_2,...,w_j=v_2)$, *and any integer t. if we shift all nodes in* $\{U(w_i), i = p+1, ...,j\}$ *by t to the right*, *then* G_k *remains straight-line embedded and internally convex*.

Proof: the proof is by induction on k. For G_1 the lemma is obvious, by inspection. Assume the lemma holds for

G_{k-1}, we will show that the lemma properties are preserved when we add V_k. As in the algorithm, before installing V_k, the contour C_{k-1}= (w_1=v_1,w_2,...,w_j=v_2) and w_p be the leftmost and w_q be the rightmost neighbors of V_k in C_{k-1}. When we are going to install V_k, from (1) by moving all the vertices $U(w_{p+1}),...,U(w_j)$ by t to the right, we have three classes of faces in G_{k-1}. First class, all vertices of the face are belong to $U(w_1),...,U(w_p)$, there is no any shift. Therefore, all faces of this type are not change, and its properties in G_k will be the same as in G_{k-1}. Second class, all vertices of the face are belong to $U(w_{p+1}),...,U(w_j)$, so, all vertices shifted by t to the right. Therefore, all faces of this type are moved by t to the right and its properties in G_k will be the same as in G_{k-1}. Third class, the vertices of a face classified two to sets, the first set are belong to $U(w_1),...,U(w_p)$, they not moved to the right, the second set are belong to $U(w_{p+1}),...,U(w_j)$, they moved to the right by t, in this case, any edge of the considerable face which has one vertex element in the first set and the second element lies in the second set will be stretched, and this will not affect its properties.
Let us assume now that V_k is a singleton, V_k ={z_1} . Let z_1 have λ neighbors among w_p,w_{p+1},..., w_q and let F_1,F_2,..., $F_{\lambda-1}$ be the faces created when adding V_k. From the algorithm all these faces preserved the lemma properties. The proof when V_k is a chain is very similar. □

4 Improving the Grid Size

Now we sketch how to modify the algorithm ConvexDraw in order to reduce the grid size to (n-3)×(n-3). First we pick V_m ={z_0} to be the neighbor of v_2 different from v_1 on the outer face of G. We construct a canonical decomposition and run the algorithm ConvexDraw for m-1 steps. In the last step, having already embedded G_{m-1}, we set $P(z_0)$:=(1,n-3) and we do not shift any vertices to the right.

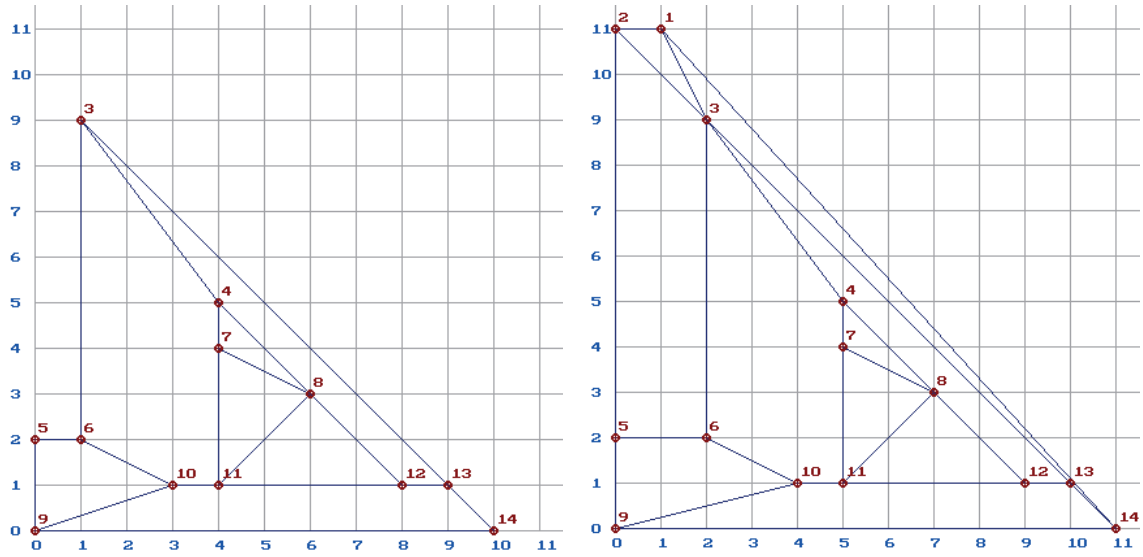
Algorithm MConvexDraw
Input: A convex graph G with β vertices and m contours.
Output: Outline graph embedded in (β-2)×(β-2) grid.
Begin
We initialize the embedding by drawing C_1 =(v_1=z_1, z_2,..., z_t= v_2), as follows :

