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Fixed Point Results in Fuzzy F Menger Metric Space

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Abstract

In this paper we prove fixed point theorems in Fuzzy F menger space with non compatible condition and rational expression.

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Key Words: Fuzzy F Menger Space, Non-Compatible Mappings, Common Fixed Points, Discontinuity, R-weak commutative of type (A_g).

1. INTRODUCTION

The study of fixed points for non-compatible mappings can be extended to the class of non-expansive or Lipschitz mapping pairs even without continuity of the mappings involved or completeness of the space. Menger [4] in 1942 introduced the notation of the probabilistic metric space. The notion of weak commutativity generalized by Junjck [3] and another generalization is given by Pant [6] as R-weak commutative of type (A_g).

The probabilistic generalization of metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds. Recently, Shrivastav,Patel and Dhagat [14] and [15] worked on fuzzy probabilistic metric space. We have proved fixed point theorems for fuzzy probabilistic metric space.

2. PRILIMINARIES

Definition 2.1 A fuzzy F menger space is an ordered pair (X,F^2_α) consisting of a nonempty set X and a mapping F^2_α from XxX into the collections of all distribution functions $F^2_\alpha \in R$ for all α. ∈ [0,1]. For x, y ∈ X we denote the distribution function F_α (x,y) by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F^2_{\alpha(x,y)}$ at u in R. The functions $F^2_{\alpha(x,y)}$ for all α. ∈ [0,1] assumed to satisfy the following conditions:

(a) $F^2_{\alpha(x,y)}(u) = 1 \ \forall\ u > 0$ iff x = y,

(b) $F^2_{\alpha(x,y)}(0) = 0 \; \forall$ x , y in X,

(c) $F^2_{\alpha(x,y)} = F_{\alpha(y,x)} \; \forall$ x , y in X,

(d) If $F^2_{\alpha(x,y)}(u) = 1$ and $F^2_{\alpha(y,z)}(v) = 1$ then $F^2_{\alpha(x,z)}(u+v) = 1 \; \forall$ x ,y, z in X

and u, v > 0

Definition 2.2 A commutative, associative and non-decreasing mapping t: $[0,1] \times [0,1] \to [0,1]$ is a t-norm if and only if t(a,1)=a for all a∈[0,1] , t(0,0)=0 and t(c,d) ≥ t(a,b) for c ≥ a, d ≥ b .

Definition 2.3 A Fuzzy F Menger space is a triplet (X, F^2_α, t), where (X, F^2_α) is a FFM-space,t is a t-norm and the generalized triangle inequality

$$F^2_{\alpha(x,z)}(u+v) \geq t\left(F^2_{\alpha(x,z)}(u), F^2_{\alpha(y,z)}(v)\right)$$

holds for all x, y, z in X u, v > 0 and α. ∈ [0,1]

The concept of neighborhoods in Fuzzy F Menger space is introduced as

Definition 2.4 Let (X, F^2_α, t) be a Fuzzy F Menger space. If x∈X, ε > 0 and λ ∈(0,1), then (ε,λ) - neighborhood of x, called U_x (ε,λ), is defined by

$$U_x(\varepsilon,\lambda) = \{y \in X : F^2_{\alpha(x,y)}(\varepsilon) > (1-\lambda)\}$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda) : x \in X, \varepsilon > 0, \alpha. \in [0,1]$ and $\lambda \in (0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy F Menger space (X, F^2_α, t) is a Housdroff space in (ε,λ)-topology.

Let (X, F^2_α, t) be a complete Fuzzy F Menger space and A⊂X. Then A is called a bounded set

$$\text{if} \quad \lim_{u \to \infty} \inf_{x,y \in A} F^2_{\alpha(x,y)}(u) = 1$$

Definition 2.5 A sequence $\{x_n\}$ in (X, F^2_α, t) is said to be convergent to a point x in X if for every ε>0and λ>0, there exists an integer N=N(ε,λ) such that $x_n \in U_x(\varepsilon,\lambda)$ for all n ≥ N or equivalently $F^2_\alpha(x_n, x; \varepsilon) > 1-\lambda$ for all n ≥ N and α∈[0,1].

Definition 2.6 A sequence $\{x_n\}$in (X,F^2_α,t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F^2_\alpha(x_n,x_m;\ \varepsilon) > 1-\lambda\ \forall\, n, m \geq N$ for all $\alpha\in[0,1]$.

Definition 2.07 A Fuzzy F Menger space (X,F^2_α,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha\in[0,1]$.

