# Higher Dimensional Image Analysis using Brunn-Minkowski Theorem, Convexity and Mathematical Morphology

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**Abstract**

The theory of deterministic morphological operators is quite rich and has been used on set and lattice theory. Mathematical Morphology can benefit from the already developed theory in convex analysis. Mathematical Morphology introduced by Serra is a very important tool in image processing and Pattern recognition. The framework of Mathematical Morphology consists in Erosions and Dilations. Fractals are mathematical sets with a high degree of geometrical complexity that can model many natural phenomena. Examples include physical objects such as clouds, mountains, trees and coastlines as well as image intensity signals that emanate from certain type of fractal surfaces. So this article tries to link the relation between combinatorial convexity and Mathematical Morphology.

Keywords: Convex bodies, convex polyhedra, homothetics, morphological cover, fractal, dilation, erosion.

## 1. Introduction

### *1.1 Types of Images*

An image is a mapping denoted as I, from a set, $N_P$, of pixel coordinates to a set, M, of values such that for every coordinate vector, p $=(\mathbf{p_1}, \mathbf{p_2})$ in $N_P$, there is a value I(p) drawn from M. $N_P$ is also called the image plane.[1]

Under the above defined mapping a real image maps an n-dimensional Euclidean vector space into the real numbers. Pixel coordinates and pixel values are real.

A discrete image maps an n-dimensional grid of points into the set of real numbers. Coordinates are n-tuples of integers, pixel values are real.

A digital image maps an n-dimensional grid into a finite set of integers. Pixel coordinates and pixel values are integers.

A binary image has only 2 values. That is, M= $\{m_{fg}, m_{bg}\}$, where $m_{fg}$, is called the foreground value and $m_{bg}$ is called the background value.

The foreground value is $m_{fg} = 0$, and the background is $m_{bg} = -\infty$. Other possibilities are $\{m_{fg}, m_{bg}\} = \{0,\infty\}$, $\{0,1\}$, $\{1,0\}$, $\{0,255\}$, and $\{255,0\}$.

### *1.2 Definition*

The foreground of binary image I is $FG\{I\} = \{I(\mathbf{p}), p = (p_1, p_2) \epsilon N_p / I(p) = m_{fg}$.

The background is the complement of the foreground and vice-versa

### *1.3 Dilation and Erosion*

Morphology uses 'Set Theory' as the foundation for many functions [1]. The simplest functions to implement are

'Dilation' and 'Erosion'

1.3.1 Definition: Dilation of the object $A$ by the structuring element $B$ is given by

$$A \oplus B = \{x : \hat{B}_x \cap A \neq \emptyset\}.$$

Usually A will be the signal or image being operated on A and B will be the Structuring Element'

1.3.2 Definition: Erosion

The opposite of dilation is known as erosion. Erosion of the object $A$ by a structuring element $B$ is given by

$$A \ominus B = \{x : B_x \subseteq A\}.$$

Erosion of A by B is the set of points x such that B translated by x is contained in A.

*1.4 Opening and Closing*

Two very important transformations are *opening* and *closing*. Dilation expands an image object and erosion shrinks it. Opening, generally smoothes a contour in an image, breaking narrow isthmuses and eliminating thin protrusions. Closing tends to narrow smooth sections of contours, fusing narrow breaks and long thin gulfs, eliminating small holes, and filling gaps in contours.

1.4.1 Definition Opening

The opening of $A$ by $B$, denoted by $A \circ B$, is given by the erosion by $B$, followed by the dilation by $B$, that is

$$A \circ B = (A \ominus B) \oplus B.$$

1.4.2 Closing

The opposite of opening is 'Closing' defined by $A \bullet B = (A \oplus B) \ominus B.$

Closing is the dual operation of opening and is denoted by $A \bullet B$. It is produced by the dilation of $A$ by $B$, followed by the erosion by $B$:

**2. Morphological Operators defined on a Lattice**

*2.1 Definition: Dilation*

Let $(L, \leq)$ be a complete lattice, with infimum and minimum symbolized by $\wedge$and $\vee$, respectively.[1],[2].[10]

A dilation is any operator $\delta : L \rightarrow L$that distributes over the supremum and preserves the least element. $\bigvee_i \delta(X_i) = \delta\left(\bigvee_i X_i\right)$, $\delta(\emptyset) = \emptyset$.

*2.2 Definition : Erosion*

An erosion is any operator $\varepsilon : L \rightarrow L$that distributes over the infimum $\bigwedge_i \varepsilon(X_i) = \varepsilon\left(\bigwedge_i X_i\right)$, $\varepsilon(U) = U$.

*2.3 Galois connections*

Dilations and erosions form Galois connections. That is, for all dilation δ there is one and only one erosion $\varepsilon$that satisfies

$X \leq \varepsilon(Y) \Leftrightarrow \delta(X) \leq Y$ for all $X, Y \in L$.

Similarly, for all erosion there is one and only one dilation satisfying the above connection.

Furthermore, if two operators satisfy the connection, then δ must be a dilation , and $\varepsilon$ an erosion.

