

On the Comparative Study of Some Mathematical Tools for Specific Sequences Design

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Abstract

In modern communication system, cryptography and automatic test patterns, some specific sequences with strictly defined properties are required to meet the application demands. [1,2,3,4,...] These requirements are:

- Good pseudorandom (PN) properties (large period length, uniform distribution...).
- Low periodic correlation property.
- Low aperiodic correlation property.
- Large linear complexity.
- Large cardinality (number of sequences in the set).

Unfortunately, there is not any set of sequence satisfying all these requirements, despite the fact that a lot of efforts have been given for design such sequences. For this purpose, different mathematical tools have been widely used such as: matrix, d-transform and trace function representations. However, to the best of our knowledge, there has been no any comparative study on these mathematical tools carried out so far!

In this contribution, we try to fill some parts of this gap by considering some typical applications of these tools. The paper is constructed as below:

- In the introduction, the basic concepts and definitions of matrix, d-transform and trace are given (briefly).
- In section II, some typical applications (for demonstration) will be shown. In this regard, we will give some discussions and suggestions for choosing the appropriate mathematical tools for each application.
- Especially, in section III, we will show the relationship (interchanging or equivalencies) between them.

Keywords: specific sequences, mathematical tool, matrix, d-transform and trace.

1. Introduction

1.1 Matrix and State-machine representation of LFSR

In this section, two typical configurations of linear feedback shift register for the same output i.e., the Galois and Fibonacci, are considered and showed as Fig 1.1a and Fig 1.1b, which can be used to generate the m-sequence for mobile communication and cryptography application [5,6]. The state at the time instant n is the contents of the shift register represented by a column vector S_n :

$$S_n = [S_{0,n} S_{1,n} \dots S_{m-2,n} S_{m-1,n}]^T \quad (1)$$

Let consider the Fig 1.1a first. Given the contents of the LFSR at the time instant n, then the content of the shift register at instant (n+1) can be obtained:

$$\begin{aligned} S_{0,n+1} &= S_{1,n} + g_1 S_{0,n} \\ S_{1,n+1} &= S_{2,n} + g_2 S_{0,n} \\ &\vdots \\ S_{m-2,n+1} &= S_{m-1,n} + g_{m-1} S_{0,n} \\ S_{m-1,n+1} &= g_r S_{0,n} \end{aligned} \quad (2)$$

Where addition and multiplication are mod2 and $g_i \in \{0,1\}$ for $i = \{1,2,\dots,m\}$. These equations can be written in matrix form as:

$$S_{n+1} = \begin{bmatrix} S_{0,n+1} \\ S_{1,n+1} \\ S_{2,n+1} \\ \vdots \\ S_{m-1,n+1} \end{bmatrix} = \begin{bmatrix} g_1 & 1 & 0 & \dots & 0 \\ g_2 & 0 & 1 & \dots & 0 \\ g_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_m & 0 & 0 & \dots & 0 \end{bmatrix} \times \begin{bmatrix} S_{0,n} \\ S_{1,n} \\ S_{2,n} \\ \vdots \\ S_{m-1,n} \end{bmatrix} \quad (3)$$

which can be written for short:

$$S_i = G^i S_0 \text{ for } i = \{1,2,\dots,n\} \quad (4)$$

Where G is a square Galois state transition matrix of order m, S_0 is the vector of initial state.

The output b_n of the shift register at the given instant n (the content of the right most shift register) can be determined as:

$$b_n = [1 \ 0 \ 0 \ \dots \ 0] \times G^n \times S_0 \quad (5)$$

Which presents the relationship between output sequence and the state transition matrix and the initial state S_0 of the LFSR.