Definition 2.08: Let (X, F^2_α,t) be a Fuzzy F Menger space. Two mappings f, g: $X\to X$ are said to be compatible if and only if $F^2_{\alpha(fgx_n,gfx_n)}(t)\to 1$ for all $t > 0$ whenever $\{x_n\}$ in X such that $fx_n, gx_n \to z$ for some $z \in X$. It follows that the maps f and g are called *non-compatible* if they are not compatible. Thus f and g are non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = z$ for some z in X but $\lim_n d_\alpha(gfx_n,fgx_n)\neq 0$ or $\lim_n d_\alpha(gfx_n,fgx_n)$ does not exists.

Definition 2.09: Two self mappings f and g of a Fuzzy F Menger Space (X, F^2_α,t) are said to be point wise R-weakly commuting if given $x \in X$, there exists $R > 0$ such that

$$F^2_{\alpha(fgx,gfx)}(t)\geq F^2_{\alpha(fx,gx)}(t/R)\ \text{ for } t > 0.$$

And f and g were R-weak commutative of type (A_g) if $F^2_{\alpha(ffx,gfx)}(t)\geq F^2_{\alpha(fx,gx)}(t/R)$

3. Main Results

Theorem 3.1. Let f and gbe non-compatible self mappings of a complete fuzzy F menger space (X, F^2_α,Δ) , where t is continuous and $\Delta(t,t)\geq t$ for all t in [0,1] such that

(I) $\overline{T(X)}\subset S(X)$ where $\overline{T(X)}$ is closure of the range of T

$$\text{(II) } F^2_{\alpha(fx,fy)}(kt)\ \geq \min\left\{\begin{array}{l} F^2_{\alpha(gx,gy)}(t), F^2_{\alpha(gx,fy)}(t), \dfrac{\Delta(F^2_{\alpha(fx,gy)}(t),F^2_{\alpha(fy,gy)}(t))}{\Delta(F^2_{\alpha(fx,gx)}(t),F^2_{\alpha(fy,gx)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(fx,gx)}(t),F^2_{\alpha(fy,gx)}(t))}{\Delta(F^2_{\alpha(fx,gy)\alpha(}(t),F^2_{\alpha(fy,gy)}(t))} \end{array}\right\}$$

If f and g be weak compatible of type A, then f and g have a unique common fixed point.

Proof : Since f and g are non-compatible mappings the there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p∈ X……… (3.1)

but either $\lim_{n\to\infty} F_{\alpha(fgxn,\, gfxn)}(t) \neq 1$ or limit does not exist. Again $\overline{f(X)} \subset g(X)$ and p∈ $\overline{f(X)}$ therefore there exists u in X such that gu = p , (where $\lim_{n\to\infty} gx_n = p$). If fu ≠ gu then by (II)

$$F^2_{\alpha fx_n, fu}(kt) \geq \min\left\{ \begin{array}{l} F^2_{\alpha gx_n, gu}(t), F^2_{\alpha gx_n, fu}(t), \dfrac{\Delta(F^2_{\alpha fx_n, gu}(t), F^2_{\alpha fu, gu}(t))}{\Delta(F^2_{\alpha fx_n, gx_n}(t), F^2_{\alpha fu, gx_n}(t))}, \\ \dfrac{\Delta(F^2_{\alpha fx_n, gx_n}(t), F^2_{\alpha fu, gx_n}(t))}{\Delta(F^2_{\alpha fx_n, gu}(t), F^2_{\alpha fu, gu}(t))} \end{array} \right\}$$

On taking limit n→∞

$$F^2_{\alpha(gu,fu)}(kt) > \min\left\{ \begin{array}{l} F^2_{\alpha(gu,gu)}(t), F_{\alpha(gu,fu)}(t), \dfrac{\Delta(F^2_{\alpha(gu,gu)}(t), F^2_{\alpha(fu,gu)}(t))}{\Delta(F^2_{\alpha(gu,gu)}(t), F^2_{\alpha(fu,gu)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(gu,gu)}(t), F^2_{\alpha(fu,gu)}(t))}{\Delta(F^2_{\alpha(gu,gu)}(t), F^2_{\alpha(fu,gu)}(t))} \end{array} \right\}$$

$$\Rightarrow F^2_{\alpha(fu,gu)}(kt) > F^2_{\alpha(fu,gu)}(t) \quad \Rightarrow gu = fu.$$

Now, f and g are weak compatible of type (A), therefore ffu = gfu.