- $P(z_1)$:=(0,0);
- $P(z_t)$:=(t-2,0);
- $P(z_i)$:=(i-2,1), for all i=2,…,t-1;
- $U(z_i)=\{z_i\},\ i=1,2,3,...,t.$

For each k= 2,…,m-1, we do the following:

- Let C_{k-1}= (v_1=w_1,w_2,...,w_j= v_2) be the contour of G_{k-1} .
- Let V_k =(z_1, z_2,..., z_t), and V_k may be a singleton or a chain.
- Let w_p be the leftmost and w_q be the rightmost neighbors of V_k in C_{k-1}.
- Calculate the shift operation.
- Install operation.
- Execute the update operation.

Finally , for k= m, we put $P(V_m$={z_0}) =(1,n-3)
End
Let us call this algorithm MConvexDraw. In order to show correctness, we only need to show that adding $z_0$will result in a correct, convex embedding. By lemma 2 and the algorithm, before adding $z_0$we have $x(w_1)$=$x(w_2)$=….=$x(w_p)$=0, and $x(w_q)$=n-3, where w_q=v_2. The edge with slope -45° from v_2 contains the point (1,n-4). This implies that all vertices w_p,...,w_q are visible from (1,n-3). The convexity of the outer face follows from the choice of z_0. Consequently, we obtain the following theorems:



(a) G_{m-2} (b) G_m

Figure 2: The drawing of the graph G

Table 1: The values of the different variables in ConvexDraw.

k	V_k	w_p	w_q	x-coordinates of vertices													
				1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	9-14	-	-									0	0	1	2	3	4
2	8	11	12								1	0	0	1	3	4	5
3	7	11	8							1	2	0	0	1	4	5	6
4	5,6	9	10					0	1	3	4	0	2	3	6	7	8
5	4	7	8				3	0	1	3	5	0	2	3	7	8	9
6	3	6	13			1	4	0	1	4	6	0	3	4	8	9	10
7	2	5	3		0	2	5	0	2	5	7	0	4	5	9	10	11
8	1	2	14	1	0	2	5	0	2	5	7	0	4	5	9	10	11
	y-coordinates			11	11	9	5	2	2	4	3	0	1	1	1	1	0

Theorem 1: *Given a 3-connected plane graph G, algorithm MConvexDraw constructs a straight-line convex embedding of G into a* (n-3) × (n-3) *grid.*

Theorem 2: *Given a plane graph G, the above algorithm MConvexDraw computes a convex embedding of G into the* (n-3) × (n-3) *grid in O*(n) *time.*

In Figure 2 an example of a drawing is given. After adding vertex 3, we have $U(w)=\{w\}$ for $w\in\{5,6,9,13,14\}$, $U(3)=\{3,4,7,8,10,11,12\}$. Therefore, when adding vertex 2, the vertices in $U(3)\cup U(6)\cup U(13)\cup U(14)$ $=\{3,4,6,7,8,10,11,12, 13,14\}$ will be shifted right. After adding vertex 2, we have $U(w)=\{w\}$ for $w\in\{5,9,13,14\}$, $U(3)=\{3,4, 7,8,10,11,12\}$ and $U(2)=\{2,6\}$. Table 1 show the values of the different variables in ConvexDraw. Notice that the drawing is not *strictly convex*, i.e. there are angles of size 180 °.

The 3-regular plane graphs are plane graphs where every vertex has exactly 3 neighbours. Especially in the mathematical literature 3-regular graphs are also called "cubic" graphs.