For the shift register configuration in Fig 1.1b, a similar result may be obtained in the same way as:

$$S_{n+1} = F \times S_n \quad (6)$$

Where:

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ g_m & g_{m-1} & g_{m-2} & g_{m-3} & \dots & g_1 \end{bmatrix} \quad (7)$$

And the output b_n at the instant of time n is:

$$b_n = [g_m \ g_{m-1} \ \dots \ g_1] \times F^n \times S_0 \quad (8)$$

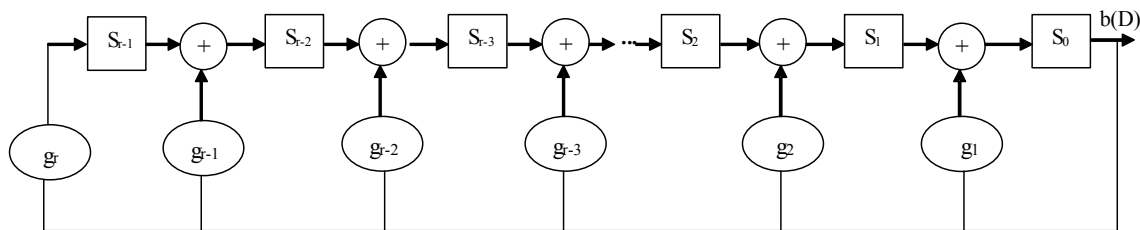


Figure 1.1a. Galois feedback generator.

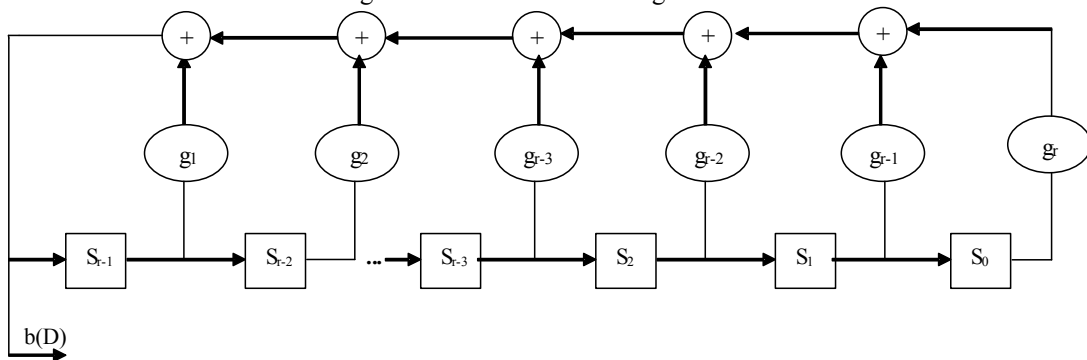


Figure 1.1b. Fibonacci feedback generator.

1.2 D-transform

The D-transform of a sequence $\{b_n\}$ over $GF(p)$ (Galois field (mod p), a primitive integer) is denoted by $D[b_n]$ [4,7,8,9]:

$$D[b_n] = \sum_{i=0}^m b_i D^i, \quad b_i \in \{GF(p)\} \quad (9)$$

Thus, the D-transform of the sequence will have the form of a polynomial in d over $GF(p)$ and has been conveniently used in signal and system analysis in data transmission and CDMA [8,9].

The D-transform of the generator sequence $\{b_n\}$ of a linear feedback shift register (LFSR) is then given by:

$$b(D) = \frac{S(D)}{g(D)} \quad (10)$$

Where $g(D)$ of degree n is the generating polynomial of a LFSR and $S(D)$ of degree $\leq n-1$ specifies the initial condition corresponding to a particular shifted version of $\{b_n\}$.

1.3 Representation the m-sequence in term of the power of primitive elements (Trace function)

Let α be the root of the primitive polynomial $g(D)$ of the degree m . The trace function of α is mapping from $GF(q^m)$ to $GF(q)$ and is defined as:

$$\text{Tr}_q^{q^m}(\alpha) = \sum_{i=0}^{m-1} \alpha^{q^i} \quad (11)$$

In other words, the trace of α over $GF(q)$ is the sum of conjugate of α with respect to $GF(q)$.

2. Application cases

2.1 Matrix representation

2.1.1 State and configure description

As clearly seen in 1.1 previous section, the matrix presentation may be best suitable to describe FSR in term of Finite State Machine (FSM). It shows the close relations in the set of four element $\{S_n, S_{n+1}, F, b_n\}$ which defines the FSM.

Observing the fig 1.1a, 1.1b one can easily get elements of the feedback tap g_n , state vector S_n and vice versa. In word: matrix presentation is no doubt the most intuitive.