If fu ≠ ffu the by (II)

$$F^2_{\alpha(fu,ffu)}(kt) > \min\left\{ \begin{array}{l} F^2_{\alpha(gu,gfu)}(t), F^2_{\alpha(gu,ffu)}(t), \dfrac{\Delta(F^2_{\alpha(fu,gfu)}(t), F^2_{\alpha(ffu,gfu)}(t))}{\Delta(F^2_{\alpha(fu,gu)}(t), F^2_{\alpha(ffu,gu)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(fu,gu)}(t), F^2_{\alpha(gu,ffu)}(t))}{\Delta(F^2_{\alpha(fu,gfu)}(t), F^2_{\alpha(ffu,gfu)}(t))} \end{array} \right\}$$

$$> \min\left\{ \begin{array}{l} F^2_{\alpha(fu,ffu)}(t), F^2_{\alpha(fu,ffu)}(t), \dfrac{\Delta(F^2_{\alpha(fu,ffu)}(t), F^2_{\alpha(ffu,ffu)}(t))}{\Delta(F^2_{\alpha(fu,fu)}(t), F^2_{\alpha(ffu,fu)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(fu,fu)}(t), F^2_{\alpha(fu,ffu)}(t))}{\Delta(F^2_{\alpha(fu,ffu)}(t), F^2_{\alpha(ffu,ffu)}(t))} \end{array} \right\}$$

$$F^2_{\alpha(fu,ffu)}(kt) > F^2_{\alpha(fu,ffu)}(t)$$

$\Rightarrow$ fu = ffu = gfu.

Hence, fu is common fixed point of g and f.

Uniqueness: Let for u $\neq$ v, fu and fv are two common fixed points of f and g. Assume that fu $\neq$ fv, then by (II)

$$F^2_{\alpha\, fu,fv}(kt)$$

$$> \min\left\{F^2_{\alpha(gu,gv)}(t), F^2_{\alpha gu,fv}(t), \frac{\Delta(F^2_{\alpha(fu,gv)}(t), F^2_{\alpha(fv,gv)}(t))}{\Delta(F^2_{\alpha(fu,gu)}(t), F^2_{\alpha(fv,gu)}(t))}, \frac{\Delta(F^2_{\alpha(fu,gu)}(t), F^2_{\alpha(gu,fv)}(t))}{\Delta(F^2_{\alpha(fu,gv)}(t), F^2_{\alpha(fv,gv)}(t))} - \right\}$$

$$= \min\left\{\begin{array}{l} F^2_{\alpha(fu,fv)}(t), F^2_{\alpha(fu,fv)}(t), \frac{\Delta(F^2_{\alpha(fu,fv)}(t), F^2_{\alpha fv,fv}(t))}{\Delta(F^2_{\alpha(fu,fu)}(t), F^2_{\alpha(fv,fu)}(t))}, \\ \frac{\Delta(F^2_{\alpha(fu,fu)}(t), F^2_{\alpha(fu,fv)}(t))}{\Delta(F^2_{\alpha(fu,fv)}(t), F^2_{\alpha(fv,fv)}(t))} \end{array}\right\}$$

$$F^2_{\alpha(fu,fv)}(kt) > F^2_{\alpha(fu,fv)}(t)$$

$\Rightarrow$ fu = fv.

In the theorem, we make modification in the condition of theorem 1 with R-weak commutative of type (A_g),we get discontinuity at common fixed point. Let S and T satisfying following condition:

$$\lim_{n\to\infty} ffx_n = fp \quad \text{and} \quad \lim_{n\to\infty} gfx_n = gp \ldots\ldots\ldots..(3.2),$$

whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X

Theorem 3.2. Let f and g be non-compatible self mappings of a Fuzzy F Menger space (X, F^2_α ,Δ) satisfying (II) of theorem 1 and the above condition (3.2). If f and g were R-weak commutative of type (A_g), then f and g have a unique common fixed point and fixed point is a fixed point of discontinuity.

Proof: Since f and g are non-compatible mappings the there exists a sequence $\{x_n\}$ in X such that by (3.1) $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$

but either $\lim_{n\to\infty} F_{\alpha\, fgxn,\, gfxn}(t) \neq 1$ or limit does not exist. Again by (3.2) we have

$\lim_{n\to\infty} ffx_n = fp$ and $\lim_{n\to\infty} gfx_n = gp$.

Further, R-weak commutative of type (A_g) yields

$F_\alpha\, ff_{xn,\, gfxn}\,(kt) \geq Ff_{xn,\, gxn}(t/R)$

On taking limit, we get fp = gp and ffp = gfp.

Now, we claim that ffp=fp. Let ffp≠fp, then by (II)

$$F^2_{\alpha(fp,ffp)}(kt) > \min\left\{\begin{array}{l} F^2_{\alpha(gp,gfp)}(t), F^2_{\alpha(gp,ffp)}(t), \dfrac{\Delta(F^2_{\alpha(fp,gfp)}(t), F^2_{\alpha(ffp,gfp)}(t))}{\Delta(F^2_{\alpha(fp,gp)}(t), F^2_{\alpha(ffp,gp)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(fp,gp)}(t), F^2_{\alpha(gp,ffp)}(t))}{\Delta(F^2_{\alpha(fp,gfp)}(t), F^2_{\alpha(ffp,gfp)}(t))} \end{array}\right\}$$

$= F^2_{\alpha\, fp,ffp}(t)$

Which is contradiction, therefore ffp=fp.