Lemma 6 *Let* (G,π) *given a 3-plane graph . Algorithm MConvexDraw constructs a straight-line convex embedding of G at most in* (2f-7) × (2f-7) *grid.*

Proof: Assume first that G is 3-plane graph. By Euler's formula, N is even, number of edges M=3N/2 and f=N/2+2. Let a canonical decomposition of G be given. Science N=2f-4, and from Theorem 1, the grid size is at most *in* (2f-7) × (2f-7). □

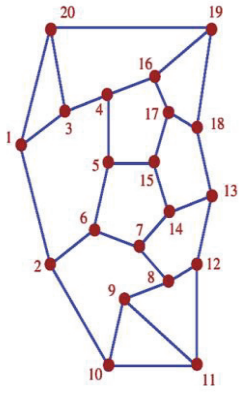
5. The Orthogonal Algorithm of 3-planar Graph

In this section we explain the *leftmost canonical* (*lmc*-) *ordering* for tri-connected planar graphs, which can be used in various ways to get better planar drawing algorithms. The *lmc-ordering* can be constructed in linear time [21]. Let an embedding of a 3-planar graph G be given. The vertices of a triconnected planar graph G can be

ordered in a sequence $v_1, v_2,..., v_n$ such that v_2 and v_n are neighbors of v_1 and share a common face, and for every k, $k>3$, one condition from the following is satisfied:

1. The singleton vertex v_k is in the exterior face of the biconnected G_k and has at least two neighbors on the outerface of G_{k-1} and v_k has at least one neighbor in $G - G_k$.
2. There exists an $t \geq 1$ such that v_k ,... , v_{k+t} is a chain in the exterior face of G_{k+t} and has exactly two neighbors on the outerface of G_{k-1}. The subgraph. G_{k+t} of G is biconnected, and the vertices $v_k,..., v_{k+t}$ have degree 2 in G_{k+t}.

This means that starting with an edge (v_1, v_2) one can add in every step either a vertex v_k or a face (which is implied by the chain $v_k,..., v_{k+t}$ and the involved vertices of G_{k-1}). We call this added face F_k. This means that at any step k during the canonical ordering, when we can add both v_i (or F_i) and v_j (or F_j), then we take this vertex or face, for which the right point w_t is minimal with respect to t.



k	V_k
1	1,3,4,5,6,2
2	7,8,9,10
3	11
4	12
5	14,13
6	15
7	16,17
8	18
9	19
10	20

Figure 3: A graph with *lmc*- ordering and corresponding variable-values at some step.

In Figure 3 an example of the *lmc*-ordering is given. By an ordered plane graph (G,π) we will understand a plane graph G with a given *lmc*-ordering $\pi=V_1,...,V_M$. By the *contour* of G_k we will mean its outer face written as C_k. We will commonly view C_k as a path $(w_1,w_2,...,w_j)$ starting with $w_1=v_1$ and ending with $w_j=v_2$, ignoring the edge $e(v_1, v_2)$. We will also view C_k as being ordered from "left" to "right", where w_1 is the leftmost and w_j is the rightmost vertex on C_k. Let w_p be the leftmost and w_q be the rightmost neighbors of v in C_k, we will say that the vertex v *covers* the vertices $w_{p+1},..., w_{q-1}$.

The vertices w_l $(p<l<q)$ of V_k on C_{k-1} are called *internal* vertices. All these edges from w_i $(p\leq i \leq q)$ to V_k are called *incoming* edges of V_k . The other edges of V_k are called *outgoing* edges of V_k. By $P(v)$ we will denote the current position of vertex v in the grid, i.e., $P(v):=(x(v),y(v))$. By $P(u,v)$ we denote the embedding of edge $e(u,v)$, that is, the line segment that connects $P(u)$ with $P(v)$. To each vertex w we assign a set of vertices, $U(w)$, that will contain certain vertices that are located below w and have to be shifted right whenever w is shifted right. The precise definition of $U(w)$ is part of the algorithm and is given throw it.