2.1.2 Interleaved structure representation

It is well known that the specific sequences must have very large length. Except for few Mersenne primitive integers, the naturally possess interleaved structure which draws a lot of attentions in the literature [1,2,3,4,10,11...]. For this case, the matrix representation can offer a very intuitive approach in represent the interleaved structure [4,10,11...] but it may look complicated due to the great length of those sequences.

Decomposition of m-sequence $\{b_n\}$ and rearrange it Matrix form [3,4,12]. It is well known that if we decimate $\{b_n\}$ by T , we obtain:

$$\{a_n\} = \{b_{nT}\} = \text{Tr}_1^n(\alpha^{Tn}) = \text{Tr}_0^n(\beta^n) \quad (12)$$

With $\beta = \alpha^T$. Since α^T is primitive element in $GF(2^m)$, $\{a_n\}$ is also an m-sequence of length $N=(2^m-1)/T$. When the decimation start at the first bit, we will obtain the subsequence: $\{a_0, a_T, \dots, a_{(2^m-2)T}\}$

Similarly, we'll obtain the subsequence $\{a_{t-1}, a_{T+t-1}, \dots, a_{(2^m-2)T+t-1}\}$ when the decimation start at the t^{th} -bit. Thus, on the time-domain, these subsequence (arrange in columns) can be considered as T time-multiplexed sequences $\{a_{nT}\} \{a_{nT+1}\} \dots \{a_{(n+1)T-1}\}$ onto T time slots:

$$\underline{\underline{M}} = \begin{bmatrix} a_0 & a_1 & \dots & a_{T-1} \\ a_T & a_{T+1} & \dots & a_{2T-1} \\ \vdots & \vdots & & \vdots \\ a_{(2^m-2)T} & a_{(2^m-2)T+1} & \dots & a_{(2^m-1)T-1} \end{bmatrix} = \{a_{nT}\} \{a_{nT+1}\} \dots \{a_{(n+1)T-1}\} \quad (13)$$

This representation is closely related to the time multiplex technique, which well understood by engineers.

