Journal of Information Engineering and Applications
ISSN 2224-5782 (print) ISSN 2225-0506 (online)
Vol.5, No.2, 2015

On Existence of Coincidence and Common Fixed Points for Weakly Compatible Self Maps in Normed Boolean Vector Space

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Abstract
In this paper we discuss existence of coincidence and common fixed points for two pairs of weakly compatible self maps in Normed Boolean Vector Space. Our results extend and improve the results of Mishra et al. [8] and others existing in the literature. Some illustrative examples to highlight the validity of obtained results are also furnished.
Keywords: Weakly compatible maps, coincidence point, common fixed point, Normed Boolean vector space, Boolean metric.

1. Introduction

In real life problems a fixed point indicates the situation where a steady state condition or equilibrium is reached. The stability of each fixed point can be determined after locating fixed points in a system. Fixed points of Boolean functions play a crucial role in medicine(diagnostic, risk assessment), in social sciences (qualitative analysis of data), in psychology (human concept learning) , in complexity theory, coding theory, the design of chips for digital computers, error correcting codes, switching circuits, the relationship between the consistency of a Boolean equation, cryptography, in the design of symmetric key algorithms, electrical engineering, reliability theory, game theory and combinatory. These applications have often provided motivation for the study of problems in fixed point theory for Boolean valued functions. However, to best of our knowledge, there are only a limited number of results available in literature (see, for instance [3], [8], [9], [10], [11], [12] and references therein). S. Rudeanu[10] presented some interesting results on the existence of a unique fixed point of Boolean transformations by using the canonical disjunctive form of a Boolean function. Rao *et al.* [9] obtained some fixed and common fixed point theorems for asymptotically regular maps in finite dimensional normed Boolean vector spaces. It is necessary to point out here that the problems concerning existence of fixed points of Boolean functions in some cases are far from being solved till now.

Recently Mishra *et al.* [8] obtained some common fixed point theorems for a pair of weakly compatible self maps on normed Boolean vector space satisfying the property (*E.A)* with containment , completeness of range subspace and the axiom $SSu = Su$ without using continuity. The purpose of this paper is to extend and improve the results of Mishra *et al*. [8] to two pairs of maps via property (*E.A.*) and its variants in Normed Boolean Vector Space besides furnishing examples which demonstrate the validity of results obtained.

2. Preliminaries

Definition 2.1.[11] Let $V = (V, +)$ be an additive abelian group and $B = (B, +, .,)$ a Boolean algebra. The set V with two operations namely 'addition' and 'scalar multiplication' is said to be a Boolean vector space over B (or simply, a B - vector space) if for all $x, y \in V$ and $a, b \in B$,
(i) $a(x + y) = ax + ay$;
(ii) $(ab)x = a(bx) = b(ax)$;
(iii) $1x = x$; and
(iv) if $ab = 0$, then $(a + b)x = ax + bx$.
The elements of V and B will be denoted respectively by x, y, z and a, b, c (with or without indices); the zero of V and also null-element of B will both be denoted by 0, while the universal element ($= 0'$) of B will be denoted by 1.
Example 2.2.[11]. Let B be any Boolean algebra and V be the additive group of the corresponding Boolean ring then V is a B -vector space if we define: for $a \in B$ and $x \in V$, ax = the (Boolean) product of a and x in B.
Example 2.3.[11]. Let R be any Ring with unity element 1 and let B denotes the set of all the central idempotent of R then it is known that $(B, \cup, \cap)$ is a Boolean algebra, where by defining $a \cup b = a + b - ab$, $a \cap b = ab$ and $a' = 1 - a$. If V is the additive group of the ring R and for $a \in B$ and $x \in V$, ax = the product of a and x in R, then V is a Boolean vector space over $(B, \cup, \cap)$.
Definition 2.4.[11]. A Boolean vector space V over a Boolean algebra B is said to be B -normed (or simply, normed) if and only if there exists a map $\|.\|$ (called norm): $V \to B$ such that
(i) $\|x\| = 0$ if and only if $x = 0$, and

(ii) $\|ax\| = a\,\|x\|$ for all $a \in$ B and $x \in V$.
In view of [[11], Corollary 3.2] we have:
Let V be a B - normed vector space and d: $V \times V \to B$ then $d(x, y) = \|x - y\|$ defines a Boolean metric on V, i.e.,
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ and
(iii) $d(x, z) < d(x, y) + d(y, z)$.