Assume first that G is triconnected. By Euler's formula, n is even, $m=3n/2$ and $f=n/2 + 2$. Let an *lmc*-ordering of G be given. There are four directions to connect an edge at v, namely, *left*, *right*, *up* and *down* of v. A direction is called free if there is no edge connected in that direction of v yet. The idea for the *OrthDraw* algorithm is as follows: we add v to G_{k-1} such that *down*(v) is not free in G_k. Let w_p and w_q be the left- and right-vertex of v. Every vertex v (except v_n) has one outgoing edge, and we connect this edge via *up*(v) at v. We start with placing the vertices $z_1= v_1, z_2, z_3,...., z_t=v_2$ of the first face are placed on the horizontal line at positions (0,1), (1,1), (2,1), …, (t-1,1), i.e. v_1 and v_2 at (0,1) and (t-1,1). edge (v_1, v_2) goes via *down*(v_1) and *down*(v_2), hence via (0,0) and (t-1,0), and z_2 via *right*(v_1), z_3 via *right*(z_2),…., and z_{t-1} via *left*(v_2).

In every step k, $1 < k <M$, let $y(w_\alpha)=max\ \{ y(w_p), y(w_{p+1}), ..., y(w_q)\}$ from the right. if $t=1$, we place $x(z_1) =x(w_p)$, $y(z_1)= y(w_\alpha)+1$, i.e. we have one bend of edge (z_1,w_q) at position $(x(w_q), y(z_1))$. Otherwise, $t>1$, we place $z_1, z_2,....$ z_{t-1}, and z_t on a horizontal line of height $y(w_\alpha)+1$, with w_p and w_q the left- and right-vertex of V_k. In this case ($t>1$) we shift the drawing such that $x(z_1) =x(w_p)$ and $x(z_t) =x(w_q)$. Let w_γ the first node of chain w_{p+1} ,... , w_{q-1} ,such that $x(w_\gamma) =...= x(w_{q-1}) = x(w_q)$.

The vertices w_l $(p<l<q)$ of V_k on C_{k-1} are called *internal* vertices. All these edges from w_i $(p\leq i \leq q)$ to V_k are called *incoming* edges of V_k . The other edges of V_k are called *outgoing* edges of V_k. Since *incoming*(V_k) =2 for $1<k<M$ and *incoming*(v_n) =3, it follows that $M = f-2$, where f the number of faces in G. Notice that $f =n/2 + 2$ (n is even).

The complete *OrthDraw* algorithm can now be described as follows:

Algorithm OrthDraw
Input: A 3-planar graph G with lmc-ordering.
Output: Orthogonal drawing of G embedded in $(\mu+1)\times(n/2)$ grid.
Begin
We initialize the embedding by drawing $C_1=(v_1=z_1, z_2,..., z_t=v_2)$, as follows:

- $P(z_i):=(i\text{-}1,1)$, for all $i=1,\ldots,t$;
- $U(z_i):=\{z_i\}$, for all $i=1,\ldots,t$.

Then, for each $k=2,\ldots,M\text{-}1$, we do the following:

- Let $C_{k-1}=(v_1=w_1,w_2,...,w_j=v_2)$ be the contour of G_{k-1} .
- Let $V_k=\{z_1, z_2,..., z_t\}$; V_k may be a singleton or a chain.
- Let w_p be the leftmost and w_q be the rightmost neighbors of V_k in C_{k-1}.

Step 1: (Shift operation)
If $(x(w_q)\text{-}x(w_p)<t\text{-}1)$ then
If $(x(w_{p+1})=x(w_p))$ then $v\in\{U(w_i), i=q, ...,j\}$ else $v\in\{U(w_i), i=p+1, ...,j\}$ do
$x(v)=x(v)+t-x(w_q)+x(w_p)-1$;
Step 2: (Install operation)
If $t=1$ then $P(z_t):=(x(w_p), y(w_\alpha)+1)$;
For each $i=1,\ldots,t\text{-}1$, let $P(z_i)$ be defied by
$P(z_i):=(x(w_p)+i\text{-}1, y(w_\alpha)+1)$; and $P(z_t):=(x(w_q), y(z_t):=y(w_\alpha)+1)$;
Step 3: (Update operation)
$U(w_p)=\{U(w_{p+1}),U(w_{p+2}), ...,U(w_{\gamma-1})\}$.
$U(w_q)=\{U(w_\gamma),U(w_{\gamma+1}), ...,U(w_{q-1})\}$.
$U(z_i)=\{z_i\}, i=1,2,...,t$
Then, for $k=M$, let the vertices w_p ,w_c , w_q are the neighbors of v_n in G_{n-1} , then we place $P(v_n)=(x(w_c), y(w_\alpha)+1)$.
End