Example 1: Decomposition of the sequence of 1023 bits long into 33 subsequences, each of length 31:

```

100100100110100110101111100110001
111100100011101111110000111000000
011111111110001110001001110110010
101110111101010001111010010101000
00101111111010101010111101000011
101001000110010110101100111101011
00011001111100101010100110011001
010011111010011100001000110110010
001010011011110111010101110011001
11011101110011101010011101000011
110110111000011000100101001011001
101000100010110100101110100110001
011000000101001001011111011110001
100011011101100001111001001110010
11000100001101111110011100011010
100101000010000100101101111101011
100010111001000011111011010101000
101111011001110011111000001110010
010101100101111001011100000101011
011001100001101011011101000101011
111101000111001101110010100011010
000001100100100010000010011011010
011110011010101100001011101101000
110000100111111101110001111000000
111011011000101000100110010000011
010010011110111110001010101101000
010100000001011011011110011110001
000111111011000111010110101000011
001101100000110000000011011011010
111010111100001010100100001011001
001100000100010010000001000000000
    
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Based on the phase shifts between the subsequences represented by the columns and define the Null sequence by ∞ we can easily get the interleaving order [4,9,12]

$$I_p^T = \{12, 21, 1, 29, 6, 17, 0, 6, 6, 2, 4, 16, 21, 7, 22, 4, 29, 16, 13, 16, 15, 8, 24, 12, 4, 5, 25, 15, 10, 18, \infty, 17, 8\}$$

Note:

- This sequences is still too short for practical application.
- One can get the interleaving order I_p^T (shift sequence e) only after knowing the sequences.

2.2 D-transform representaion

2.2.1 State and configure description

Refer to (10) we see that in D-transform the information about configuration $g(D)$, state $S(D)$ and signal $b(D)$ are given. It is short and effective to handle the relation between these three parameters. If $S(D)$ and $g(D)$ are known, we can find out the corresponding output sequence by the long division algorithm [5,13] as in example 2.

Example 2: Apply the long division algorithm to determinate shift register output when $S(D)=1$, $g(D)=1+D+D^4$

Solution:

The shift register output is:

$$b(D) = \frac{1}{1+D+D^4}$$

$$b(D) = 1+D+D^2+D^3+D^5+D^7+D^8+\dots \text{ and } b_n = 1, 1, 1, 1, 0, 1, 1, \dots$$

2.2.2 Interleaved structure representation

Since interleaving process and D-transform are both sort of time multiplexing [1, 3, 4, 9...] one can easily derive the interleaving order I_p^T straightforwardly from D-transform of the sequences. In fact, there are two methods for derive I_p^T , namely: expanding and decomposition. For the sake of simplicity we just show the result as in example 3.

Example 3: Let $m=3$, $n=6$ and let α be a primitive element of $GF(2^6)$ with primitive polynomial $b(D)=D^6+D^5+1$ over $GF(2)$. Let $\{b_n\}$ denote the m-sequence generated by $b(D)$

$\{b_n\} = \{0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\}$.

Decimation of $\{b_n\}$ by $T=9$, we obtain $\{a_n\} = \{b_{n9}\}$ and rearrange is as:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that the columns are shift equivalent and $I_p^T = \{\infty, 5, 3, 5, 6, 3, 3, 2, 5\}$, where ∞ represents Null sequence.

2.2.3 Hardware implementation of interleaving

D-transform can offer this representation, since it is time multiplexing from nature! It is clear that all m-subsequences are shift versions generated by same LFSR. Therefore, the software controlled switch S just picks up these shift versions and deliver them in the order determined by $I_p^T \dots$. This approach have been successfully have been successfully simulated and implemented in [14,15,16].

