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On Quadruple Random Fixed Point Theorems in Partially Ordered Metric Spaces

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1. Introduction

Bhaskar and Lakshmikantham in [15] introduced the concept of coupled fixed point of a mapping $F: X \times X \to X$ and investigated the existence and uniquencess of a coupled fixed point theorem in partially ordered complete metric space. Lakshmikantham and Ciric [16] defined mixed g-monotone property and coincidence point in partially ordered metric space. V. Berinde and M. Borcut[18] introduced the concept of triple fixed point and proved some related theorems. Following this trand, Karapinar[19] introduced the nation of quadruple fixed point. The object of this note is to prove quadruple random fixed point theorem in partially ordered metric spaces.

2. Preliminaries

Definition 2.1[19]. Let $(X, \leq)$ be a partially ordered set and $F: X^4 \to X$. The map F has the mixed monotone property if $F(x, y, z, t)$ is monotone nondecreasing in x and z and is monotone nonincreasing in y, t; that is, for any $x, y, z, t \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y, z, t) \leq F(x_2, y, z, t)$$
$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1, z, t) \geq F(x, y_2, z, t)$$
$$z_1, z_2 \in X, \quad z_1 \leq z_2 \Rightarrow F(x, y, z_1, t) \leq F(x, y, z_2, t)$$
$$t_1, t_2 \in X, \quad t_1 \leq t_2 \Rightarrow F(x, y, z, t_1) \geq F(x, y, z, t_2)$$

Definition 2.2[19]. An element $(x, y, z) \in X^4$ is called a quadruple fixed point of a mapping $F: X^4 \to X$ if

$$F(x, y, z, t) = x, \qquad F(y, z, t, x) = y,$$
$$F(z, t, x, y) = z \qquad F(t, x, y, z) = t$$

Definition 2.3[20]. Let $(X, \leq)$ be a partially ordered set and $F: X^4 \to X$ and $g: X \to X$. Then the map F has the mixed g-monotone property if $F(x, y, z, t)$ is monotone g-non-decreasing in x and z and is monotone g-non-increasing in y and t; that is, for any $x, y \in X$.

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z, t) \leq F(x_2, y, z, t)$$
$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z, t) \geq F(x, y_2, z, t)$$
$$z_1, z_2 \in X, \quad g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1, t) \leq F(x, y, z_2, t)$$
$$t_1, t_2 \in X, \quad g(t_1) \leq g(t_2) \Rightarrow F(x, y, z, t_1) \geq F(x, y, z, t_2)$$

Definition 4[20]. An element $(x, y, z, t) \in X^4$ is called a quadruple coincidence point of a mappings $F: X^4 \to X$ and $g: X \to X$ if

$$F(x, y, z, t) = g(x), \qquad F(y, z, t, x) = g(y),$$
$$F(z, t, x, y) = g(z), \qquad F(t, x, y, z) = g(t)$$

Definition 5[20]. Let $F: X^4 \to X$ and $g: X \to X$ be mappings. We say F and g are commutative if

$$g(F(x, y, z, t)) = F(g(x), g(y), g(z), g(t)) \qquad \text{for all } x, y, z, t \in X.$$

Let Φ denote the all functions $\varphi: [0, \infty) \to [0, \infty)$ which are continuous and satisfy that

(i) $\varphi(t) < t$,

(ii) $\lim_{r \to t^+} \varphi(t) < t$ for each $t > 0$.

Let (Ω, Σ) be a measurable space with Σ, a sigma algebra of subsets of Ω and let (X, d) be a metric space. A mapping $T: \Omega \to X$ is called measurable if for open subset U of X, $T^{-1}(U) = \{\omega: T(\omega) \in U\} \in \Sigma$. A mapping $T: \Omega \times X \to X$ is said to be random mapping if for each fixed $x \in X$, the mapping $T(., x): \Omega \to X$ is measurable. A measurable mapping $\xi: \Omega \to X$ is called a random fixed point of the random mapping $T: \Omega \times X \to X$ if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi: \Omega \to X$ is called a random coincidence of $T: \Omega \times X \to X$ and $g: \Omega \times X \to X$ if $T(\omega, \xi(\omega)) = g(\omega, \xi(\omega))$ foe each $\omega \in \Omega$.