Definition 2.5. [11]. Let B be a σ -complete (or countably complete) Boolean algebra. If $\{a_n\}$ is a sequence of elements of B, we define: $\lim\inf a_n = \cup_{k\geq 1}\cap_{n\geq k} a_n$ and $\lim\sup a_n = \cap_{k\geq 1}\cup_{n\geq k} a_n$; and if $\lim\inf a_n = a = \lim\sup a_n$, then we say that a_n converges to a, and is written as $a_n \to a$. A sequence $\{a_n\}$ in B is a Cauchy sequence if and only if $d^*(a_n, a_m) \to 0$, where d^* is the Boolean metric on B defined by $d^*(a, b) = a'b + ab'$.

Definition 2.6.[11]. If $\{x_n\}$ is a sequence of elements of V, we say that $x_n \to x$ ($x \in V$) if and only if $\|x_n - x\| \to 0$; and a sequence $\{x_n\}$ in V is Cauchy if and only if $\|x_n - x_m\| \to 0$.

Definition 2.7.[6]. A point x in a normed Boolean vector space V is coincidence point of maps T and S if $Tx = Sx = w$ (say), $w \in V$. In this case, w is a point of coincidence of T and S.

Definition 2.8.[6]. Self maps T and S of a normed Boolean vector space V are weakly compatible if T and S commute at coincidence points.

Aamri *et al.* [1] generalized the concept of noncompatibility by defining the notion of property (*E.A.*). The following definition is the consequence of their definitions.

Definition 2.9. Self maps T and S of a normed Boolean vector space V satisfy the property (*E.A*) if there exists a sequence $\{x_n\}$ in V such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in V$.

Example 2.10.[1] Let A be a non-empty set and B the class of all subsets of A. Then the class B with three set operation +, •, ʻ (union, intersection, and complement) defines a Boolean algebra. Further, this class B with the set operation "exclusive-or addition" ⊕ (symmetric difference of sets) defines a Boolean ring. Let $V = (V, \oplus)$ be the additive abelian group of this Boolean ring. For a in B and x in V, we define $ax = a \bullet x$ (the Boolean product of a and x in B). Then V is a Boolean vector space over B. Define self maps T and S on $V = [0, \infty)$ as $Sx = k$ and $Tx = x$ for all $x \in V$ and k is any fixed element of V. Consider a sequence $\{x_n\} = \{k\}$ .Clearly $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = k$. Hence, self maps T and S satisfy the property (*E.A*).

It is well known that weak compatibility and property (*E.A*) are independent of each other. Evidently the class of maps satisfying property (*E.A*) contains the class of the well-known compatible maps as well as non-compatible maps which is the motivation to use property (E.A)) instead of compatibility or non-compatibility. Liu *et al.* [7] further improved it by common property (*E.A*) .Also on the lines of Liu *et al.* [7], one may have the following:

Definition 2.11. Pairs of self maps (A, S) and (B, T) of a normed Boolean vector space V satisfy the common property (*E.A*) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in V such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in V$.

Example 2.12. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Sx = Tx = k$ and $Bx = Ax = x$ for all $x \in V$ and k is any fixed element of V. Then with sequences $\{x_n\} = \{y_n\} = \{k\}$ in V,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = k.$$

Therefore, pairs of self maps (A, S) and (B, T) satisfy the common property (*E.A.*). One may notice that SV and TV are closed subspaces of V.

It was pointed out that the property (*E.A*) and common property (*E.A.*) allow replacing the completeness requirement of the space to more natural condition of closedness of range and relaxes the continuity of maps. Also the notion of common property (*E.A.*) relaxes containment requirement of range of one map into the range of other, needed to construct the sequence of joint iterates (also containment requirement of range of maps is not needed for a pair of maps which satisfy property (*E.A*)) besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence [4].

Recently, Imdad et al. [5] introduced the notion of CLR_{ST} property and Chauhan et al.[2] introduced the notion of $JCLR_{ST}$ property which does not require even closedness of range for the existence of coincidence and common fixed point. On the same lines, one may have the following.

Definition 2.13. The pairs (A, S) and (B, T) on a normed Boolean vector space V satisfy CLR_{ST} property (common limit in the range with respect to maps S and T) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in V such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$$

where $z \in SV \cap TV$.

Example 2.14. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Ax = Bx = k$ and $Sx = Tx = x$ for all $x \in V$ and k is any fixed element of V. Then with sequences $\{x_n\} = \{y_n\} = \{k\}$ in V,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = k \in SV \cap TV.$$

So pairs of self maps (A, S) and (B, T) satisfy CLR_{ST} property. One may notice that neither SV nor TV is closed subspace of V.