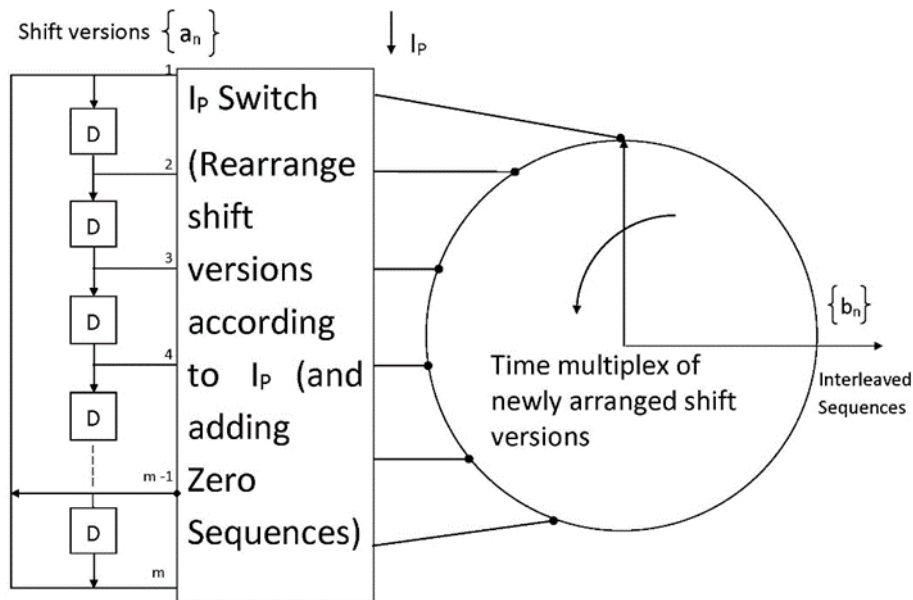


Figure 2. Shift versions for interleaving

2.3 Trace function representation

2.3.1 Field structure and sequences

As seen in the definition, this is also very short and compact representation: the information about the extension Field and subfield is given (mapping between them). One can derive the sequence from it trace function (see example 4).

Example 4: The binary m-sequence $\{a_n\}$ with period 2^m-1 can be represented through the trace function as:

$$\{a_n\} = \{a_0, a_1, a_2, \dots, a_{2^m-1}\} = \text{Tr}_1^m(\alpha^0), \text{Tr}_1^m(\alpha^1), \dots, \text{Tr}_1^m(\alpha^{2^m-1}) = \text{Tr}_1^m(\alpha^n)$$

It is well known that, the shift version of sequence $\{a_n\}$ can be created by its decimation.

The sequence $\{a_{kn}\}$ obtained by sampling every k^{th} bit of $\{a_n\}$, starting from the first bit is referred to as decimation of $\{a_n\}$. A decimation by k of $\{a_n\}$ results in the sequence: $\{a_{nk}\} = \text{Tr}_1^m(\alpha^n)$.

Let $\{a_n\}$ be m-sequence generator by $g(D) = D^5 + D^2 + 1$ over $GF(2)$.

The corresponding binary form is: 10010.11001.11110.00110.11101.010000

Below are few sequences result from decimation of $\{a_n\}$ by difference k [17]

k	Sequence	Characteristic Polynomial (in octal)
1	1001011001111100011011101010000	45 47 61 56 50 0
3	1111101110001010110100001100100	76 70 53 20 62 0
5	1110100010010101100001110011011	72 11 26 07 15 4
7	1001001100001011010100011101111	44 60 55 21 67 4
11	1110110011100001101010010001011	73 16 06 51 05 4
15	1000010101110110001111100110100	41 27 30 76 32 0

2.3.2 Interleaved structure representation

By nature, trace function shows the “interleaving” of subfields inside the extension field. Therefore it is convenient to describe the interleaved structure.

Base on trace function one can easily derive the interleaving order directly as in example 5.

Example 5: Let trace function from $GF(3^4)$ onto $GF(3^2)$ with primitive polynomial 10021: $f(x)=x^4+2x+1$ with $n=4$, $L=3^4-1=80$, $m=2$, $N=3^2-1=8$, $S=L/N=10$.

Calculating the trace function of x from $GF(3^4)$ into $GF(3^2)$ we obtained:

$$Tr_m^n(\alpha) = \sum_{k=0}^{n/m-1} \alpha^{3^{2k}} = \alpha + \alpha^9$$

α^{Si} table:

$$i = 0 \Rightarrow \alpha^0 = 1$$

$$i = 1 \Rightarrow \alpha^{10} = 1 + 2\alpha + \alpha^2 + \alpha^3$$

$$i = 2 \Rightarrow \alpha^{20} = \alpha + 2\alpha^2 + 2\alpha^3$$

$$i = 3 \Rightarrow \alpha^{30} = 1 + \alpha + 2\alpha^2 + 2\alpha^3$$

$$i = 4 \Rightarrow \alpha^{40} = 2$$

$$i = 5 \Rightarrow \alpha^{50} = 2 + \alpha + 2\alpha^2 + 2\alpha^3$$

$$i = 6 \Rightarrow \alpha^{60} = 2\alpha + \alpha^2 + \alpha^3$$

$$i = 7 \Rightarrow \alpha^{70} = 2 + 2\alpha + \alpha^2 + \alpha^3$$

With j run from 0 to $S-1$ we get:

$$j = 0 \Rightarrow Tr(\alpha^0) = Tr(1) = 1 + 1 = 2 = \alpha^{40} \Rightarrow I_p^0 = 4$$

$$j = 1 \Rightarrow Tr(\alpha^1) = \alpha + \alpha^9 = 2\alpha + \alpha^2 + \alpha^3 = \alpha^{60} \Rightarrow I_p^1 = 6$$

$$j = 2 \Rightarrow Tr(\alpha^2) = \alpha^2 + \alpha^{18} = 2\alpha + \alpha^2 + \alpha^3 = \alpha^{60} \Rightarrow I_p^2 = 6$$

$$j = 3 \Rightarrow Tr(\alpha^3) = \alpha^3 + \alpha^{27} = \alpha + 2\alpha^2 + 2\alpha^3 = \alpha^{20} \Rightarrow I_p^3 = 2$$

$$j = 4 \Rightarrow Tr(\alpha^4) = \alpha^4 + \alpha^{36} = 2 + \alpha + 2\alpha^2 + 2\alpha^3 = \alpha^{50} \Rightarrow I_p^4 = 5$$

$$j = 5 \Rightarrow Tr(\alpha^5) = \alpha^5 + \alpha^{45} = 0 \Rightarrow I_p^5 = \infty$$

$$j = 6 \Rightarrow Tr(\alpha^6) = \alpha^6 + \alpha^{54} = \alpha + 2\alpha^2 + 2\alpha^3 = \alpha^{20} \Rightarrow I_p^6 = 2$$

