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Centralizing Higher Left Centralizers on Prime Gamma Rings

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Abstract
In this paper we study the commutativity of prime Γ- rings satisfying certain identities involving higher left centralizer on it .
Keywords: prime Γ- rings , higher left centralizer .

1.Introduction

Let M and Γ be two additive abelian groups . then M is called Γ- ring if there exist a mapping $M \times \Gamma \times M \to M$ which satisfies the following conditions

i) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$

ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

holds for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, in the sense of Barnes [2]

throughout this paper M is denote to a Γ- ring and it is center will denoted by $Z(M)$ which equal to the set of all element $x \in M$ such that $x\alpha y = y\alpha x$ for all $y \in M$.

now for any $x, y \in M$ and $\alpha \in \Gamma$, the symbols $[x,y]_\alpha$ and $\langle x,y\rangle_\alpha$ will denoted to $x\alpha y - y\alpha x$ and $x\alpha y + y\alpha x$ respectively which are called commutator and anti- commutator respectively .[4] .

A Γ- ring M is called commutative if $[x,y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$ [1].

Throughout this paper we consider the Γ- ring M satisfy the following condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ which will represented by $(*)$

The above commutator and anti- commutator satisfies the following[4]:

1) $[x\beta y, z]_\alpha = [x,z]_\alpha \beta y + x\beta[y,z]_\alpha$
2) $[x, y\beta z]_\alpha = y\beta[x,z]_\alpha + [x,y]_\alpha \beta z$
3) $\langle x, y\beta z\rangle_\alpha = \langle x,y\rangle_\alpha \beta z - y\beta[x,z]_\alpha = y\beta\langle x,y\rangle_\alpha + [x,y]_\alpha \beta z$
4) $\langle x\beta y, z\rangle_\alpha = x\beta\langle y,z\rangle_\alpha - [x,z]_\alpha \beta y = \langle x,z\rangle_\alpha \beta y + x\beta[y,z]_\alpha$

A Γ- ring M is called prime if $x\Gamma M\Gamma y = \{0\}$ implies that $x = 0$ or $y = 0$ and it is called semi-prime if $x\Gamma M\Gamma x = \{0\}$ implies that $x = 0$.

An additive mapping $F: M \to M$ is called centralizing on asubset S of Γ- ring M if $[F(x), x]_\alpha \in Z(M)$ and it is called commuting if $[F(x), x]_\alpha = 0$ for all $x \in S$.[6].

An additive mapping $T: M \to M$ is called left(right) centralizer on a Γ- ring M if

$T(x\alpha y) = T(x)\alpha y$ holds for all $x, y \in M$ and $\alpha \in \Gamma$[5] .

Many authors covers the concept of left centralizer and study the relation between the commutativity of ring and left centralizers .

In [3] K.K.Dey and A.C. Paul proved that a mapping $T: M \to M$ is centralizer if and only if it is centralizing left centralizer . they also showed that if T_1 *and* T_2 are two left centralizers on a prime Γ- ring M such that $T_1(x)\alpha x + x\alpha T_2(x) \in Z(M)$ for all $x \in M$ and $\alpha \in \Gamma$ .then both T_1 and T_2 are centralizers on M .

K.K.Dey and A.C. Paul in [4] study the commutativety of Γ- ring in which satisfying certain identities involving left centralizers .

In this paper , we obtain the commutativity of Γ- ring satisfying certain identities involving higher left centralizers on Γ- ring . This work motivated from the work of K.K.Dey and A.C.Paul [4] .

We introduce a new definition of higher left centralizer on a Γ- ring M as the following

Definition 1.1 :let M be a Γ- ring and let $T = (T_i)_{i \in N}$ be a family of left centralizers on M . Then $T_n: M \to M$ is called higher left centralizer on M if

$T_n(x\alpha y) = \sum_{i=1}^{n} T_i(x)\alpha y$
holds for all $x, y \in M$ and $\alpha \in \Gamma$.

2. Commutativity of Prime Gamma Rings

In this section we study when the gamma rings is commutative by using the higher left centralizers on prime gamma rings.