3. Main Result

Theorem: Let (X, d) be a complete separable metric space, and let (Ω, Σ) be a measurable space and $\varphi \in \Phi$. Let $F: \Omega \times X^4 \to X$ and $g: \Omega \times X \to X$ be mappings such that

(1) $F(\omega, .), g(\omega, .)$ are continuous for all $\omega \in \Omega$,

(2) $F(.,u), g(.,v)$ are measurable for all $u \in X^4$ and $v \in X$ respectively,

(3) $F: \Omega \times X^4 \to X$ and $g: \Omega \times X \to X$ are such that F has the mixed g-monotone property and

$$d\left(F(\omega,(x,y,z,s)), F(\omega,(u,v,r,t))\right) \leq \varphi\left[\max\left\{\begin{array}{l} d(g(\omega,x), g(\omega,u)), d(g(\omega,y), g(\omega,v)), \\ d(g(\omega,z), g(\omega,r)), d(g(\omega,s), g(\omega,t)) \end{array}\right\}\right] \qquad (1)$$

For all $x, y, z, s, u, v, r, t \in X$ for which $g(\omega,x) \leq g(\omega,u), g(\omega,y) \geq g(\omega,v), g(\omega,z) \leq g(\omega,r)$ and $g(\omega,s) \geq g(\omega,t)$ for all $\omega \in \Omega$. Suppose $g(\omega \times X) = X$ for each $\omega \in \Omega$ And g is continuous and commutes with F and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) If a non decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all n,

(ii) If a non increasing sequence $\{y_n\} \to y$ then $y_n \geq y$ for all n.

If there exist measurable mappings $\xi_0, \eta_0, \zeta_0, \rho_0: \Omega \to X$ such that

$$\left.\begin{array}{l} g(\omega,\xi_0(\omega)) \leq F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right), \\ g(\omega,\eta_0(\omega)) \geq F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right), \\ g(\omega,\zeta_0(\omega)) \leq F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right), \\ g(\omega,\rho_0(\omega)) \geq F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{array}\right\} \quad \text{For all } \omega \in \Omega.$$

Then there are measurable mappings $\xi, \eta, \zeta, \rho: \Omega \to X$ such that

$$\left.\begin{array}{l} F\left(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\right) = g(\omega,\xi(\omega)), \\ F\left(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))\right) = g(\omega,\eta(\omega)), \\ F\left(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))\right) = g(\omega,\zeta(\omega)), \\ F\left(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))\right) = g(\omega,\rho(\omega)) \end{array}\right\} \quad \text{For all } \omega \in \Omega.$$

that is, F and g have a quadruple random coincidence point .

Proof . Let $\Theta = \{\xi: \Omega \to X\}$ be a family of measurable mappings. Define a function $h: \Omega \times X \to R^+$ as follows: $h(\omega,x) = d(x, g(\omega,x))$. Since $x \to g(\omega,x)$ is continuous for all $\omega \in \Omega$, we conclude that $h(\omega,.)$ is continuous for all $\omega \in \Omega$. Also, since $\omega \to g(\omega,x)$ is measurable for all $x \in \Omega$, we conclude that $h(\omega,.)$ is measurable for all $\omega \in \Omega$ (see Wagner [11],page 868).Thus, $h(\omega,x)$ is the Caratheodory function. Therefore, if $\xi: \Omega \to X$ is a measurable mapping, then $\omega \to (h(\omega,\xi(\omega))$ is also measurable (see [9]). Also, for each $\xi \in \Theta$, the function $\eta: \Omega \to X$ defined by $\eta(\omega) = g(\omega,\xi(\omega))$ is measurable; that is, $\eta \in \Theta$.

Now, we will construct four sequences of measurable mappings $\{\xi_n\}, \{\eta_n\}, \{\zeta_n\}$ and $\{\rho_n\}$ *in* Θ and four sequences $\{g(\omega,\xi_n(\omega))\}, \{g(\omega,\eta_n(\omega))\}, \{g(\omega,\zeta_n(\omega))\}$, and $\{g(\omega,\rho_n(\omega))\}$ in X as follows:

Let $\xi_0, \eta_0, \zeta_0, \rho_0 \in \Theta$ such that

$$\left.\begin{array}{l} g(\omega,\xi_0(\omega)) \leq F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right) \\ g(\omega,\eta_0(\omega)) \geq F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right) \\ g(\omega,\zeta_0(\omega)) \leq F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right) \\ g(\omega,\rho_0(\omega)) \geq F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{array}\right\} \quad \text{for all } \omega \in \Omega. \qquad (2)$$

Since $F(\omega \times X^4) \in X = g(\omega \times X)$, then by a sort of filippov measurable implicit function theorem [1,5,6,24] , we can choose $\xi_1, \eta_1, \zeta_1, \rho_1 \in \Theta$ such that

$$\left.\begin{array}{l} g(\omega,\xi_1(\omega)) = F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right) \\ g(\omega,\eta_1(\omega)) = F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right) \\ g(\omega,\zeta_1(\omega)) = F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right) \\ g(\omega,\rho_1(\omega)) = F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) \end{array}\right\} \quad \text{for all } \omega \in \Omega. \qquad (3)$$

Again taking into account that $F(\omega \times X^4) \in X = g(\omega \times X)$ and continuing this process, we can construct sequences $\{\xi_n\}$, $\{\eta_n\}$, $\{\zeta_n\}$ and $\{\rho_n\}$ in X such that

$$\left.\begin{aligned} g(\omega,\xi_{n+1}(\omega)) &= F\left(\omega,(\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega))\right),\\ g(\omega,\eta_{n+1}(\omega)) &= F\left(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega))\right),\\ g(\omega,\zeta_{n+1}(\omega)) &= F\left(\omega,(\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega))\right),\\ g(\omega,\rho_{n+1}(\omega)) &= F\left(\omega,(\rho_n(\omega),\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega))\right) \end{aligned}\right\} \quad \text{for all } \omega \in \Omega. \qquad (4)$$