Lemma 4 *Using OrthDraw algorithm, the number of bends is at most* $(n/2)+2$.
Proof: From the initialization step, the first face has 2 bends between v_1 and v_2. As a result adding the last vertex v_n, the last three faces will be added and 2 bends will be produced. Adding the remaining f-4 faces (adding any one of them will produce at most one bend) will produces at most f-4 bends. i.e. the total number of bends = $2+2+(f\text{-}4)=f=(n/2)+2$. □

Lemma 5 *Let* (G,π) *given with* μ (*the number of* V_i *with* $t_i=1$). *The grid size is at most* $(\mu+1)\times(n/2)$.
Proof: First, the y-direction: Edge (v_1,v_2) gives 1 unit in the y-direction. Then adding $(n/2)$-2 times a face with $t\geq 1$ vertices, increasing the y-direction by 1 unit. Adding v_n increases the y-direction by 1 unit. Counting this together leads to at most $n/2$ units in y-direction.
Secondary, the x-direction: Edge (v_1,v_2) gives t_1-1 units, (where t_1 is the number of vertices in the first face) *in* x-direction. Then adding all the faces f_i , for all $2\leq i\leq(n/2)-1$
gives (t_i-1 units for every face in the x-direction) at most in the x-direction . Counting this together leads to $\sum_{i=2}^{(n/2)-1}(t_i-1)=\sum_{i=1}^{(n/2)-1}t_i-\sum_{i=1}^{(n/2)-1}1=(n-1)-((n/2)-1)=n/2$. at most $n/2$ units in x-direction.
Let μ be the number of V_i which has a singleton nodes, then the number of V_i with $t_i\geq 2$ is $(n/2)$- μ . in the case when $t_i\geq 2$, really we add exactly t_i-2 units in the x-direction instead of at most t_i-1(except for the first face) units which considered above. Then, the grid size is exactly $(n/2)$ -$[(n/2)-\mu-1]=\mu+1$ in the x-direction. □

Figure 4 describes OrthDraw algorithm steps using a given example in Figure 3.

