 

# An Unconstrained Q - G Programming Problem and its Application

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**Abstract**
In practice there exist many methods to solve unconstrained, constrained and mixed quadratic and geometric programming problems. In this paper an attempt is made to develop unconstraint Q – G programming problem by combining quadratic programming problem and geometric programming problem. This model is solved using the technique of geometric programming problem. A hypothetical example is considered to illustrate the model.
**Keywords:** Unconstrained Quadratic Programming problems, Geometric Programming problem, Primal and Dual problem, Orthogonality and normality conditions.

## 1. Introduction

In practice various techniques are available to solve unconstrained quadratic problem. The objective of quadratic programming is to maximize or to minimize the quadratic objective function. Let decision variables $\underline{x}$ and the coefficients of objective function, $\underline{C} \in R^n$ and $D$ be symmetric matrix of real numbers of order $n \times n$ then the unconstrained quadratic programming problem is define as follows:

$$Minimize \;\; f(x) = \underline{C}' \cdot \underline{x} + \frac{1}{2} \underline{x}' \cdot D \cdot \underline{x}$$

Here $\underline{x}' \cdot D \cdot \underline{x}$ is in quadratic form and $D = (d_{ij})_{n \times n}$ is a symmetric matrix.

Geometric programming is a technique for solving a special case of nonlinear problems. Duffin, Peterson and Zener [2] published a book "*Geometric Programming: Theory and Applications"* that started the field of Geometric Programming as a branch of nonlinear optimization with many useful theoretical and computational properties of Geometric Programming, to a large extent the scope of linear programming applications and is naturally applied to several important nonlinear systems in science and engineering. Several important developments of Geometric Programming are in the area of mechanical and civil engineering, chemical engineering, probability and statistics, finance and economics, control theory, circuit design, information technology, coding and signal processing, wireless networking, etc. took place in the late 1960s to early 1970s. There are several books on nonlinear optimization that have a section on Geometric Programming, e.g., M. Avriel, [5], C. S. Beightler [1], G. Hadley [4], Taha [6], etc. However, many researchers felt that most of the theoretical, algorithmic and application aspects of Geometric Programming had been exhausted by the early 1980's, the period of 1980–98 was relatively quiet. After the revolution in the electronic field, over the last few years, Geometric Programming started to receive renewed attention from the operations research community.

In 1964, R. Duffin and C. Zener [3], have defined unconstrained Geometric Programming in the following manner:

$$Z = f(\underline{x}) = \sum_{k=1}^{N} U_k$$

$$Where,$$

$$U_k = C_k \prod_{i=1}^{n} x_i^{a_{ik}} \;\; for \;\; k = 1,2,3,......,N \tag{1}$$

Here it is assumed that the coefficient $c_k > 0$ and N is finite. The exponents $a_{ik}$ are unrestricted in sign i.e. it may be positive or negative.

A model is considered in which the concept of quadratic programming problem and geometric programming is combined, is defined as Q – G model.

## 2.Assumptions

In the present study following assumptions are made to derive a solution to the quadratic-geometric programming.

1. The coefficients are unrestricted in sign i.e. $c_i \geq 0 \;\; or \;\; c_i \leq 0 \;\; ; i = 1, 2, .....n$
2. N is finite i.e. number of terms is finite.
3. The number of terms $N = n + 1$ where $n$ is number of variables.
4. $D = (d_{ij})_{n \times n}$ is a symmetric matrix.