We shall show that

$$\left.\begin{aligned} g(\omega,\xi_n(\omega)) \le g(\omega,\xi_{n+1}(\omega)),\ g(\omega,\eta_{n+1}(\omega)) \le g(\omega,\eta_n(\omega)),\\ g(\omega,\zeta_n(\omega)) \le g(\omega,\zeta_{n+1}(\omega)),\ g(\omega,\rho_{n+1}(\omega)) \le (\omega,\rho_n(\omega)) \end{aligned}\right\} \text{ for } n = 0,1,2,.. \qquad (5)$$

For this purpose, we will use mathematical induction. By using (2) and (3), we obtain

$$\begin{aligned} g(\omega,\xi_0(\omega)) &\le F\left(\omega,(\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega))\right) = g(\omega,\xi_1(\omega))\\ g(\omega,\eta_0(\omega)) &\ge F\left(\omega,(\eta_0(\omega),\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega))\right) = g(\omega,\eta_1(\omega))\\ g(\omega,\zeta_0(\omega)) &\le F\left(\omega,(\zeta_0(\omega),\rho_0(\omega),\xi_0(\omega),\eta_0(\omega))\right) = g(\omega,\zeta_1(\omega))\\ g(\omega,\rho_0(\omega)) &\ge F\left(\omega,(\rho_0(\omega),\xi_0(\omega),\eta_0(\omega),\zeta_0(\omega))\right) = g(\omega,\rho_1(\omega)) \end{aligned}$$

For all $\omega \in \Omega$.
Therefore (5) hold for $n = 0$.
Suppose that (5) hold for some $n > 0$. Then since F has the mixed g-monotone property and by (4) we have

$$\begin{aligned} g(\omega,\xi_{n+1}(\omega)) &= F\left(\omega,(\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega))\right)\\ &\le F\left(\omega,(\xi_{n+1}(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega))\right)\\ &\le F\left(\omega,(\xi_{n+1}(\omega),\eta_n(\omega),\zeta_{n+1}(\omega),\rho_n(\omega))\right)\\ &\le F\left(\omega,(\xi_{n+1}(\omega),\eta_{n+1}(\omega),\zeta_{n+1}(\omega),\rho_n(\omega))\right)\\ &\le F\left(\omega,(\xi_{n+1}(\omega),\eta_{n+1}(\omega),\zeta_{n+1}(\omega),\rho_{n+1}(\omega))\right) = g(\omega,\xi_{n+2}(\omega))\\ g(\omega,\eta_{n+2}(\omega)) &= F\left(\omega,(\eta_{n+1}(\omega),\zeta_{n+1}(\omega),\rho_{n+1}(\omega),\xi_{n+1}(\omega))\right)\\ &\le F\left(\omega,(\eta_{n+1}(\omega),\zeta_n(\omega),\rho_{n+1}(\omega),\xi_{n+1}(\omega))\right)\\ &\le F\left(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_{n+1}(\omega),\xi_{n+1}(\omega))\right)\\ &\le F\left(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_{n+1}(\omega),\xi_n(\omega))\right)\\ &\le F\left(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega))\right) = g(\omega,\eta_{n+1}(\omega))\\ g(\omega,\zeta_{n+1}(\omega)) &= F\left(\omega,(\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega))\right)\\ &\le F\left(\omega,(\zeta_{n+1}(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega))\right)\\ &\le F\left(\omega,(\zeta_{n+1}(\omega),\rho_{n+1}(\omega),\xi_n(\omega),\eta_n(\omega))\right)\\ &\le F\left(\omega,(\zeta_{n+1}(\omega),\rho_{n+1}(\omega),\xi_{n+1}(\omega),\eta_n(\omega))\right)\\ &\le F\left(\omega,(\zeta_{n+1}(\omega),\rho_{n+1}(\omega),\xi_{n+1}(\omega),\eta_{n+1}(\omega))\right) = g(\omega,\zeta_{n+2}(\omega)) \end{aligned}$$

$$\begin{aligned} g(\omega,\rho_{n+2}(\omega)) &= F\left(\omega,(\rho_{n+1}(\omega),\xi_{n+1}(\omega),\eta_{n+1}(\omega),\zeta_{n+1}(\omega))\right)\\ &\le F\left(\omega,(\rho_{n+1}(\omega),\xi_n(\omega),\eta_{n+1}(\omega),\zeta_{n+1}(\omega))\right) \end{aligned}$$

$$\leq F\left(\omega,\left(\rho_n(\omega),\xi_n(\omega),\eta_{n+1}(\omega),\zeta_{n+1}(\omega)\right)\right)$$
$$\leq F\left(\omega,\left(\rho_n(\omega),\xi_n(\omega),\eta_{n+1}(\omega),\zeta_n(\omega)\right)\right)$$
$$\leq F\left(\omega,\left(\rho_n(\omega),\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega)\right)\right) = g\left(\omega,\rho_{n+1}(\omega)\right)$$