$$j = 7 \Rightarrow Tr(\alpha^7) = \alpha^7 + \alpha^{63} = 1 = \alpha^0 \Rightarrow I_p^7 = 0$$

$$j = 8 \Rightarrow Tr(\alpha^8) = \alpha^8 + \alpha^{72} = 2 + \alpha + 2\alpha^2 + 2\alpha^3 = \alpha^{50} \Rightarrow I_p^8 = 5$$

$$j = 9 \Rightarrow Tr(\alpha^9) = \alpha^9 + \alpha^{81} = 2\alpha + \alpha^2 + \alpha^3 = \alpha^{60} \Rightarrow I_p^9 = 6$$

The interleaving order is:

$$I_p^T = \{4, 6, 6, 2, 5, \infty, 2, 0, 5, 6\}$$

The subsequences can be interleaved like this [3]

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 2 & 0 & 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 2 & 0 & 2 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

Even though trace function is very effective, it is abstract (pure mathematic) and one cannot expect it to be intuitive. Note: It has shown that the interleaving order (or shift sequence) calculated by D-transform and by trace representation are identical for the same composite sequence [4,9,12].

3. The links between above mentioned tools

3.1 Between Matrix and D-transform

In order to determinate $S(D)$ (in D-transform) which specifies the corresponding memory content of LFSR we have use the matrix equation system. This fact can be best demonstrated by an example. Unfortunately the matrix representation for specific sequences which have very large length would take a lot of spaces, let alone the manipulation with these matrices. Therefore we consider here only the subsequence (shorter length) P_n generated by:

$$g(D)=1+D^5+D^6$$

Example 6: The D-transform of P_n is $W_N(D)$ with $N=2^m-1=63$, so, the length of P_n is then=63 and

$$W_i(d^T) = \frac{S_i(d^T)}{g_s(d^T)}$$

With T denotes the number of subsequences in the specific sequence.

The corresponding $S_i(d^T)$ of memory state: S_i can be calculated as:

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ w_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-1} & w_{N-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{pmatrix}$$

And $[g_i]$ denote the feedback taps of the LFSR characterized by $g(D)$.

Table i the corresponding phase shifts of sequences specified by $S_i(d^T)$

phase shifts	$S_i(d^T)$	phase shifts	$S_i(d^T)$
0	$1+D^5$	32	$1+D^3+D^5$
1	D^5	33	D^3+D^5
2	D^4	34	$D+D^4$
3	D^3	35	$1+D^3$
4	D^2	36	$D^2+D^4+D^5$
5	D	37	$D+D^3+D^5$
6	1	38	$1+D^2+D^3$
7	D^4+D^5	39	$D+D^2+D^4+D^5$
8	D^3+D^4	40	$1+D+D^3+D^4$
9	D^2+D^3	41	$1+D^2+D^3+D^4+D^5$
10	$D+D^2$	42	$D+D^2+D^3+D^5$
11	$1+D$	43	$1+D+D^2+D^4$
12	$1+D^4+D^5$	44	$1+D+D^3+D^4+D^5$
13	D^3+D^5	45	$1+D^2+D^3+D^5$
14	D^2+D^4	46	$D+D^2+D^5$
15	D^1+D^3	47	$1+D+D^4$
16	$1+D^2$	48	$1+D^3+D^4+D^5$
17	$D+D^4+D^5$	49	$D^2+D^3+D^5$
18	$1+D^3+D^4$	50	$D+D^2+D^4$
19	$D^2+D^3+D^4+D^5$	51	$1+D+D^3$
20	$D+D^2+D^3+D^4$	52	$1+D^2+D^4+D^5$
21	$1+D+D^2+D^3$	53	$D+D^3+D^5$
22	$1+D+D^2+D^4+D^5$	54	$1+D^2+D^4$
23	$1+D+D^3+D^5$	55	$D+D^3+D^4+D^5$
24	$1+D^2+D^5$	56	$1+D^2+D^3+D^4$
25	$D+D^5$	57	$D+D^2+D^3+D^4+D^5$
26	$1+D^4$	58	$1+D+D^2+D^3+D^4$
27	$D^3+D^4+D^5$	59	$1+D+D^2+D^3+D^4+D^5$
28	$D^2+D^3+D^4$	60	$1+D+D^2+D^3+D^5$
29	$D+D^2+D^3$	61	$1+D+D^2+D^5$
30	$1+D+D^2$	62	$1+D+D^5$
31	$1+D+D^4+D^5$		

Note that T may be few thousands or more to create the specific sequences used in the practice!

On the other hand from $g(D)$ we can easily get the feedback tap of fig 1.1a, 1.1b and therefore G_i of (4), the state transition matrix F of (7). If $S(D)$ is known, then we can get the output b_n .

3.2 Between D-transform and trace

The equivalence can be proven as follows:

Let consider the Decomposition of m, sequences $\{b_n\}$

Let $\{b_n\}$ be an m-sequence generated by $g(D)$ of degree $n = lm$, and denote $T=(2^n-1)/(2^m-1)$

From $\{b_n\}$ sequences, we try to find out $\{a_n\}$ and I_p^T