*2.4 Definition :Adjunctions :*

Pairs of erosions and dilations satisfying the above connection are called "adjunctions", and the erosion is said to be the adjoint erosion of the dilation, and vice-versa.

*2.5 Opening and Closing*

For all adjunction $(\varepsilon, \delta)$, the morphological opening $\gamma : L \to L$ and morphological closing $\phi : L \to L$ are defined as follows:[2]

$\gamma = \delta\varepsilon$, and $\phi = \varepsilon\delta$.

The morphological opening and closing are particular cases of algebraic opening (or simply opening) and algebraic closing(or simply closing). Algebraic openings are operators in *L* that are idempotent, increasing, and anti-extensive. Algebraic closings are operators in *L* that are idempotent, increasing, and extensive.

*2.6 Particular cases*

Binary morphology is a particular case of lattice morphology, where *L* is the power set of *E* (Euclidean space or grid), that is, *L* is the set of all subsets of *E*, and $\leq$ is the set inclusion. In this case, the infimum is set intersection, and the supremum is set union.

Similarly, grayscale morphology is another particular case, [2] where *L* is the set of functions mapping *E* into $\mathbb{R} \cup \{\infty, -\infty\}$, and $\leq$, $\vee$, and $\wedge$, are the point-wise order, supremum, and infimum, respectively. That is, is *f* and *g* are functions in *L*, then $f \leq g$ if and only if $f(x) \leq g(x), \forall x \in E$; the infimum $f \wedge g$ is given by $(f \wedge g)(x) = f(x) \wedge g(x)$; and the supremum $f \vee g$ is given by $(f \vee g)(x) = f(x) \vee g(x)$.[1]

Let $f(n)$ be the signal and $\begin{matrix} g(m) \\ g(n) \end{matrix}$ be the structuring element over a domain then the dilation and erosion are defined as

$(f \oplus g)(n) = \max_{m \in D}[f(n-m) + g(m)], D$ is any domain and

$(f \ominus g)(n) = \min_{m \in D}[f(n+m) - g(m)], D$ is any domain. $\oplus$ and $\ominus$ are dilation and erosion operators.[2]

*2.7 Morphogenetic field*

Let X$\neq \varphi$ and W$\subseteq P(X)$ such that i) $\phi$, X $\in$ W , ii) If B $\in$ W then its complement $\overline{B}$ $\in$W iii) If

$B_i \in W$ is a sequence of signals defined in X, then $\bigcup_{n=1}^{\infty} Bi \in$ W.

Let A= $\{\phi : W \to U / \phi(\cup A_i) = \vee \phi(A_i) \,\&\, \phi(\wedge A_i) = \wedge \phi(A_i)\}$ . Then $W_U$ is called Morphogenetic field [16]where the family $W_u$ is the set of all image signals defined on the continuous or discrete image Plane X and taking values in a set U .The pair ( $W_u$, A ) is called an operator space where A is the collection of operators defined on X.

*2.8 Morphological space*

The triplet (X, $W_u$, A ) consisting of a set X, a morphogenetic field $W_u$ and an operator A(or collection of operators) defined on X is called a Morphological space [8].

Note: If X = $Z^2$ then it is called Discrete Morphological space

## 3 . Morphological Cover

Let $(X, W_u, A)$ be a morphological space. Let $\varepsilon$ be the size of the structuring element B, then

$C_B(\varepsilon) = \sum_{n=1}^{N} (f \oplus B_\varepsilon - f \ominus B_\varepsilon)(n)$ is called the morphological cover of f.[13]

## 4. Brunn-Minkowski Theorem

Let $(X, W_u, A)$ be a morphological space .Let 0<t<1 and let Ho and H1 be any two compact convex structuring bodies in $R^n$.Consider the compact convex body $H_t = (1-t)H_0 + tH_1$.Then the following inequality holds true:

$(v(H_t))^{1/n} \geq (1-t)(v(H_0))^{1/n} + t(v(H_1))^{1/n}$ where $v$ denotes any volume element in $R^n$.

This inequality is reduced to the equality iff $H_0$ and $H_1$ are homothetic.[14]

*4.1 Lemma*:

Let 0<t<1 and let $C_0$ and $C_1$ be two compact convex bodies in $R^n$ such that $v(C_0) = v(C_1) = 1$. Then, for the compact convex body $C_t = (1-t)C_0 + tC_1$ the inequality $v(C_t) \geq 1$ holds true. Furthermore, the inequality $v(C_t) = 1$ is fulfilled iff $C_0$ is a translate of $C_1$ .(ie. $C_0 = h(C_1)$ for some translation h of $R^n$.[14]

 

*4.2 Result*: Let $(X, W_u, A)$ be a morphological space .Let U= E be a vector space and let dim(E) =n. Let P be an n-dimensional convex body structuring element in X taking values in E. Then $\exists$ a finite family of structuring elements $(P_i)_{i \in I}$ of n dimensional simplices such that

1) $P = \bigcup_{i \in I} P_i$
2) The interior of these structuring elements (simplices) are pairwise disjoint.
3) For each $i \in I$ ,the set of vertices of $P_i$ is contained in the set of vertices of P.