Thus (5) holds for all $n \geq 0$.
Assume, for some $n \in N$, that
$g(\omega,\xi_n(\omega)) = g(\omega,\xi_{n+1}(\omega))$, $g(\omega,\eta_n(\omega)) = g(\omega,\eta_{n+1}(\omega))$,
$g(\omega,\zeta_n(\omega)) = g(\omega,\zeta_{n+1}(\omega))$, $g(\omega,\rho_n(\omega)) = g(\omega,\rho_{n+1}(\omega))$.
Then, by (4), $(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))$ is a quadruple coincidence point of F and g. From now on, assume for any $n \in N$ that at least
$g(\omega,\xi_n(\omega)) \neq g(\omega,\xi_{n+1}(\omega))$, $g(\omega,\eta_n(\omega)) \neq g(\omega,\eta_{n+1}(\omega))$,
$g(\omega,\zeta_n(\omega)) \neq g(\omega,\zeta_{n+1}(\omega))$, $g(\omega,\rho_n(\omega)) \neq g(\omega,\rho_{n+1}(\omega))$.
Due to (1) and (4), we have

$$d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right)$$
$$= d\left(F\left(\omega,\left(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\zeta_{n-1}(\omega),\rho_{n-1}(\omega)\right)\right),F\left(\omega,\left(\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right),d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right), \\ d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right),d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right) \end{array}\right\}\right] \quad (6)$$

$$d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right)$$
$$= d\left(F\left(\omega,\left(\eta_{n-1}(\omega),\zeta_{n-1}(\omega),\rho_{n-1}(\omega),\xi_{n-1}(\omega)\right)\right),F\left(\omega,\left(\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right),d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right), \\ d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right),d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right) \end{array}\right\}\right] \quad (7)$$

$$d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right)$$
$$= d\left(F\left(\omega,\left(\zeta_{n-1}(\omega),\rho_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega)\right)\right),F\left(\omega,\left(\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right),d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right), \\ d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right),d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right) \end{array}\right\}\right] \quad (8)$$

$$d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right)$$
$$= d\left(F\left(\omega,\left(\rho_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega),\zeta_{n-1}(\omega)\right)\right),F\left(\omega,\left(\rho_n(\omega),\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right),d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right), \\ d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right),d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right) \end{array}\right\}\right] \quad (9)$$

Having in mind that $\varphi(t) < t$ for all > 0 , so from (6)-(9) we obtain that
$$0 < \max\left\{\begin{array}{l} d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right),d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right), \\ d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right),d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right) \end{array}\right\}$$

$$\leq \varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right), d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right), \\ d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right), d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right) \end{array}\right\}\right] \quad (10)$$

$$< \max\left\{\begin{array}{l} d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right), d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right), \\ d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right), d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right) \end{array}\right\}$$

It follows that

$$\max\left\{\begin{array}{l} d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right), \\ d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right), \\ d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right), \\ d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right) \end{array}\right\} < \max\left\{\begin{array}{l} d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right), \\ d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right), \\ d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right), \\ d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right) \end{array}\right\}$$

Thus, $\max\left\{\begin{array}{l} d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right), d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right), \\ d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right), d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right) \end{array}\right\}$ is a positive decreasing sequence. Hence there exist $r \geq 0$ such that

$$\lim_{n\to\infty}\max\left\{\begin{array}{l} d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right), d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right), \\ d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right), d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right) \end{array}\right\} = r$$

Suppose that $r > 0$. Letting $n \to \infty$ in (10),we obtain that

$$0 < r \leq \lim_{n\to\infty}\varphi\left[\max\left\{\begin{array}{l} d\left(g(\omega,\rho_{n-1}(\omega)),g(\omega,\rho_n(\omega))\right), d\left(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))\right), \\ d\left(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))\right), d\left(g(\omega,\zeta_{n-1}(\omega)),g(\omega,\zeta_n(\omega))\right) \end{array}\right\}\right]$$

$$\leq \lim_{t\to r^+}\varphi(t) < r$$

It is contraction. We deduce that

$$\lim_{n\to\infty}\max\left\{\begin{array}{l} d\left(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))\right), d\left(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))\right), \\ d\left(g(\omega,\zeta_n(\omega)),g(\omega,\zeta_{n+1}(\omega))\right), d\left(g(\omega,\rho_n(\omega)),g(\omega,\rho_{n+1}(\omega))\right) \end{array}\right\} = 0 \quad (11)$$