**3. Mathematical Model and Procedure:**

Let $\left(\dfrac{1}{\underline{x}}\right)$ and $\underline{C} \in R^n$ and $D$ is any real $n \times n$ matrix then unconstraint Q - G programming problem is defined as under:

$$Minimise \quad Z = f(\underline{x}) = \underline{C}' \left(\frac{1}{\underline{x}}\right) + \frac{1}{2}\left(\frac{1}{\underline{x}}\right)' \cdot D \cdot \left(\frac{1}{\underline{x}}\right)$$

$$= \sum_{i=1}^{n} \frac{c_i}{x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d_{ij}}{x_i x_j} \qquad [2]$$

The above problem can be converted in to geometric programming problem as under:

$$Z = f(\underline{x}) = \sum_{k=1}^{N} U_k$$

*Where,*

$$U_k = c_k \prod_{i=1}^{n} x_i^{a_{ik}} \quad for \quad k = 1, 2, 3, ......, N$$

This problem will be considered as the primal problem. Here $Z$ is in the polynomial form and it is assumed that all variables $x_i$ are strictly positive or negative. The requirement $x_i \neq 0$ plays an essential role in the derivation of the results.

For minimum value of the objective function, the first order partial derivative of $Z$ must be zero, now differentiate $Z$ with respect to $x_k \;\; ; k = 1, 2, ....n$

$$\frac{\partial z}{\partial x_r} = \sum_{k=1}^{N} \frac{\partial U_k}{\partial x_k}$$

$$= \sum_{k=1}^{N} c_k \cdot a_{ik} (x_i')^{a_{ik}-1} \cdot \prod_{i \neq r}^{n} x_i^{a_{ik}} = 0, \quad for \quad r, i = 1, 2, 3, ....., n \qquad [3]$$

Since, each $x_i \neq 0$

$$\frac{\partial z}{\partial x_r} = 0 = \frac{1}{x_r} \sum_{k=1}^{N} a_{ik} U_k, \quad for \quad k = 1, 2, 3, ....., N, i = 1, 2, ...., n \qquad [4]$$

Let $z^*$ be the minimum value of $Z$ It is necessary that $z^* \neq 0,$ since each $x_r^* \neq 0$ and $Z$ is a polynomial defined as

$$y_k = \frac{U_k^*}{z^*} \qquad [5]$$

Which shows that $y_k \neq 0$ and

$$\sum_{k=1}^{N} y_k = 1 \qquad \left(\because \sum_{k=1}^{N} y_k^* = z^*\right) \qquad [6]$$

Thus the value of $y_k$ represents the relative combination of the $k^{th}$ term $U_k$ to the optimum value of the objective function $z^*$.

Now the necessary conditions can be written as

$$\sum_{k=1}^{N} a_{rk} \cdot y_k = 0 \quad (z^* > 0,\ x_r \neq 0,\ r = 1,2,....n) \tag{7}$$

and

$$\sum_{k=1}^{N} y_k = 1, \qquad (y_k \neq 0,\ k = 1,2,3,....N) \tag{8}$$

These conditions [7] and [8] are known as orthogonality and normality conditions. By using matrix inversion method, these conditions will give a unique solution for $y_k$, if (n+1) = N and all the equations are independent. If N > (n+1) then the problem becomes more complex because the values of $y_k$ are not unique. However, it is possible to determine $y_k$ uniquely for the purpose of minimizing $Z$.

Now, suppose that $y_k^*$ are the unique values determined from the equations given in the results [7] and [8]. These values are used to determined the values of $z^*$ and $x_r^*$ for $r = 1,2,3,....n$ as under,

Consider,

$$z^* = \left(z^*\right)^{\sum_{k=1}^{N} y_k^*} \quad \left(\because \sum_{k=1}^{N} y_k^* = 1\right) \tag{9}$$

$$z^* = \prod_{r=1}^{n} \left( \frac{C_r \cdot \prod_{i=1}^{n} x_i^{a_{ir}}}{y_r^*} \right)^{y_r^*} \tag{10}$$

$$z^* = \prod_{k=1}^{N} \left( \frac{C_k}{y_k^*} \right)^{y_k^*} \left[ \prod_{k=1}^{N} \left( \prod_{i=1}^{n} x_i^{a_{ik}} \right)^{y_k^*} \right] \tag{11}$$

$$z^* = \prod_{n=1}^{N} \left( \frac{C_k}{y_k^*} \right)^{y_k^*} \left[ \prod_{i=1}^{n} x_i^{\sum_{k=1}^{N} a_{ik} \cdot y_k^*} \right] \tag{12}$$

$$z^* = \prod_{k=1}^{N} \left( \frac{C_k}{y_k^*} \right)^{y_k^*} \qquad \left(\because \sum_{k=1}^{N} a_{ik} \cdot y_k^* = 0\right) \tag{13}$$

Thus, the value of $z^*$ is determined from result [13] as soon as all $y_k^*$ are determined.