Decimate $\{b_n\}$ by T, we will obtain: $\{a_n\} = Tr_1^n(\alpha^{Tn}) = Tr_0^n(\beta^n)$

Since (α^{Tn}) and (β^n) are primitive element in $GF(2^n)$, $\{a_n\}$ is also an m-sequence of length N with $N=(2^n-1)/T$.

Let arrange $\{a_n\}$ in the column of the decimation matrix

$$M = \begin{pmatrix} a_0 & a_1 & \dots & a_{T-1} \\ a_T & a_{T+1} & \dots & a_{2T-1} \\ a_{2T} & a_{2T+1} & \dots & \\ \dots & \dots & \dots & \dots \\ a_{(2^m-2)T} & a_{(2^m-2)T+1} & \dots & a_{(2^m-1)T-1} \end{pmatrix}$$

Each column is a decimation of $\{b_n\}$ by T, an m-subsequence [17].

On other hand, from d-transform property: $S(d^T)/g(d^T)$ represents a decimations of b_n by $T=(2^n-1)/(2^m-1)$, which turn out to be $\{a_n\}$, an m-sequence also because: $S(d^T)$ and $g(d^T)$ are relative prime.

So, the subsequences in the interleaving are identical. Now, we need only to check I_p^T . It can be seen that I_p^T

calculated by [3] or lookup table [12] (similar to table 1, example 1) are identical also (they must be the same because they are derived from matrix M). That implies D-transform and trace function are equivalent in creating the sequences with interleaved structure.

4. Conclusion remarks and future works

In this contribution we try to compare some mathematical Tools widely used in representation of specific sequences applied in such areas like: advanced communications, cryptography and automatic testing.

The matrix representation is very intuitive and easily understandable in describing the configuration, state transition. It can also be used for determination the particular memory content when the D-transform of either the sequences or the corresponding initial state are given. However, for the sequences of great length, the dimension of the Matrix is unbearable great, which makes the manipulation difficult.

The trace function is compact and very comfortable for investigation of the m-sequences in general and interleaved m-sequences in particular. Since the length of sequences defined in trace function is $L=p^m-1$ only, it cannot be used for an arbitrary interleaved sequences ($L \neq p^m-1$) without some complicated modification as in D-transform case. Furthermore, one cannot get the information about states and configuration of LFSR in trace representation. However, from shift sequence ($e=I_p^T$) one can also figure out the hardware implementation (interleaving order).

The D-transform representation is also short and easy to handle. Furthermore, it contains information about the state (S(D)) and configuration, g(D) as well. Like Trace function representation, it also indicates the hardware implementation (I_p^T). We notice once more that D-transform can be applied for any periodic sequence without any new concept. We hope to prove this statement in coming papers!

The link between those tools are also pointed out, for the first time.

As mentioned in our introduction and [1,2,3,4,...] all the above requirements need to be carefully evaluated before application. The static properties of the sequences (correlation functions, symbol distribution...) have to satisfy some random criteria so that the processed signal can be seen as pseudo noise and the linear complexities have to be large enough to make the sequence estimation difficult [4,11,12,18,19,20]. The evaluation process is obviously more complicated so that the trace function and D-transform need to be modified correspondingly. However, due to the limited scope of this paper we concentrate only on the comparison of representation methods and leave other issues such as analysis and sequence selection for the future contributions.

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