We shall show that there exists $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ and $\{g(\omega,\rho_n(\omega))\}$ are Cauchy sequences. Assume the contrary, that is one of the sequence $\{g(\omega,\xi_n(\omega))\}$,$\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ or $\{g(\omega,\rho_n(\omega))\}$ is not a cauchy sequence, that is,

$$\lim_{m,n\to\infty} d\left(g(\omega,\xi_m(\omega)),g(\omega,\xi_n(\omega))\right) \neq 0 \quad \text{or} \quad \lim_{m,n\to\infty} d\left(g(\omega,\eta_m(\omega)),g(\omega,\eta_n(\omega))\right) \neq 0$$

Or

$$\lim_{m,n\to\infty} d\left(g(\omega,\zeta_m(\omega)),g(\omega,\zeta_n(\omega))\right) \neq 0 \quad \text{or} \quad \lim_{m,n\to\infty} d\left(g(\omega,\rho_m(\omega)),g(\omega,\rho_n(\omega))\right) \neq 0$$

This means that there exist $\varepsilon > 0$, for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\max\left\{\begin{array}{l} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right), d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right), \\ d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right), d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right) \end{array}\right\} \geq \varepsilon \quad (12)$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (12). Then

$$\max\left\{\begin{array}{l} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k-1}(\omega)\right)\right), d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k-1}(\omega)\right)\right), \\ d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k-1}(\omega)\right)\right), d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k-1}(\omega)\right)\right) \end{array}\right\} < \varepsilon \quad (13)$$

By triangular inequality and (13), we have

$$d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)$$
$$\le d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big)+d\Big(g\big(\omega,\xi_{n_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)$$
$$<\varepsilon+d\Big(g\big(\omega,\xi_{n_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)$$

Letting $k\to\infty$ and using (11), we get

$$\lim_{k\to\infty} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big)\le\varepsilon$$

Similarly , we have

$$\lim_{k\to\infty} d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k-1}(\omega)\big)\Big)\le\varepsilon$$
$$\lim_{k\to\infty} d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k-1}(\omega)\big)\Big)\le\varepsilon$$
$$\lim_{k\to\infty} d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k-1}(\omega)\big)\Big)\le\varepsilon$$

Again by (13) ,we have

$$\begin{aligned}
d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big) &\le d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{m_k-1}(\omega)\big)\Big)\\
&\quad+\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big)+d\Big(g\big(\omega,\xi_{n_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)\\
&\le d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{m_k-1}(\omega)\big)\Big)+\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{m_k}(\omega)\big)\Big)\\
&\quad+\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big)+d\Big(g\big(\omega,\xi_{n_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)\\
&< d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{m_k-1}(\omega)\big)\Big)+\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{m_k}(\omega)\big)\Big)\\
&\quad+\varepsilon+d\Big(g\big(\omega,\xi_{n_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)
\end{aligned}$$

Letting $k\to\infty$ and using (11), we get

$$\lim_{k\to\infty} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big)\le\varepsilon \qquad (14)$$

Similarly , we have

$$\lim_{k\to\infty} d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\eta_{m_k-1}(\omega)\big),g\big(\omega,\eta_{n_k-1}(\omega)\big)\Big)\le\varepsilon \qquad (15)$$
$$\lim_{k\to\infty} d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\zeta_{m_k-1}(\omega)\big),g\big(\omega,\zeta_{n_k-1}(\omega)\big)\Big)\le\varepsilon \qquad (16)$$
$$\lim_{k\to\infty} d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k}(\omega)\big)\Big)\le\lim_{k\to\infty} d\Big(g\big(\omega,\rho_{m_k-1}(\omega)\big),g\big(\omega,\rho_{n_k-1}(\omega)\big)\Big)\le\varepsilon \qquad (17)$$

Using (12) and (14)-(17), we have

$$\begin{aligned}
&\lim_{k\to\infty}\max\left\{\begin{array}{l} d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big),d\Big(g\big(\omega,\eta_{m_k}(\omega)\big),g\big(\omega,\eta_{n_k}(\omega)\big)\Big),\\ d\Big(g\big(\omega,\zeta_{m_k}(\omega)\big),g\big(\omega,\zeta_{n_k}(\omega)\big)\Big),d\Big(g\big(\omega,\rho_{m_k}(\omega)\big),g\big(\omega,\rho_{n_k}(\omega)\big)\Big)\end{array}\right\}\\
&=\lim_{k\to\infty}\max\left\{\begin{array}{l} d\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{n_k-1}(\omega)\big)\Big),d\Big(g\big(\omega,\eta_{m_k-1}(\omega)\big),g\big(\omega,\eta_{n_k-1}(\omega)\big)\Big),\\ d\Big(g\big(\omega,\zeta_{m_k-1}(\omega)\big),g\big(\omega,\zeta_{n_k-1}(\omega)\big)\Big),d\Big(g\big(\omega,\rho_{m_k-1}(\omega)\big),g\big(\omega,\rho_{n_k-1}(\omega)\big)\Big)\end{array}\right\}\\
&=\varepsilon \qquad (18)
\end{aligned}$$