Theorem 2.1 : let M be a prime Γ- ring and I be anon –zero ideal of M. suppose that M admits a family of non – zero higher left centralizers T = (T_i $)_{i \in n}$ such that $\sum_{i=1}^{n} T_i\ (x) \neq x$ for all $x \in I$ and $i \in N$. if $T_n([x,y]_\alpha - [x,y]_\alpha = 0$ for all x,y $\in I$ and α∈ Γ then M is commutative .

Proof : Given that T = (T_i $)_{i\in N}$ afamily of left centralizens of M such that.

$T_n([x,y]_\alpha - [x,y]_\alpha = 0$ …….. (1)

for all $x, y \in I, \alpha \in \Gamma$

Then $T_n(\ x\alpha y - y\alpha x\ ) - (x\alpha y - y\alpha x) = 0$

So that

$(\ \sum_{i=1}^{n} T_i\ (x)\alpha y - \sum_{i=1}^{n} T_i\ (y)\alpha x)$ - $(x\alpha y - y\alpha x\ ) = 0$

Which leads to

$\sum_{i=i}^{n}(\ T_i\ (\ x) - \ x\ )\ \alpha y - \sum_{i=1}^{n} T_i\ \ (y) - y\ )\alpha x = 0$ …………….(2)

Replace x by $x\beta r$ in (2) we get

$\sum_{i=i}^{n}(\ T_i\ (\ x\beta r) - \ x\beta r\ )\ \alpha y - \sum_{i=1}^{n} T_i\ \ (y) - y\ )\alpha x\beta r = 0$

Hence

$\sum_{i=i}^{n}(\ T_i\ (\ x) - \ x\ )\beta r\alpha y - \sum_{i=1}^{n} T_i\ \ (y) - y\ )\alpha x\beta r = 0$ …………(3)

For all $x, y\ \in I\ , r\ \in M$ and $\alpha, \beta\ \in \Gamma$

Using the condition $(*)$ we get

$\sum_{i=1}^{n}(\ T_i\ (\ x) - \ x\ )\beta r\alpha y - \sum_{i=1}^{n} T_i\ \ (y) - y\ )\beta x\alpha r = 0$ ……..(4)

For all $x, y\ \in I\ , r\ \in M$ and $\alpha, \beta\ \in \Gamma$

Using (2) in (4) to simplify , we obtain

$\sum_{i=1}^{n}(\ T_i\ (\ x) - \ x\ )\ \beta\ [\ r, y]_\alpha\ = 0$ …………….(5)

For all $x, y\ \in I\ , r\ \in M$ and $, \beta\ \in \Gamma$.

Again replacing r by $r\lambda s$ in (5) and using $(*)$ we get

$\sum_{i=1}^{n}(\ T_i\ (\ x) - \ x\ )\ \beta r\lambda\ [s, y]_\alpha\ = 0$

For all $, y\ \in I$, $r, s\ \in M$ and $\alpha, \beta \in \Gamma$

$\sum_{i=1}^{n}(\ T_i\ (\ x) - \ x\ )\ \Gamma M \Gamma\ \ [s, y\ ]_\alpha\ \ = 0$

for all $x, y\ \in I$,$s \in M$, $\alpha\ \in \Gamma$

by primness' of m and since $\sum_{i=i}^{n}(\ T_i\ (x) - \ x\ ) \neq 0$

hence [$s\ , y\ ]_\alpha = 0$ for all $y \in I$,$s\ \in M$,$\alpha \in \Gamma$

there fore $I\ \subset\ Z(M)$ and hence M is commutative . ■

Corollary 2.2 : In theorem 2.1 , if the family T of higher left centralizers is zero then M is commutative

proof : suppose that T_n $(\ [x\ , y\ ])$-$[\ x\ , y\ ]_\alpha$ = 0 for any $x\ , y\ \in I\ , \alpha \in \Gamma$

if T_n = 0 then $[\ x\ , y\ ]_\alpha\ = 0$ for all $x\ , y\ \in I$ and$\alpha\ \in \Gamma$