Definition 2.15. Two pairs of self maps (A, S) and (B, T) on a Boolean vector space V satisfy the $JCLR_{ST}$ property (with respect to mappings S and T) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in V such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz = Tz$ where $z \in V$.

Example 2.16. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Bx = Sx = k$ and $Ax = Tx = x$ for all $x \in V$ and k is any fixed element of V. Then with sequences $\{x_n\} = \{y_n\} = \{k\}$ in V,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = k = Sk = Tk.$$

So pairs of self maps (A, S) and (B, T) satisfy $JCLR_{ST}$ property. It should be noted that neither AV nor TV is closed subspace of V.

3. Main Result

Let Φ be the set of all continuous functions $\Psi : B \to B$ satisfying $\Psi(a) < a'$ for all a in B.

Theorem 3.1. Let A, B, S and T be four self maps of a normed Boolean vector space V such that:

(i) there exists $\psi \in \Phi$ such that
$d(Ax, By) = \psi(max\{ d(Sx,Ty), d(Ax, Sx), d(By,Ty), d(Ax, Ty), d(By,Sx) \})$, for all $x, y \in V$;
(ii) $BV \subseteq SV$ or $AV \subseteq TV$;
(iii) pair (B, T) or (A,S) satisfies property $(E.A.)$;
(iv) TV or SV is a closed subspace of V.

Then pairs (B, T) and (A, S) have point of coincidence in V. If both the pairs (A, S) and (B, T) are weakly compatible on V then the maps A, B, S and T have a unique common fixed point in V.

Proof. Let the pair (B, T) satisfies property $(E.A.)$, there exists a sequence $\{x_n\}$ in V such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in V$. Since $BV \subseteq SV$, there exists a sequence $\{y_n\}$ in V such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = z$.

First we claim that $\lim_{n\to\infty} Ay_n = z$. For proving this, take $x = y_n$ and $y = x_n$ in (i), we get
$d(Ay_n,Bx_n) = \psi(max\{ d(Sy_n,Tx_n), d(Ay_n,Sy_n), d(Bx_n,Tx_n), d(Ay_n, Tx_n), d(Bx_n,Sy_n) \})$.
On making $n \to \infty$, we get

$$d(\lim_{n\to\infty} Ay_n, z) = \psi(max\{ d(z, z), d(\lim_{n\to\infty} Ay_n, z), d(z ,z), d(\lim_{n\to\infty} Ay_n, z), d(z, z) \})$$
$$= \psi(d(\lim_{n\to\infty} Ay_n, z))$$
$$< d(\lim_{n\to\infty} Ay_n, z)'.$$

This gives $\lim_{n\to\infty} Ay_n = z$.

Let TV be a closed subspace of V, we have $z = Tv$ for some $v \in V$.
Next we claim that $Bv = z$. By (i), we have
$d(Ay_n, Bv) = \psi(max\{ d(Sy_n, Tv), d(Ay_n, Sy_n), d(Bv, Tv), d(Ay_n, Tv), d(Bv, Sy_n) \})$.
On making $n \to \infty$, we get
$d(z, Bv) = \psi(max\{ d(z, z), d(z, z), d(Bv, z), d(z, z), d(Bv, z) \})$

$= \psi(d(Bv, z))$
$< d(Bv, z)$'.
This gives, $Bv = z = Tv$, i.e., B and T have a point of coincidence.
Since $BV \subseteq SV$, there exists a $u \in V$ such that $z = Su$. Lastly we claim that $Au = z$.
By (i), we have
$d(Au, z) = d(Au, Bv)$
$= \psi(\max\{ d(Su, Tv), d(Au, Su), d(Bv, Tv), d(Au, Tv), d(Bv, Su)\})$
$= \psi(\max\{ d(z, z), d(Au, z), d(z, z), d(Au, z), d(z, z)\})$
$= \psi(d(Au, z))$
$< d(Au, z)$'.
This gives $Au = z$. Hence $Su = Au = z$, i.e., A and S have a point of coincidence.
Similarly, if the pair (A, S) satisfies property (*E.A.*), $AV \subseteq TV$ and SV is a closed subspace of V then also pair (A, S) and (B, T) has a point of coincidence in V.
Since the pair (A, S) is weakly compatible, $ASu = SAu$, i.e. $Az = Sz$. Also, since the pair (B, T) is weakly compatible, $BTv = TBv$. i.e. $Bz = Tz$.
Next we claim that $Az = Bz$.
By (i), we have
$d(Az, Bz) = \psi(max\{d(Az, Bz), d(Az, Az), d(Bz, Bz), d(Az, Bz), d(Bz, Az) \})$
$= \psi(d(Az, Bz))$
$< d(Az, Bz)$'.
Therefore, $Az = Sz = Bz = Tz$.
Lastly, we claim that $Az = z$. For proving this, again by (i), we have,
$d(Az, Bv)= \psi(\max\{ d(Sz, Tv), d(Az, Sz), d(Bv, Tv), d(z, Tv), d(Bv, Sz) \})$
$d(Az, z) = \psi(max\{ d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(z, z) \})$
$= \psi(d(Az, z))$
$< d(Au, z)$',
i.e. $Az = z$.
Therefore z is common fixed point of A, B, S and T.
The uniqueness of fixed point z easily follows from inequality (i).