Now using inequality (1) we obtain

$$\begin{aligned}
&d\Big(g\big(\omega,\xi_{m_k}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big)\\
&=d\Big(F\Big(\omega,\big(\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega)\big)\Big),F\Big(\omega,\big(\xi_{n_k}(\omega),\eta_{n_k}(\omega),\zeta_{n_k}(\omega),\rho_{n_k}(\omega)\big)\Big)\Big)\\
&\le\varphi\left[\max\left\{\begin{array}{l} d\Big(g\big(\omega,\xi_{m_k-1}(\omega)\big),g\big(\omega,\xi_{n_k}(\omega)\big)\Big),d\Big(g\big(\omega,\eta_{m_k-1}(\omega)\big),g\big(\omega,\eta_{n_k}(\omega)\big)\Big),\\ d\Big(g\big(\omega,\zeta_{m_k-1}(\omega)\big),g\big(\omega,\zeta_{n_k}(\omega)\big)\Big),d\Big(g\big(\omega,\rho_{m_k-1}(\omega)\big),g\big(\omega,\rho_{n_k}(\omega)\big)\Big)\end{array}\right\}\right] \qquad (19)
\end{aligned}$$

$$d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)$$
$$= d\left(F\left(\omega,\left(\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\eta_{n_k}(\omega),\zeta_{n_k}(\omega),\rho_{n_k}(\omega),\xi_{n_k}(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{matrix} d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right)\end{matrix}\right\}\right] \quad (20)$$

$$d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)$$
$$= d\left(F\left(\omega,\left(\zeta_{m_k-1}(\omega),\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\zeta_{n_k}(\omega),\rho_{n_k}(\omega),\xi_{n_k}(\omega),\eta_{n_k}(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{matrix} d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right)\end{matrix}\right\}\right] \quad (21)$$

$$d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)$$
$$= d\left(F\left(\omega,\left(\rho_{m_k-1}(\omega),\xi_{m_k-1}(\omega),\eta_{m_k-1}(\omega),\zeta_{m_k-1}(\omega)\right)\right),F\left(\omega,\left(\rho_{n_k}(\omega),\xi_{n_k}(\omega),\eta_{n_k}(\omega),\zeta_{n_k}(\omega)\right)\right)\right)$$
$$\leq \varphi\left[\max\left\{\begin{matrix} d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right)\end{matrix}\right\}\right] \quad (22)$$

From (19) – (22) we deduce that

$$\max\left\{\begin{matrix} d\left(g\left(\omega,\xi_{m_k}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{matrix}\right\}$$
$$\leq \varphi\left[\max\left\{\begin{matrix} d\left(g\left(\omega,\xi_{m_k-1}(\omega)\right),g\left(\omega,\xi_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\eta_{m_k-1}(\omega)\right),g\left(\omega,\eta_{n_k}(\omega)\right)\right),\\ d\left(g\left(\omega,\zeta_{m_k-1}(\omega)\right),g\left(\omega,\zeta_{n_k}(\omega)\right)\right),d\left(g\left(\omega,\rho_{m_k-1}(\omega)\right),g\left(\omega,\rho_{n_k}(\omega)\right)\right)\end{matrix}\right\}\right] \quad (23)$$

Letting $k \to \infty$ in (23) and having in mind (18) we get that

$$0 < \varepsilon \leq \lim_{t\to\varepsilon^+}\varphi(t) < \varepsilon,$$

It is contraction. Thus $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\zeta_n(\omega))\}$ and $\{g(\omega,\rho_n(\omega))\}$ are Cauchy sequences in (X,d).

Since (X,d) is complete and $g(\omega\times X) = X$ then there exist $\theta_p,\vartheta_p,\mu_p,\upsilon_p \in \Theta$ such that

$$\left.\begin{aligned}\lim_{n\to\infty} g(\omega,\xi_n(\omega)) = g(\omega,\theta_p(\omega)),\ \lim_{n\to\infty} g(\omega,\eta_n(\omega)) = g(\omega,\vartheta_p(\omega)),\\ \lim_{n\to\infty} g(\omega,\zeta_n(\omega)) = g(\omega,\mu_p(\omega)),\ \lim_{n\to\infty} g(\omega,\rho_n(\omega)) = g(\omega,\upsilon_p(\omega)).\end{aligned}\right\} \tag{24}$$

Since $g(\omega,\theta_p(\omega)), g(\omega,\vartheta_p(\omega)), g(\omega,\mu_p(\omega))$ and $g(\omega,\upsilon_p(\omega))$ are measurable, then the function $\xi(\omega),\eta(\omega),\zeta(\omega)$ and $\rho(\omega)$, defined by

$$\left.\begin{aligned}\xi(\omega) = g(\omega,\theta_p(\omega)),\ \eta(\omega) = g(\omega,\vartheta_p(\omega)),\\ \zeta(\omega) = g(\omega,\mu_p(\omega)),\ \rho(\omega) = g(\omega,\upsilon_p(\omega))\end{aligned}\right\} \tag{25}$$