Now, for known values of $y_k^*$ and $z^*$ the value of $U_k^*$ can be determined from $U_k^* = y_k^* \cdot z^*$

Since $U_k^* = C_k \prod_{i=1}^{n} \left(x_i^*\right)^{a_{ik}} \ for\ \ k = 1,2,3,....,N$ simultaneously solution of these equations should give $x_i^* \ for\ \ i = 1,2,3,...n.$

The procedure described hare shows that the solution to the original polynomial $Z$ can be transformed in to the solution of a set of linear equations in $y_k$. Observed that all $y_k^*$ are determined from the necessary conditions for a minimum. However, it can be shown that, these conditions are also sufficient.

 

**5. Conclusion**
**Hypothetical Problem:**
Consider the following problem of Q - G Programming with three decision variables and four terms;

$$Minimise \quad Z = \frac{2}{x_1} + \frac{3}{x_2^2} - \frac{5}{x_1 x_2}$$

The above problem can be written as

$$\begin{aligned} Minimise\, Z &= 2 \cdot x_1^{-1} \cdot x_2^{0} + 3x_1^{0} \cdot x_2^{-2} - 5x_1^{-1} \cdot x_2^{-1} \\ &= U_1 + \mathrm{U}_2 + \mathrm{U}_3 \\ &= c_1 \cdot x_1^{a_{11}} \cdot x_2^{a_{21}} + c_2 \cdot x_1^{a_{12}} \cdot x_2^{a_{22}} + c_3 \cdot x_1^{a_{13}} \cdot x_2^{a_{23}} \end{aligned}$$

Then by comparison, following matrices are obtained.

$$\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & -1 \end{pmatrix}$$

Here $i = 1,2$ and $k = 1,2,3$ so the case in which N = (n+1) is to be considered. Using orthogonality and normality conditions

$$\sum_{k=1}^{3} a_{ik} \cdot y_k = 0 \quad \text{and} \quad \sum_{k=1}^{3} y_k = 1,$$

Following equations are obtained

$$\begin{aligned} &a_{11} \cdot y_1 + a_{12} \cdot y_2 + a_{13} \cdot y_3 = 0 \\ &a_{21} \cdot y_1 + a_{22} \cdot y_2 + a_{23} \cdot y_3 = 0 \\ &and \quad y_1 + y_2 + y_3 + y_4 = 1 \end{aligned}$$

The above equations can be represented as

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, by using matrix inversion method the values of dual variables can be determined as under,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ -2 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

This is a unique solution given as

$$y_1^* = 2, \quad y_2^* = 1, \quad y_3^* = -2$$

Now,

$$\begin{aligned} z^* &= \prod_{k=1}^{3} \left( \frac{c_k}{y_k^*} \right)^{y_k^*} \\ &= \left( \frac{2}{2} \right)^2 \cdot \left( \frac{3}{1} \right)^1 \cdot \left( \frac{-5}{-2} \right)^{-2} \\ &= 0.48 \end{aligned}$$

From the equation $u_k^* = y_k^* \cdot z^*$ it can be deduced that,

$$u_1 = 0.96,\ u_2 = 0.48,\ u_3 = -0.96$$

Which will give the optimum solution to the primal problem as under:

$$x_1^* = 2.0833,\ x_2^* = 2.5\ \ and\ \ z^* = 0.48.$$

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