We now furnish an example in support of our result.
Example 3.2. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Bx = Tx = k$ and $Sx = Ax = x$ for all $x \in V$ and k is any fixed element of V. Then, with the sequence $\{x_n\} = \{k\}$ in V, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = k$. So pair (A, S) satisfies the property (*E.A.*). Clearly, $BV \subseteq SV$, $AV \not\subset TV$ and TV is a closed subspace of V. In addition, the self maps A, B, S and T satisfy the inequality (i) and the pairs of self maps (A, S) and (B, T) are weakly compatible. Thus, all the hypotheses of Theorem 3.1 are satisfied and k is a unique common fixed point of A, B, S and T.

Now we attempt to drop containment requirement of range of maps in the above theorem using common property (*E.A.*).
Theorem 3.3. Let A, B, S and T be four self maps of a normed Boolean vector space V satisfying the inequality (i) of Theorem 3.1 such that

(v) the pairs (A, S) and (B,T) satisfy a common property (*E.A.*);
(vi) SV and TV are closed subspaces of V.

Then pairs (B, T) and (A, S) have a point of coincidence in V. If both the pairs (A, S) and (B, T) are weakly compatible on V then the maps A, B, S and T have a unique common fixed point in V.
Proof. As the pairs (A, S) and (B, T) satisfy a common property (*E.A.*), then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in V such that $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in V$.
Since SV and TV are closed subspaces of V. Then $z = Su = Tv$ for some $u, v \in V$.
We first claim that $Bv = z$. So using inequality (i),
$d(Ay_n, Bv)= \psi(max\{ d(Sy_n, Tv), d(Ay_n, Sy_n), d(Bv, Tv), d(Ay_n, Tv), d(Bv, Sy_n) \})$.
On making $n \to \infty$, we get
$d(z, Bv)= \psi(max\{ d(z, z), d(z, z), d(Bv, z), d(z, z), d(Bv, z) \})$
$= \psi(d(Bv, z))$
$< d(Bv, z)$'
which gives $z = Bv = Tv$, i.e. B and T have a point of coincidence.

Next we claim that $Au = z$. By (i), we have
$d(Au, z) = d(Au, Bv)$
$= \psi(max\{ d(Su,Tv), d(Au, Su), d(Bv, Tv), d(Au, Tv), d(Bv, Su) \})$
$= \psi(max\{ d(z, z), d(Au, z), d(z, z), d(Au, z), d(z, z) \})$
$= \psi(d(Au, z))$
$< d(Au, z)$'.
This gives $Au = z$. Hence $Su = Au = z$, i.e., A and S have a point of coincidence.
The rest of the proof is same as Theorem 3.1.
Now we furnish example in support of our result.

Example 3.4. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Sx = Tx = k$ and $Bx = Ax = x$ for all $x \in V$ and k is any fixed element of V. Then, with sequences $\{x_n\} = \{y_n\} = \{k\}$ in V,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = k.$$

So, pairs of self maps (A, S) and (B, T) satisfy the common property (*E.A.*). Clearly, SV and TV are closed subspaces of V. In addition, the self maps A, B, S and T satisfy the inequality (i) and the pairs (A, S) and (B, T) are weakly compatible. Thus, all the hypotheses of Theorem 3.3 are satisfied and k is a unique common fixed point of A, B, S and T. Moreover $AV \not\subset TV$ and $BV \not\subset SV$.

It is well known that property (*E.A.*) and common property (*E.A.*) always require closedness of subspace for the existence of common fixed point. We now attempt to drop closedness of subspace using CLR_{ST} property.