Are measurable too. Thus

$$\left.\begin{aligned}\lim_{n\to\infty} g(\omega,\xi_n(\omega)) = \xi(\omega),\ \lim_{n\to\infty} g(\omega,\eta_n(\omega)) = \eta(\omega),\\ \lim_{n\to\infty} g(\omega,\zeta_n(\omega)) = \zeta(\omega),\ \lim_{n\to\infty} g(\omega,\rho_n(\omega)) = \rho(\omega)\end{aligned}\right\} \tag{26}$$

Since g is continuous, (26) implies that

$$\left.\begin{aligned}\lim_{n\to\infty} g(\omega,g(\omega,\xi_n(\omega))) = g(\omega,\xi(\omega)),\ \lim_{n\to\infty} g(\omega,g(\omega,\eta_n(\omega))) = g(\omega,\eta(\omega)),\\ \lim_{n\to\infty} g(\omega,g(\omega,\zeta_n(\omega))) = g(\omega,\zeta(\omega)),\ \lim_{n\to\infty} g(\omega,g(\omega,\rho_n(\omega))) = g(\omega,\rho(\omega)).\end{aligned}\right\} \tag{27}$$

by using the fact that F and g are commutative, From (4)

$$\begin{aligned}&F\big(\omega,\big(g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)),g(\omega,\zeta_n(\omega)),g(\omega,\rho_n(\omega))\big)\big)\\ &\qquad= g\big(\omega,F\big(\omega,(\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega))\big)\big)\\ &\qquad= g(\omega,g(\omega,\xi_{n+1}(\omega)))\end{aligned} \tag{28}$$

$$\begin{aligned}&F\big(\omega,\big(g(\omega,\eta_n(\omega)),g(\omega,\zeta_n(\omega)),g(\omega,\rho_n(\omega)),g(\omega,\xi_n(\omega))\big)\big)\\ &\qquad= g\big(\omega,F\big(\omega,(\eta_n(\omega),\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega))\big)\big)\\ &\qquad= g(\omega,g(\omega,\eta_{n+1}(\omega)))\end{aligned} \tag{29}$$

$$\begin{aligned}&F\big(\omega,\big(g(\omega,\zeta_n(\omega)),g(\omega,\rho_n(\omega)),g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega))\big)\big)\\ &\qquad= g\big(\omega,F\big(\omega,(\zeta_n(\omega),\rho_n(\omega),\xi_n(\omega),\eta_n(\omega))\big)\big)\\ &\qquad= g(\omega,g(\omega,\zeta_{n+1}(\omega)))\end{aligned} \tag{30}$$

$$\begin{aligned}&F\big(\omega,\big(g(\omega,\rho_n(\omega)),g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)),g(\omega,\zeta_n(\omega))\big)\big)\\ &\qquad= g\big(\omega,F\big(\omega,(\rho_n(\omega),\xi_n(\omega),\eta_n(\omega),\zeta_n(\omega))\big)\big)\\ &\qquad= g(\omega,g(\omega,\rho_{n+1}(\omega)))\end{aligned} \tag{31}$$

Now we will show that if the assumption (a) and (b) hold, then

$$\left.\begin{aligned}F(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))) = g(\omega,\xi(\omega)),\\ F(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))) = g(\omega,\eta(\omega)),\\ F(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))) = g(\omega,\zeta(\omega)),\\ F(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))) = g(\omega,\rho(\omega)),\end{aligned}\right\} \quad \text{For all } \omega\in\Omega .$$

Suppose (a) hold from (26), (27), (28) and the continuity of F, we obtain

$$\begin{aligned}g(\omega,\xi(\omega)) &= \lim_{n\to\infty} g(\omega,g(\omega,\xi_{n+1}(\omega)))\\ &= \lim_{n\to\infty} F\big(\omega,\big(g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)),g(\omega,\zeta_n(\omega)),g(\omega,\rho_n(\omega))\big)\big)\\ &= F\big(\omega,\big(\lim_{n\to\infty} g(\omega,\xi_n(\omega)),\lim_{n\to\infty} g(\omega,\eta_n(\omega)),\lim_{n\to\infty} g(\omega,\zeta_n(\omega)),\lim_{n\to\infty} g(\omega,\rho_n(\omega))\big)\big)\end{aligned}$$

$$= F\Big(\omega,\big(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega)\big)\Big)$$