$\Rightarrow$ fp = ffp = gfp. Hence, fp is common fixed point of f and g.

Now, we show that g and f are discontinuous at the common fixed point fp=gp=p. If possible, suppose f is continuous. Then considering sequence $\{x_n\}$ of (1) we get $\lim_{n\to\infty} ffx_n = fp = p$. R-weak commutativity of type (A_g) implies that $F^2_{\alpha(ffx_n, gfx_n)}(kt) \geq F_{\alpha(fx_n, gx_n)}(t/R)$.

On taking limit n→∞ this yields $\lim_{n\to\infty} gfx_n = fp = p$.

Therefore, $\lim_{n\to\infty} F_{\alpha(fgx_n, gfx_n)}(t) = 1$. This contradiction the fact that either $\lim_{n\to\infty} F_{\alpha(fgx_n, gfx_n)}(t) \neq 1$ or nonexistent for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at the fixed point. Next, suppose that g is continuous. Then for the sequence $\{x_n\}$ of (1), we get

$$\lim_{n\to\infty} gfx_n = gp = p \text{ and } \lim_{n\to\infty} ggx_n = gp = p.$$

In view of these limits, the inequality

$$F^2_{\alpha(fp,fgx_n)}(kt) \geq \min\left\{ \begin{array}{l} F^2_{\alpha(gp,ggx_n)}(t), F^2_{\alpha(gp,fgx_n)}(t), \dfrac{\Delta(F^2_{\alpha(fp,SSx_n)}(t), F^2_{\alpha(fx_n,SSx_n)}(t))}{\Delta(F^2_{\alpha(fp,gp)}(t), F^2_{\alpha(fgx_n,gp)}(t))}, \\ \dfrac{\Delta(F^2_{\alpha(fp,gp)}(t), F^2_{\alpha(fgx_n,gp)}(t))}{\Delta(F^2_{\alpha(fp,ggx_n)}(t), F^2_{\alpha(fgx_n,ggx_n)}(t))} \end{array} \right\}$$

yields a contradiction unless $\lim_{n\to\infty} fgx_n = fp = gp$. But $\lim_{n\to\infty} fgx_n = gp$ and $\lim_{n\to\infty} gfx_n = gp$ contradicts the fact that either $\lim_{n\to\infty} F_{\alpha\ fgxn,\ Sgfn}(t) \neq 1$ or nonexistent for the sequence $\{x_n\}$ of (1). Hence both f and g are discontinuous at their common fixed point.

4.Examples

Ex.4.1: Let X = [1,10] and $F_{\alpha\, x,y}(t) = \dfrac{\alpha t}{\alpha t + d(x,y)}$ and d is usual metric on X. Define g,f:X→X by

$$fx = \begin{cases} 1 & \text{if } 1 \leq x < 3 \\ \dfrac{2+x}{4} & \text{if } x \geq 3 \end{cases} \quad \text{and} \quad gx = \begin{cases} \dfrac{x^2+1}{2} & \text{if } 1 \leq x < 2 \\ \dfrac{2x+1}{5} & \text{if } x \geq 2 \end{cases}$$

Also consider the sequence $x_n = 2+(1/n)$. $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = 1$.

$\lim_{n\to\infty} fgx_n = 3/4$ and $\lim_{n\to\infty} gfx_n = 3/5$.

Clearly, f & g are noncompatible but f & g are weak compatible of type (A).

$\Rightarrow$ ff2=gf2 and f2 = g2 = 1 (also ff1=gf1 & f1=g1=1).

We observe that g and f satisfying the conditions of **theorem 1** and hence **1** is the fixed point.

Ex. 4.2: Let X = [1,10] and $F^2_{\alpha\, x,\, y}(t) = \dfrac{\alpha t}{\alpha t + d^2(x, y)}$ [by 1] and d is usual metric on X.

Define g,f:X→X by $fx = \begin{cases} 1 & if\ x = 1\ or\ x > 3 \\ 4 & if\ 1 < x \le 3 \end{cases}$ and $gx = \begin{cases} 1 & if\ x = 1 \\ 5 & if\ 1 < x \le 3 \\ \dfrac{x+1}{4} & if\ x > 3 \end{cases}$.

Also consider the sequence $\{x_n = 3+(1/n): n\geq 1\}$.

$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = 1$. $\lim_{n} fgx_n = 4$ and $\lim_{n} gfx_n = 1$.

Clearly, f & g are noncompatible but it can be verify f & g are R-weak commutative of type (A_g) . We observe that g and f satisfying the conditions of **theorem 2** and hence **1** is the fixed point.

Conclusion: We ensure the unique fixed point without compatible mappings but weak compatible of type A with Lipschiz type analogue of a plane contractive in Fuzzy F Menger space which not convex but tend to convexity.

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