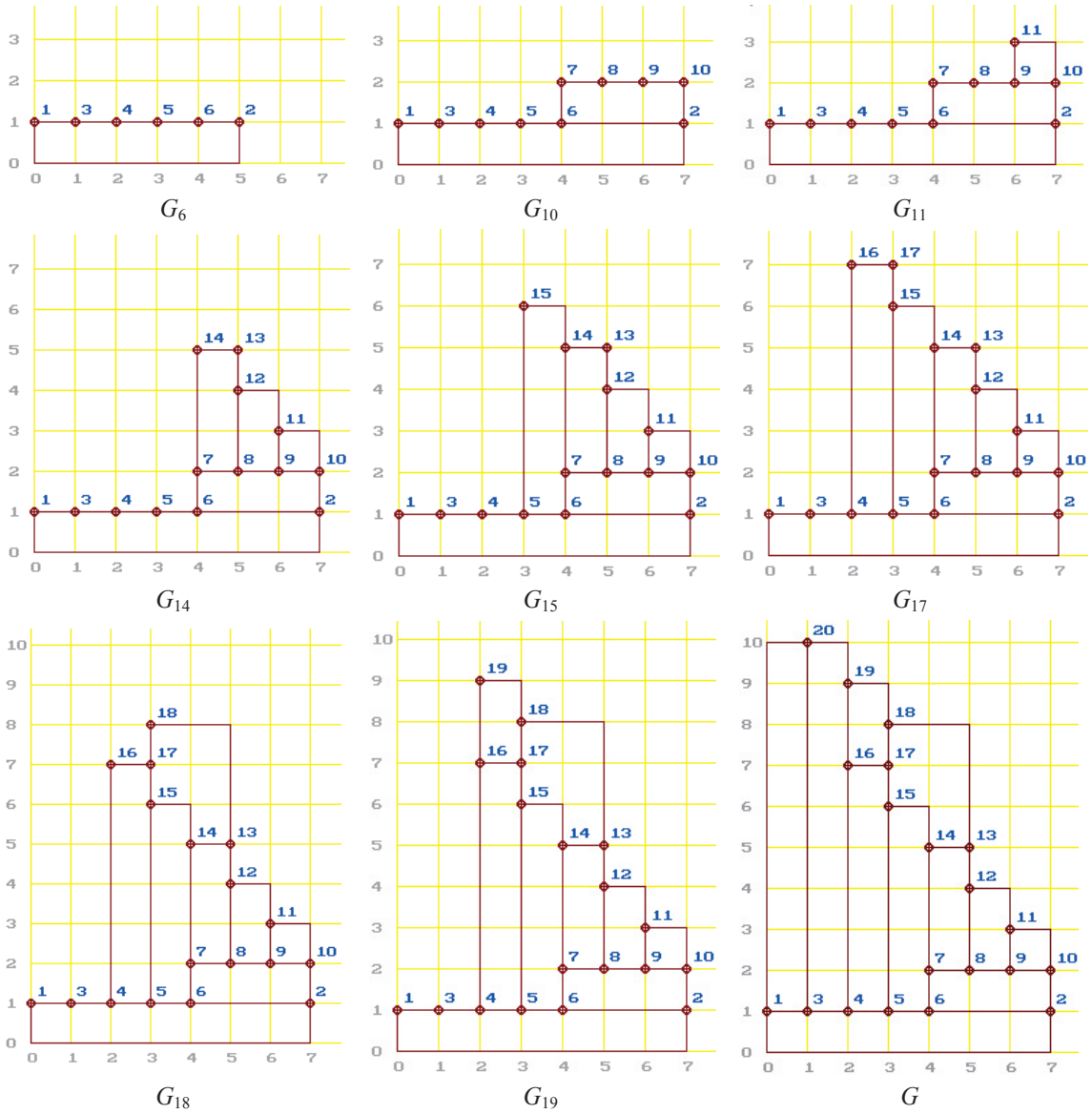


Figure 4: Graph layout steps to illustrates the OrthDraw algorithm

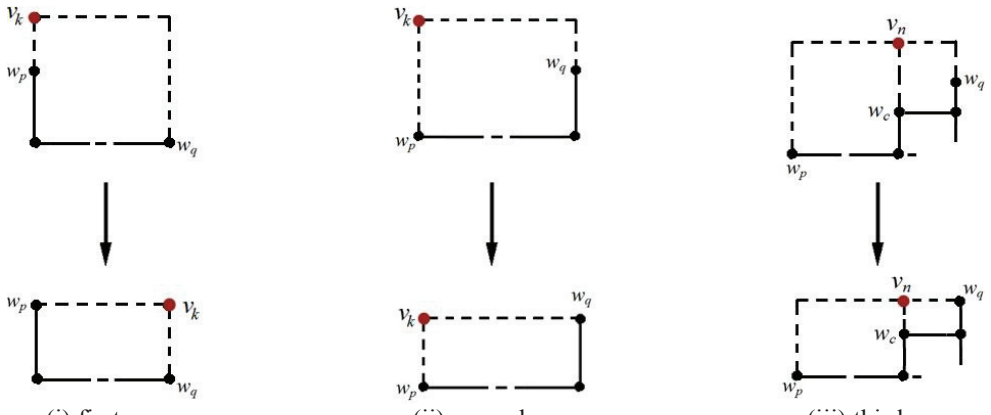


Figure 5: The improvement cases of OrthDraw algorithm

6. Modified Orthogonal 3-planar Graph Algorithm

To minimize the resultant area and number of bends in OrthDraw algorithm. There are three cases of adding the singleton vertex in which we can reduce the Y-direction and the number of bends. See Figure 5.
The first case, when $p=\alpha$ and $x(w_{p+1}) = x(w_p)$, the vertex in position ($x(w_q)$, $y(w_\alpha)$) is added, this will reduce the y-direction by 1 unit and the number of bends by 1.
The second case, when $q=\alpha$ and $x(w_q)=x(w_{q-1})$,we adding the vertex in position ($x(w_p)$, $y(w_\alpha)$), and this also will reduce the y-direction by 1 unit and the number of bends by 1.
The third case, for v_n, let the vertices w_p ,w_c , w_q are the neighbors of v_n in G_{M-1}, when $w_\alpha \neq w_c$ and $x(w_{q-1}) = x(w_q)$, we place $P(v_n) = (x(w_c), y(w_\alpha))$, and this also will reduce the y-direction by 1 unit and 1 bend.