and similarly

$$\begin{aligned}
g(\omega,\eta(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega,\eta_{n+1}(\omega))\big)\\
&= \lim_{n\to\infty} F\Big(\omega,\big(g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\lim_{n\to\infty} g(\omega,\eta_n(\omega)), \lim_{n\to\infty} g(\omega,\zeta_n(\omega)), \lim_{n\to\infty} g(\omega,\rho_n(\omega)), \lim_{n\to\infty} g(\omega,\xi_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega)\big)\Big)\\
g(\omega,\zeta(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega,\zeta_{n+1}(\omega))\big)\\
&= \lim_{n\to\infty} F\Big(\omega,\big(g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\lim_{n\to\infty} g(\omega,\zeta_n(\omega)), \lim_{n\to\infty} g(\omega,\rho_n(\omega)), \lim_{n\to\infty} g(\omega,\xi_n(\omega)), \lim_{n\to\infty} g(\omega,\eta_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega)\big)\Big)\\
g(\omega,\rho(\omega)) &= \lim_{n\to\infty} g\big(\omega, g(\omega,\rho_{n+1}(\omega))\big)\\
&= \lim_{n\to\infty} F\Big(\omega,\big(g(\omega,\rho_n(\omega)), g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\lim_{n\to\infty} g(\omega,\rho_n(\omega)), \lim_{n\to\infty} g(\omega,\xi_n(\omega)), \lim_{n\to\infty} g(\omega,\eta_n(\omega)), \lim_{n\to\infty} g(\omega,\zeta_n(\omega))\big)\Big)\\
&= F\Big(\omega,\big(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega)\big)\Big)
\end{aligned}$$

Thus , we proved that $\big(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega)\big) \in X^4$ is a quadruple random coincidence of F and g.

Suppose, now the assumption (b) holds. Since

$$g(\omega,\xi_n(\omega)) \le g(\omega,\theta_0(\omega)) = \xi(\omega),$$

$$g(\omega,\eta_n(\omega)) \ge g(\omega,\theta_0(\omega)) = \eta(\omega),$$

$$g(\omega,\zeta_n(\omega)) \le g(\omega,\mu_0(\omega)) = \zeta(\omega),$$

$$g(\omega,\rho_n(\omega)) \ge g(\omega,\upsilon_0(\omega)) = \rho(\omega).$$

Therefore, by the triangle inequality

$$\begin{aligned}
&d\Big(g(\omega,\xi(\omega)), F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) \le d\Big(g(\omega,\xi(\omega)), g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big)\Big)\\
&\qquad\qquad + d\Big(g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big), F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big)\\
&\le d\Big(g(\omega,\xi(\omega)), g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big)\Big)\\
&+ d\Big(F\big(\omega,(g(\omega,\xi_n(\omega)), g(\omega,\eta_n(\omega)), g(\omega,\zeta_n(\omega)), g(\omega,\rho_n(\omega)))\big), F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big)\\
&\le d\Big(g(\omega,\xi(\omega)), g\big(\omega, g(\omega,\xi_{n+1}(\omega))\big)\Big)\\
&+ \varphi\left[\max\left\{\begin{matrix} d\Big(g\big(\omega, g(\omega,\xi_n(\omega))\big), g(\omega,\xi(\omega))\Big), d\Big(g\big(\omega, g(\omega,\eta_n(\omega))\big), g(\omega,\eta(\omega))\Big), \\ d\Big(g\big(\omega, g(\omega,\zeta_n(\omega))\big), g(\omega,\zeta(\omega))\Big), d\Big(g\big(\omega, g(\omega,\rho_n(\omega))\big), g(\omega,\rho(\omega))\Big) \end{matrix}\right\}\right]
\end{aligned}$$

And since $\varphi(t) < t$, we have

$$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big) < d\Big(g(\omega,\xi(\omega)),g\big(\omega,g(\omega,\xi_{n+1}(\omega))\big)\Big)$$

$$+\max\left\{\begin{matrix} d\Big(g\big(\omega,g(\omega,\xi_n(\omega))\big),g(\omega,\xi(\omega))\Big),d\Big(g\big(\omega,g(\omega,\eta_n(\omega))\big),g(\omega,\eta(\omega))\Big), \\ d\Big(g\big(\omega,g(\omega,\zeta_n(\omega))\big),g(\omega,\zeta(\omega))\Big),d\Big(g\big(\omega,g(\omega,\rho_n(\omega))\big),g(\omega,\rho(\omega))\Big) \end{matrix}\right\}$$

Letting $n\to\infty$ and by (27), we get

$$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big)\le 0$$

But $$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big)\ge 0$$

Hence $$d\Big(g(\omega,\xi(\omega)),F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)\Big)= 0$$

Hence $$F\big(\omega,(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\big)=g(\omega,\xi(\omega))$$

Similarly, we can show that

$$F\big(\omega,(\eta(\omega),\zeta(\omega),\rho(\omega),\xi(\omega))\big)=g(\omega,\eta(\omega)),$$

$$F\big(\omega,(\zeta(\omega),\rho(\omega),\xi(\omega),\eta(\omega))\big)=g(\omega,\zeta(\omega)),$$

$$F\big(\omega,(\rho(\omega),\xi(\omega),\eta(\omega),\zeta(\omega))\big)=g(\omega,\rho(\omega)).$$

For all $\omega\in\Omega$.

Thus we showed that $(\xi(\omega),\eta(\omega),\zeta(\omega),\rho(\omega))\in X^4$ is a quadruple random coincidence of F and g.

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