There fore I is commutative hence M is commutative . ■

Theorem 2.3 : let M be a prime Γ- ring and I be a non – zero ideal of M suppose that M admits a family T of non – zero higher left centralizer T = (T_i $)_{i \in n}$ such that $\sum_{i=1}^{n} T_i(x)\ \neq - x$ for all $x\ \in I$ and $i\ \in N$, for ther if $T_n\ ([x,y]_\alpha) +\ [\ x\ , y\ ]_\alpha = 0$ for all $x, y\ \in I$, aneh $\alpha\ \in\ \Gamma$ then M is commutative .

Proof : Given that T = (T_i $)_{i\in N}$ is a family of higher left centralizers of M such that .

$T_n\ (\ [\ x\ , y\ ]_\alpha\ ) +\ [\ x, y\ ]_\alpha$ = 0 for all $x\ , y\ \in I\ , \alpha \in \Gamma$ ………(1)

Then

$T_n\ (\ x\ \alpha\ y - y\ \alpha\ x\ ) +\ (\ x\alpha y - y\ \alpha\ x) = 0$

So that

$$(\sum_{i=1}^{n} T_i\ (x)\alpha y - \sum_{i=1}^{n} T_i\ (y)\alpha x) + (x\alpha y - y\alpha x\ ) = 0$$

Which leads to

$$\sum_{i=1}^{n}(T_i (x) + x) \alpha y - \sum_{i=1}^{n} T_i\ (y) + y)\alpha x = 0 \quad \text{.........(.2)}$$

In (2) replace x by $x\beta r$ to get

$$\sum_{i=1}^{n}(T_i (x\beta r) + x\beta r) \alpha y - \sum_{i=1}^{n} T_i\ (y) + y)\alpha x\beta r = 0$$

Hence

$$(\sum_{i=1}^{n}(T_i (x) + x)\beta r\alpha y - (\sum_{i=1}^{n} T_i\ (y) + y)\alpha x\beta r = 0 \quad \text{......(.3)}$$

For all $x , y \in I , r \in M$ and $\alpha , \beta \in \Gamma$

Using the condition $(*)$ we get

$$(\sum_{i=1}^{n}(T_i (x) + x)\beta r\alpha y - (\sum_{i=1}^{n} T_i\ (y) + y)\beta x\alpha r = 0 \quad \text{.........(4)}$$

For all $x , y \in I , r \in M$ and $\alpha , \beta \in \Gamma$

Using (2) in (4) to simplify , we obtain

$$(\sum_{i=1}^{n} T_i(x) + x)\ \beta\ [r , y]_\alpha = 0 \quad \text{.........(5)}$$

For all $x , y \in I , r \in M$ and $\alpha , \beta \in \Gamma$

Replace r by $r\lambda s$ in (5) and using $(*)$, to get

$$(\sum_{i=1}^{n} T_i(x) + x)\ \beta r\lambda\ [s , y]_\alpha = 0$$

For all $x , y \in I , r , s \in M$ and $\alpha , \beta, \lambda \in \Gamma$

in other words

$$(\sum_{i=1}^{n} T_i(x) + x)\ \Gamma M\Gamma\ [s , y]_\alpha = o$$

For all $x , y \in I , s \in M$ and $\alpha \in \Gamma$

By primness of M and since $\sum_{i=i}^{n} (t_i\ (x) + x) \neq 0$

we get $[s , y]_\alpha = 0$ for all $y \in I , s \in M , \alpha \in \Gamma$

Therefore $I \subset Z(M)$ and hence M is commutative . ■

Theorem 2.4 : - let M be aprime Γ- ring and I be anon – zero ideal of M . suppose that M adimits afamily of non – zero higher left centralizers . $T = (T_i)_{i\in n}$ such that $\sum_{i=1}^{n} T_i\ (x) \neq x$ for all $x \in I$ and $i \in N$, further if

$$T_n\ (< x , y >_\alpha) = \langle x , y\rangle_\alpha$$

For all $x, y \in I , \alpha \in \Gamma$ then M is commutative .