In the following, we rewrite OrthDraw algorithm after modification which will be called *MOrthDraw* Algorithm.
Algorithm MOrthDraw
Input: A 3-planar graph G with lmc-ordering.
Output: Orthogonal drawing of G embedded in $(\mu+1)\times (n/2)$ grid.
Begin
We initialize the embedding by drawing $C_1 =(v_1=z_1, z_2,..., z_t= v_2)$, as follows:

- $P(z_i):=(i\text{-}1,1)$, for all $i=1,\ldots,t$;
- $U(z_i):=\{z_i\}$, for all $i=1,\ldots,t$.

Then, for each $k= 2,\ldots,M\text{-}1$, we do the following:

- Let $C_{k-1} = (v_1=w_1,w_2,...,w_j= v_2)$ be the contour of G_{k-1} .
- Let $V_k =\{ z_1, z_2,..., z_t \}$; V_k may be a singleton or a chain.
- Let w_p be the leftmost and w_q be the rightmost neighbors of V_k in C_{k-1}.

Step 1: (Shift operation)
If $(x(w_q)\text{-}x(w_p) < t\text{-}1$) then
If $(x(w_{p+1}) = x(w_p))$ then $v \in \{U(w_i), i = q, ...,j\}$ else $v \in \{U(w_i), i = p+1, ...,j\}$ do
$x(v) = x(v) + t - x(w_q) + x(w_p) - 1$;
Step 2: (Install operation)
If $t>1$ then Begin
For i=1 to t-1 do $p(z_i):=(x(w_p)+i\text{-}1, y(w_\alpha)+1)$; and $p(z_t):=(x(w_q) , y(w_\alpha)+1)$;
End Else
Begin
If $y(w_p) > y(w_q)$ then
If $(p=\alpha)$ and $(x(w_{p+1})=x(w_p))$ then $p(z_1):=(x(w_q), y(w_\alpha))$ Else $p(z_1):=(x(w_q), y(w_\alpha) +1)$
Else If $y(w_p) < y(w_q)$ then
If $(q=\alpha)$ and $(x(w_{q-1})=x(w_q))$ then $p(z_1):=(x(w_p), y(w_\alpha))$ Else $p(z_1):=(x(w_p), y(w_\alpha) +1)$
Else $p(z_1):=(x(w_p), y(w_\alpha) +1)$
End;
Step 3: (Update operation)
$U(w_p) = \{U(w_{p+1}), U(w_{p+2}), ..., U(w_{\gamma-1})\}$.
$U(w_q) = \{U(w_\gamma), U(w_{\gamma+1}), ..., U(w_{q-1})\}$.
$U(z_i) = \{z_i\}, i = 1,2,...,t$
Then, for $k = M$, let the vertices w_p , w_c , w_q are the neighbors of v_n in G_{M-1}, then if $(w_\alpha \neq w_c$) and $(x(w_{p+1})=x(w_p)$ or $x(w_{q-1})=x(w_q))$, we place $P(v_n)=(x(w_c), y(w_\alpha))$. Other wise, $P(v_n)=(x(w_c), y(w_\alpha)+1)$.
End

Theorem 4 *There is a linear time space algorithm to draw a 3-planar graph orthogonal on at most* $(\mu+1)\times\lfloor n/2\rfloor$ *grid and* $\lfloor n/2\rfloor+1$ *bends as an upper bound. And at least* $(\mu+1) \times \lfloor n/2 - \mu\rfloor$ *grid and* $\lfloor n/2\rfloor+1-\mu$ *bends as a lower bound. In this orthogonal drawing, there is a spanning tree of n-1 straight-line edges, all m-n+1 non-tree edges have at most one bend* ($n>6$).