Theorem 3.5. Let A, B, S and T be four self maps of a normed Boolean vector space V satisfying the inequality (i) of Theorem 3.1 such that the pairs (A, S) and (B,T) satisfy CLR_{ST} property (common limit in the range with respect to S and T). Then pairs (B, T) and (A, S) have a point of coincidence in V. If both the pairs (A, S) and (B, T) are weakly compatible on V then the maps A, B, S and T have a unique common fixed point in V.

Proof. As the pairs (A, S) and (B, T) satisfy CLR_{ST} property, then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in V such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in SV \cap TV$.
Since $z \in TV$, there exist a point $u \in V$ such that $Tu = z$.
First we assert that $Bu = Tu$. So
$d(Ay_n, Bu) = \psi(max\{ d(Sy_n, Tu), d(Ay_n, Sy_n), d(Bu, Tu), d(Ay_n, Tu), d(Bu, Sy_n) \})$.
On making $n \to \infty$, we get
$d(z, Bu) = \psi(max\{ d(z, Tu), d(z, z), d(Bu, Tu), d(z, z), d(Bu, z) \})$,
i.e., $d(Tu, Bu) = \psi(d(Bu, Tu)) < d(Bu, Tu)$'
which gives $Bu = Tu$, i.e. B and T have a point of coincidence.
Since $z \in SV$, there exist a point $v \in V$ such that $Sv = z$. Similarly, we can prove that $Av = Sv$, i.e., A and S have a coincidence. The rest of the proof is same as Theorem 3.1

Example 3.6. Let $V = (V, \oplus)$ be the additive abelian group as defined in Example 2.10. Define self maps A, B, S and T on V as $Ax = Bx = x$ and $Sx = Tx = k$ for all $x \in V$ and k is any fixed element of V. Then with sequences $\{x_n\} = \{y_n\} = \{k\}$ in V,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = k \in SV \cap TV.$$

So the pairs (A, S) and (B, T) satisfy the common limit in the range (CLR_{ST}) property. In addition, the self maps A, B, S and T satisfy the inequality (i) and the pairs of self maps (A, S) and (B, T) are weakly compatible. Thus, all the hypotheses of Theorem 3.4 are satisfied and k is a unique common fixed point of A, B, S and T. It should be noted that $AV \not\subset TV$ and $BV \not\subset SV$. Also, neither SV nor TV is closed subspace of V.
On taking $A = B$ and $S = T$ in Theorem 3.1 we get the following interesting result which is improved version of Theorem 2.2 of Mishra et al.[8] without containment, completeness of range subspace and the axiom $SSu = Su$.

Corollary 3.7. Let A and S be two self maps of a normed Boolean vector space V such that

(vii) there exists $\psi \in \Phi$ such that for all $x, y \in V$,
$d(Ax, Ay) = \psi(max\{ d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx) \})$,

(viii) pair (A,S) satisfies property (*E.A.*);

(ix) *SV* is a closed subspace of *V;*

Then pair (*A, S*) has coincidence in *V*. If the pair (*A, S*) is weakly compatible in *V*, maps *A* and *S* have a unique common fixed point in *V*.

Remark 3.8. Result similar to Theorem 3.5 may also be obtained by replacing *CLRST* property by *JCLRST* property.

Remark 3.9. Theorem 3.1 is also true if we replace (i) by the conditions:

(x) there exists $\psi \in \Phi$ such that for all $x, y \in V$,

$d(Ax, By) = \psi(max\{ d(Sx,Ty), d(Ax, Sx), d(By,Ty), [d(Ax, Ty) + d(By,Sx)]/2 \})$.

(xi) there exists $\psi \in \Phi$ such that

$d(Ax, By) = \psi(max\{ d(Sx,Ty), d(Ax, Sx), d(By,Ty), d(Ax, Ty) \})$, for all $x, y \in V$.

(xii) there exists $\psi \in \Phi$ such that

$d(Ax, By) = \psi(max\{ d(Sx,Ty) + d(Ax, Sx) + d(By,Ty) + d(Ax, Ty) \})$, for all $x, y \in V$.

Remark 3.10. From the above results, it is asserted that for the existence of common fixed point of two pairs of self maps in normed Boolean vector space satisfying CLR_{ST} property and $JCLR_{ST}$ property the following conditions are never required:

(a) the containment of ranges amongst the involved maps;

(b) the completeness of the whole space/subspace;

(c) the closedness of space/subspaces;

(d) continuity requirement amongst the involved maps.

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