Proof : - Given that

$$T_n\ (< x , y >_\alpha) - < x , y >_\alpha = 0 \quad \text{...........(1)}$$

For all $x , y \in I , \alpha \in \Gamma$

This implies that

$$\sum_{i=1}^{n}(T_i (x) - x) \alpha y + \sum_{i=1}^{n}(T_i\ (y) - y)\alpha x = 0 \quad \text{.......(.2)}$$

Replace $x$by $x\beta r$ in (2) we obtain .

$$\sum_{i=1}^{n}(T_i (x\beta r) - x\beta r) \alpha y + \sum_{i=1}^{n}(T_i\ (y) - y)\alpha x\beta r = 0$$

Hence

$$\sum_{i=1}^{n}(T_i (x) - x)\beta r\alpha y + \sum_{i=1}^{n}(T_i\ (y) - y)\alpha x\beta r = 0 \quad \text{..........(3)}$$

For all $x, y \in I, r \in M and\ \ \alpha , \beta \in \Gamma$

Using (2) and condition $(*)$ we get

$$\sum_{i=1}^{n}(T_i (x) - x)\beta r\alpha y + \sum_{i=1}^{n}(T_i\ (x) - x)\beta y\alpha r = 0 \quad \text{...............(4)}$$

For all $x, y \in I, r \in M and\ \ \alpha , \beta \in \Gamma$

That is

$$\sum_{i=1}^{n}(T_i (x) - x)\beta[r , y]_\alpha = 0 \quad \text{............(5)}$$

For all $x, y \in I, r \in M and\ \ \alpha , \beta \in \Gamma$

Replace r by $r\lambda s$ in (5) and by using condition$(*)$ so from (5) we get .

$$\sum_{i=1}^{n}(T_i (x) - x)\beta r\lambda[s , y]_\alpha = 0 \quad \text{........(6)}$$

For all $x, y \in I, r, s \in M and\ \ \alpha , \beta\ \lambda \in \Gamma$

i.e : $\sum_{i=1}^{n}(T_i (x) - x)\Gamma M\Gamma[s , y]_\alpha = 0$

By primness of M and since $\sum_{i=1}^{n}(T_i (x) - x) \neq 0$

Then $[s , y]_\alpha = 0$ for all $y \in I$

Hence $I \subset Z(M)$ there for M is commutative ■

Theorem 2.5 : - let M be aprime Γ- ring and I be anon – zero ideal of M . suppose that M adimits afamily of non – zero higher left centralizers . $T = (T_i)_{i \in n}$ such that $\sum_{i=1}^{n} T_i(x) \neq -x$ for all $x \in I$ and $i \in N$, further if

$$T_n(<x,y>_\alpha) + \langle x,y \rangle_\alpha = 0$$

For all $x, y \in I, \alpha \in \Gamma$ then M is commutative

Proof : - Given that $T = (T_i)_{i \in n}$ be a family of non – zero higher left centralizers of M such that

$T_n(<x,y>_\alpha) + <x,y>_\alpha = 0$ ……..(1)

For all $x, y \in I, \alpha \in \Gamma$

Then

$$\sum_{i=1}^{n}(T_i(x\alpha y + y\alpha x)) + (x\alpha y + y\alpha x) = 0$$

Hence $\sum_{i=1}^{n} T_i(x)\alpha y + \sum_{i=1}^{n} T_i(y)\alpha x + (x\alpha y + y\alpha x) = 0$

$\sum_{i=1}^{n}(T_i(x) + x)\alpha y + \sum_{i=1}^{n}(T_i(y) + y)\alpha x) = 0$ …….(.2)

In the above relation eplace x by $x\beta r$ we obtain .

$$\sum_{i=1}^{n}(T_i(x\beta r) + x\beta r)\alpha y + \sum_{i=1}^{n}(T_i(y) + y)\alpha x\beta r = 0$$

So we get

$\sum_{i=1}^{n}(T_i(x) + x)\beta r\alpha y + \sum_{i=1}^{n}(T_i(y) + y)\alpha x\beta r = 0$ …….(3)