Remark: In view of the above result, we can state the following result.

*4.3 Result:* Let $(X, W_u, A)$ be a morphological space. All convex structuring elements defined on a morphological space can be decomposed.

*4.4 Proposition*: If $\bigcup_i B_i$ denotes the decomposition of a convex structuring element $B$ then $B = \bigcup_i B_i$ .

*4.5 Proposition*: A morphological space $(X, W_u, A)$ is compact iff for every family of closed convex sets defined by $v$ has a non empty intersection.

*4.6 Proposition:* Let $(X, W_u, A)$ be a morphological space. If $B_1, B_2, \dots B_n$ are decomposition of a convex structuring element B ,then $f \oplus B = f \oplus \bigcup_i B_i$

*4.7 Lemma* ::Let $(X, W_u, A)$ be a morphological space .Let A and B be any two compact convex structuring elements in the space X = $R^n$ and let the relation $S_B(A) = n(v(A))^{(n-1)/n}.(v(B))^{1/n}$ be satisfied. Then A and B are homothetic. In particular, if B is a ball then A is a ball too, where S denotes an (n-1) dimensional volume in $R^n$.

*4.8* Definition : The cross section $X_t(f)$ of f at level t is the set defined by

$$X_t(f) = \{x / f(x) \geq t\} \quad \text{where } -\infty < t < \infty$$

*4.9* Proposition:[In view of the above lemma]

Let $(X, W_u, A)$ be a morphological space .Let $X_{t1}(f)$ and $X_{t2}(f)$ be two cross sections of a compact convex bodies in $R^n$(or image in $R^n$) and let the relation $S_{X_{t_2}}(X_{t_1}(f)) = n(v(X_{t_1}(f)))^{(n-1)/n}.(v(X_{t_2}(f)))^{1/n}$ be satisfied. Then $X_{t1}(f)$ and $X_{t2}(f)$ are homothetic.

**5 Morphological Equivalence**

If two objects are homothetic then they are morphologically equivalent.

In particular, $X_{t1}(f)$ and $X_{t2}(f)$ are Morphologically equivalent.

*5.1 Proposition*:

Let $(X, W_u, A)$ be a morphological space .Let $B_0, B_1$ be the given compact planar structuring set in $R^2$.

Let $\epsilon B_0, \epsilon B_1$ be positive homothetic. i.e. $H_0 = \varepsilon B_0 = \{\varepsilon b / b \in B_0\}$ & $H_1 = \varepsilon B_1 = \{\varepsilon b / b \in B_1\}$ and also $H_t = (l-t)H_0 + tH_1$ for a compact convex body. Then the following equality holds: $(v(H_t))^{1/n} = (1-t)(v(H_0))^{1/n} + t(v(H_1))^{1/n}$.[14]

*5.2 Proposition:*

Let $(X, W_u, A)$ be a morphological space .Let $C_0 = C_1 + 2$ where $C_1 = \epsilon B$ , $\varepsilon B = \{\varepsilon b / b \in B\}$ for any Basic structuring body B, then $v(C_t) = 1$ where $C_t = (l-t)C_0 + tC_1$ be a compact convex body. Also $v(C_0) = 1$ and $v(C_1) = 1$.[14]

*5.3 Lemma*

Let $(X, W_u, A)$ be a morphological space .Let M be a smooth surface in $R^3$ supplied with the induced metric. That is lengths are measured along the surface: For any $x \epsilon M$ let $D_r(x)$ be the disk of radius r centred at x. Then $(v(H_t))^{1/n} = (1-t)(v(H_0))^{1/n} + t(v(H_1))^{1/n}$ where $H_t = (l-t)H_0 + tH_1$ and homothetic $H_0$ and $H_1$ are defined as $H_0 = \bigcup D_{r_1}(x), \forall x$ and $H_1 = \bigcup D_{r_2}(x), \forall x$ .[14]

*5.4 Result*: Morphological cover of any smooth surface M=f is compact.

*5.5 Morphological floating space*: Let $(X, W_u, A)$ be a morphological space. . Let $\varepsilon$ be the size of any structuring element B and the morphological cover of f be $C_B(\varepsilon) = \sum_{n=1}^{N} (f \oplus B_\varepsilon - f \ominus B_\varepsilon)(n)$ .Then the infima or suprema of all morphological covers for every $\varepsilon$ defines a Morphological floating space. So, there exist two floating spaces which are known as upper and lower floating spaces.

*5.6 Complete Morphological space*

A Morphological space is said to be complete if there exist upper and lower floating spaces.

5.7 Morphological Projection

If a Morphological space $(X, W_u, A)$ is complete then there exist a projection of f into the floating space.

## 6 Conclusion

Convex sets and convex structures play an important role in higher dimensional analysis. Convex sets can be defined in a vector space or more generally in a Morphological space. An attempt towards the extension of work done in lower dimension is explained in this paper. Image processing in higher dimension is more complicated. We hope that this work will help to reduce the difficulties at least in a smaller scale.

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