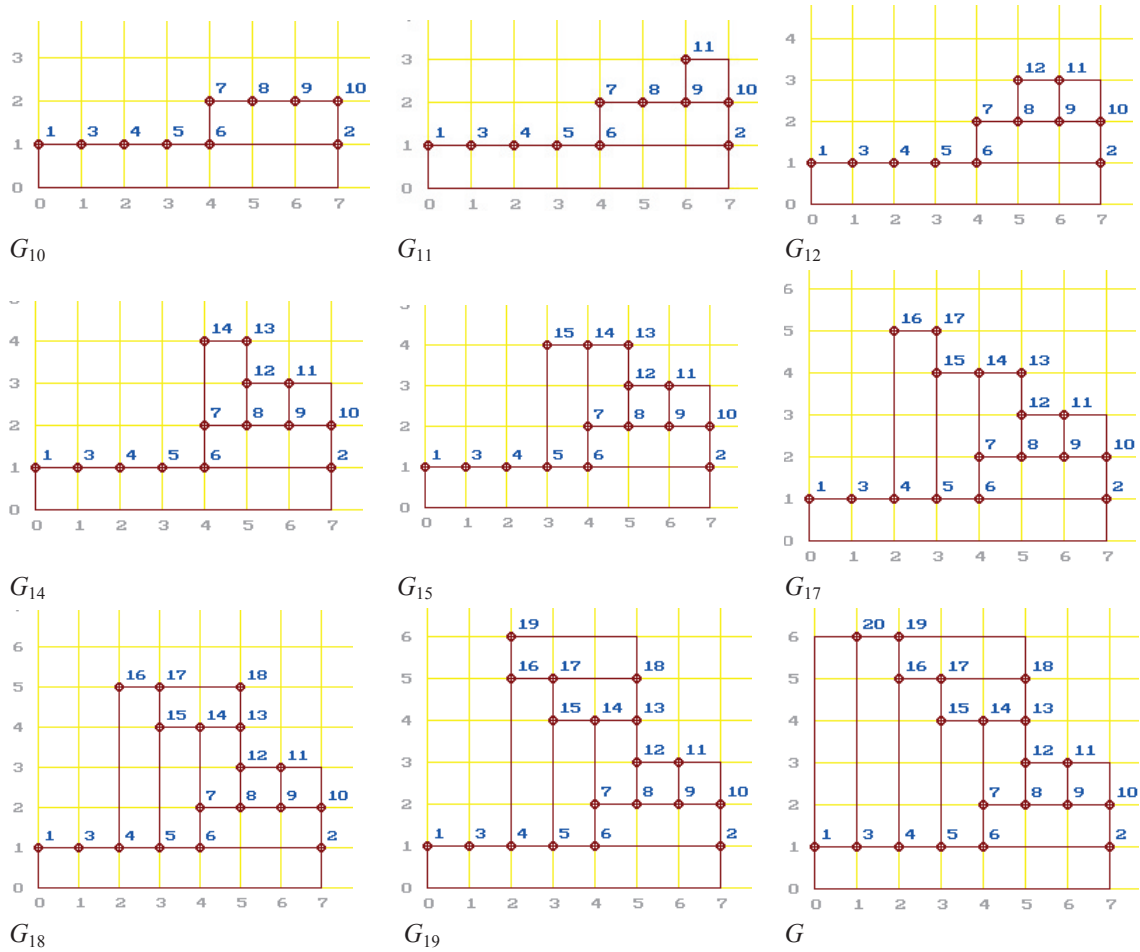


Figure 6: Graph layout steps to illustrates the MOrthDraw algorithm

7. Conclusion

In this paper, we described a new technique for graph layout that attempts to satisfy edge length constraints. This technique uses a modified Kant approach of convex drawing, one part for the edge lengths, the other to guide the relative placement of other node pairs. It can be transformed to a form that is suitable for solution using convex programming. The results produced are good and the algorithm is scalable to large graphs. In addition, the paper present a different techniques for orthogonal drawing of 3- planar graph aiming to improve them to get the optimal upper and lower area bounds. The algorithm improves the lower and upper bounds of grid area and minimizes the number of bends over the previous algorithms.

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