For all $x, y \in I, r \in M and \ \alpha, \beta \in \Gamma$

Substitute (2) in (3) and use the condition $(*)$ to get

$\sum_{i=1}^{n}(T_i(x) + x)\beta[r,y]_\alpha = 0$ ……(4)

For all $x, y \in I, r \in M and \ \alpha, \beta \in \Gamma$

Now again replace r by $r\lambda s$ in (4) we have

$\sum_{i=1}^{n}(T_i(x) + x)\beta r\lambda[s,y]_\alpha = 0$ ……..(5)

For all $x, y \in I, r, s \in M and \ \alpha, \beta \ \lambda \in \Gamma$

i.e : $\sum_{i=1}^{n}(T_i(x) + x)\Gamma M\Gamma[s,y]_\alpha = 0$

By primness of M and since $\sum_{i=1}^{n}(T_i(x) + x) \neq 0$

We have $[s,y]_\alpha = 0$ for all $y \in I, s \in M \ and \ \alpha \in \Gamma$

Hence $I \subset Z(M)$ there for M is commutative ■

Corollary 2 . 6 : in theorem 2.4 and 2.5 if a higher left centralizers T_n is zero . then M is commutative .

Proof : for any $x, y \in I$, we have

$$T_n(<x,y>_\alpha = <x,y>_\alpha$$

if $T_n = 0$ then $<x,y>_\alpha = 0$ for all $x, y \in I \ and \ \alpha \in \Gamma$

replace x by $x\beta z$ and using the fact

$y\alpha x = -x\alpha y$ we conclude that

$x\beta[z,y]_\alpha = \{0\}$ for all $x, y, z \in I, and \ \alpha, \beta \in \Gamma$

In other words we have

$I\Gamma M\Gamma[z,y]_\alpha = 0$ for all $y, z \in I \ and \ \alpha \in \Gamma$

Since M is prime and $I \neq \{0\}$

So that $[z,y]_\alpha = 0$ for all $y, z \in I \ and \ \alpha \in \Gamma$

then I is commutative and hence M is commutative . ■

Theorem 2.7 :- let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T = (T_i)_{i \in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, further if $T_n(x\alpha y) \mp (x\alpha y) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$ then M is commutative .

proof :- for any $x, y \in I$ and $\alpha \in \Gamma$ we have $T_n(x\alpha y) = (x\alpha y)$

this implies that $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$

and hance by theorem 2.1 we have M is commutative

on the other hand if M is satisfy the condition $T_n(x\alpha y) + (x\alpha y) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

$then \ for \ any \ x, y \in I \ and \ \alpha \in \Gamma \ we \ have \ T_n(x\alpha y + y\alpha x) = -(x\alpha y + y\alpha x)$

So that $T_n(\langle x,y \rangle_\alpha) + (\langle x,y \rangle_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

Then by theorem 2.5 we have M is commutative ■

Corollary 2.8 : -let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T=(T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq \mp x$ for all $x \in I$ and for all $i \in N$, further if $T_n(x\alpha y) \mp (y\alpha x) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$ then M is commutative .

proof : for any $x, y \in I$ and $\alpha \in \Gamma$ we have $T_n(x\alpha y) \mp (y\alpha x) = 0$

now if $T_n(x\alpha y) = (y\alpha x)$ this implies that $T_n([x,y]_\alpha) - ([y,x]_\alpha) = T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$

then by theorem 2.5 we have M is commutative

now when $T_n(x\alpha y) + (y\alpha x) = 0$ then $T_n([x,y]_\alpha) + ([y,x]_\alpha) = 0$ this implies that

$T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$ and hance by theorem 2.1 we have M is commutative ■

3.The Main Results:

In this section we introduce the main results of this paper we start by the following theorem .

Theorem 3.1: :let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T=(T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, then the following conditions are equivalent :

(i) $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

(ii) $T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

(iii) for all $x, y \in I$ and $\alpha \in \Gamma$, either $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ or

$$T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$$

(iv) M is commutative

Proof :

(i)→(iv) suppose that $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$

Then by theorem 2.1 we have M is commutative

(iv)→(i) suppose that M is commutative then $[x,y]_\alpha = 0$

and hence $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$

(ii) →(iv) suppose that $T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

Then by theorem 2.3 we have M is commutative

(iv)→(ii) suppose that M is commutative then $[x,y]_\alpha = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

And hence $-[x,y]_\alpha = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

Which implies that $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

(iii) →(iv) suppose that for all $x, y \in I$ and $\alpha \in \Gamma$, either $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ or

$$T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$$

Then by theorem2.1 or theorem 2.3 we have M is commutative

(iv)→(iii) suppose that M is commutative

For each fixed $y \in I$ and $\alpha \in \Gamma$ we set

$$I_1 = \{x \in I \mid T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0\}$$
$$I_2 = \{x \in I \mid T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0\}$$

Then I_1 and I_2 are additive subgroups of I such that $I = I_1 \cup I_2$.

But a group cannot be the set theoretic union of two proper subgroups , hance we have either $I_1 = I$ or $I_2 = I$.

Further , using a similar argument , we obtain

$I = \{y \in I \mid I_1 = I\}$ or $I = \{y \in I \mid I_2 = I\}$

Thus we obtain that either $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

or $T_n([x,y]_\alpha) + ([x,y]_\alpha) = 0$ for all $x, y \in I$ and $\alpha \in \Gamma$

Hence M is commutative in both cases by theorem 2.1 (respectively theorem 2.3) ■

Theorem 3.2 : :let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T=(T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, further if $T_n(x\alpha y) - (x\alpha y) \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$ then M is commutative .

Proof : for any $x, y \in I$ and $\alpha \in \Gamma$ we have

$T_n(x\alpha y) - (x\alpha y) \in Z(M)$ ……………….(1)

This can be written as $\sum_{i=1}^{n} T_i(x)\alpha y - x\alpha y \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$ ……….(2)

That is $[(\sum_{i=1}^{n} T_i(x) - x)\alpha y, r] = 0$ for all $x, y \in I, r \in M$ and $\alpha, \beta \in \Gamma$ ………. (3)

Which implies that

$(\sum_{i=1}^{n} T_i(x) - x)\alpha[y,r]_\beta + [\sum_{i=1}^{n} T_i(x) - x, r]_\beta \alpha y = 0$ ………..(4)

for all $x, y \in I, r \in M$ and $\alpha, \beta \in \Gamma$

in (4) replace x by $x\delta z$, we have

$(\sum_{i=1}^{n} T_i(x) - x)\delta z\alpha[y,r]_\beta + [(\sum_{i=1}^{n} T_i(x) - x)\delta z, r]_\beta \alpha y = 0$ ……….(5)

for all $x, y, z \in I, r \in M$ and $\alpha, \beta, \delta \in \Gamma$

from (3) we get that (5) becomes

$(\sum_{i=1}^{n} T_i(x) - x)\delta z\alpha[y,r]_\beta = 0$ for all $x, y, z \in I, r \in M$ and $\alpha, \beta, \delta \in \Gamma$

This yields that

$(\sum_{i=1}^{n} T_i(x) - x)\Gamma M\Gamma I\Gamma[y,r]_\beta = \{0\}$ for all $x, y \in I, r \in M$ and $\beta, \in \Gamma$

The primness of M implies that

$I\Gamma[y,r]_\beta = \{0\}$ or $\sum_{i=1}^{n} T_i(x) - x = 0$

and since $I \neq \{0\}$ and $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$

we get that I is central and hence M is commutative ■

Theorem 3.3: let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T = (T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq -x$ for all $x \in I$ and for all $i \in N$, further if $T_n(x\alpha y) - (x\alpha y) \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$ then M is commutative .

proof : suppose that $T = (T_i)_{i\in N}$ be a family of non –zero higher left centralizers satisfying the property $T_n(x\alpha y) - (x\alpha y) \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$

then the non- zero higher left centralizers $(-T)$ satisfies the condition

$(-T_n)(x\alpha y) - (x\alpha y) \in Z(M)$ for all $x, y \in I$ and $\alpha \in \Gamma$

Hence by theorem 3.2 we have M is commutative. ■

Remark In theorem 3.3 if the h, βigher left centralizer is zero , then M is commutative .

Theorem 3.4 : let M be a prime Γ- ring and I be anon zero ideal of M . suppose that M admits a family of non –zero higher left centralizers $T = (T_i)_{i\in N}$ such that $\sum_{i=1}^{n} T_i(x) \neq x$ for all $x \in I$ and for all $i \in N$, further if $T_n(x\alpha y) - (y\alpha x) \in Z(M)$ *for all* $x, y \in I$ *and* $\alpha \in \Gamma$ then M is commutative .

Proof : we are given that a higher left centralizer of M such that

$T_n(x\alpha y) - (y\alpha x) \in Z(M)$

for all $x, y \in I$ and $\alpha \in \Gamma$

this implies that $[T_n(x\alpha y) - (y\alpha x), r]_\beta = 0$ ………………….(1)

holds for all $x, y \in I, r \in M$ and $\alpha, \beta \in \Gamma$

which implies that

$[\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x, r]_\beta = 0$ …………………..(2)

for all $x, y \in I, r \in M$ and $\alpha, \beta \in \Gamma$

replacing y by $y\delta x$ in the above relation and use it hence

$[\sum_{i=1}^{n} T_i(x)\alpha y\delta x - y\alpha x\delta x, r]_\beta = 0$ ……………………(3)

for all $x, y \in I, r \in M$ and $\alpha, \beta, \delta \in \Gamma$

we find that

$(\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x)\delta[x,r]_\beta = 0$ …………………….(4)

for all $x, y \in I, r \in M$ and $\alpha, \beta, \delta \in \Gamma$

again replace r by $r\mu s$ in (4) to get

$(\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x)\delta r\mu[x,s]_\beta + (\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x)\delta[x,r]_\beta \mu s = 0$ ……..(5)

for all $x, y \in I, r, s \in M$ and $\alpha, \beta, \delta, \mu \in \Gamma$

From (4) the relation (5) becomes

$(\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x)\delta r\mu[x,s]_\beta = 0$ ……………….(6)

for all $x, y \in I, r, s \in M$ and $\alpha, \beta, \delta, \mu \in \Gamma$

i.e : $(\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x)\Gamma M \Gamma [x,s]_\beta = 0$

for all $x, y \in I, s \in M$ and $\alpha, \beta \in \Gamma$

the primness of M implies that either $[x,s]_\beta = 0$ or $\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x = 0$

for all $x, y \in I, s \in M$ and $\alpha, \beta \in \Gamma$

now put $I_1 = \{x \in I | [x,s]_\beta = 0 \text{ for all } s \in M \text{ and } \beta \in \Gamma\}$

$I_2 = \{x \in I | \sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x = 0 \text{ for all } x, y \in I \text{ and } \alpha \in \Gamma\}$

Then clearly that I_1 and I_2 are additive subgroups of . moreover by the discussion given I is the set-theoretic union of I_1 and I_2 but can not be the set- theoretic of two proper subgroups .

Hence $I_1 = I$ or $I_2 = I$.

If $I_1 = I$, then $[x,s]_\beta = 0$ for all $x \in I, s \in M$ and $\beta \in \Gamma$ and hence M is commutative .

On the other hand if $I_2 = I$ then $\sum_{i=1}^{n} T_i(x)\alpha y = y\alpha x$ for all for all $x, y \in I$ and $\alpha \in \Gamma$.

That is $\sum_{i=1}^{n} T_i(x)\alpha y - y\alpha x = 0$ for all for all $x, y \in I$ and $\alpha \in \Gamma$.

This implies that $T_n([x,y]_\alpha) - ([x,y]_\alpha) = 0$ for all for all $x, y \in I$ and $\alpha \in \Gamma$

Hence apply theorem 2.1 yields the required result . ■

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