Journal of Information Engineering and Applications
ISSN 2224-5782 (print) ISSN 2225-0506 (online)
Vol.6, No.10, 2016

Numerical Treatment of Non-Linear Fuzzy Integral Equations by Homotopy Perturbation Method

Jamshad Ahmad[1] Hira Nosher[2]
1.Department of Mathematics, Faculty of Sciences, University of Gujrat, Pakistan
2.Department of Mathematics, NCBA&E (Gujrat Campus), Pakistan

Abstract
The main purpose of this paper is to present an approximation method for solving fuzzy integral equation. The solution of various types of non-linear fuzzy integral equations like non-linear fuzzy Volterra integral equation, non-linear fuzzy Fredholm integral equation and non-linear able fuzzy integral equation is determined by an advanced iterative approach the homotopy perturbation method. The method is discussed in details and it is illustrated by solving some numerical examples.
Keywords: Homotopy perturbation method, non-linear fuzzy Volterra integral equations, non-linear fuzzy Fredholm integral equation, non-linear Abel fuzzy integral equations.

1. Introduction

The topics of fuzzy integral equations have been growing rapid in recent few years [1-6]. The basic concept of fuzzy was introduced by Zadeh [7, 8]. Later, Dubois and Prade [9, 10] presented the concept of fuzzy calculus, then as well as time pass the concept of fuzzy integral was introduced by M. Sugeno [11, 12], then it's becoming a research oriented topic. Homotopy perturbation method is a coupling of perturbation method and homotopy technique was firstly introduced by He JH in 1999 [13, 14], then it was farther developed by him [15, 16]. HPM is one of the most advanced and affective method to find the solution of non-linear fuzzy integral equations. In this paper we shall discuss the analysis of HPM for fuzzy integral equation and approximate solution of non-linear fuzzy integral equations by HPM.

2. Non-Linear Fuzzy Integral Equation

An integral equation

$$u(x,r) = f(x,r) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t,r)dt, \tag{1}$$

is called non-linear fuzzy integral equation of second kind if the unknown function appearing inside the integral is of non-linear nature such that $u^2(t,r), u^3(t,r), e^{u(t,r)}, \ln u(t,r)$ etc. Where $u(x,r)$ and $f(x,r)$ are fuzzy functions, r is the fuzzy parameter whose value lies between $[0,1]$ i.e. $0 \le r \le 1$, λ is constant parameter, $k(x,t)$ is known function of two variables x and t called kernel of fuzzy integral equation, $a(x)$ and $b(x)$ are limits of fuzzy integral equation, if both of limits $a(x)$ and $b(x)$ are constant, then integral equation is known as Fredholm fuzzy integral equation, if one of limit can say $a(x)$ is constant and one of limit say $b(x)$ is variable then equation is called fuzzy Volterra integral equation.
The parametric representation of Eq. (1) is as follows,

$$\begin{cases} \underline{u}(x,r) = \underline{f}(x,r) + \lambda \int_{a(x)}^{b(x)} \underline{k(x,t)u(t,r)}dt \\ \overline{u}(x,r) = \overline{f}(x,r) + \lambda \int_{a(x)}^{b(x)} \overline{k(x,t)u(t,r)}dt \end{cases}, \quad 0 \le r \le 1,$$

where $u(x,r) = (\underline{u}(x,r), \overline{u}(x,r))$, $f(x,r) = (\underline{f}(x,r), \overline{f}(x,r))$ and

$$\begin{cases} \underline{k(x,t)u(t,r)} = k(x,t)\underline{u}(t,r) & k(x,t) \ge 0 \\ \overline{k(x,t)u(t,r)} = k(x,t)\overline{u}(t,r) & k(x,t) \ge 0 \end{cases}.$$

3. Analysis of HPM to Fuzzy Integral Equations

To solve Eq. (1) by HPM 1st we construct following homotopy

$$\begin{cases} H(\underline{v}, p, r) = (1-p)\left[\underline{v}(x,r) - \underline{u}_0(x,r)\right] + p\left[\underline{v}(x,r) - \underline{f}(x,r) - \int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,r)dt\right] = 0 \\ H(\overline{v}, p, r) = (1-p)\left[\overline{v}(x,r) - \overline{u}_0(x,r)\right] + p\left[\overline{v}(x,r) - \overline{f}(x,r) - \int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,r)dt\right] = 0 \end{cases}. \tag{2}$$

Thus the initial approximation is taken a

$$\begin{cases} \underline{u}_0(x,r) = \underline{f}(x,r) \\ \overline{u}_0(x,r) = \overline{f}(x,r) \end{cases}. \tag{3}$$

Substituting Eq. (3) in Eq. (2) reduces to

$$\begin{cases} \underline{v}(x,r) = \underline{f}(x,r) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,r)dt \\ \overline{v}(x,r) = \overline{f}(x,r) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,r)dt \end{cases}. \tag{4}$$

The solution of Eq. (2) is assumed as

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \tag{5}$$

Where $(\underline{v}_i, \overline{v}_i)$ are unknown to determined.
Now by putting Eq. (5) in Eq. (4) and by comparing coefficient like power of p we get
The following iterations

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) \\ \overline{v}_0(x,r) = \overline{f}(x,r) \end{cases}, \tag{6}$$

$$p^1 : \begin{cases} \underline{v}_1(x,r) = \int_{a(x)}^{b(x)} k(x,t)\underline{v}_0(t,r)dt \\ \overline{v}_1(x,r) = \int_{a(x)}^{b(x)} k(x,t)\overline{v}_0(t,r)dt \end{cases}, \tag{7}$$

$\vdots,$

Thus the solution of FIE-2 is given as

$$\begin{cases} \underline{u}(x,r) = \lim_{p\to 1} \underline{v}(x,r) = \sum_{i=0}^{\infty} \underline{v}_i(x,r) \\ \overline{u}(x,r) = \lim_{p\to 1} \overline{v}(x,r) = \sum_{i=0}^{\infty} \overline{v}_i(x,r) \end{cases}. \tag{8}$$

4. Numerical Applications

Example 4.1 Consider the non-linear fuzzy Volterra integral equation of 2nd kind

$$u(x,r) = f(x,r) + \int_0^x u^2(t,r)dt, \tag{9}$$

where

$$\lambda = 1,\ 0 \le x \le 1,\ 0 \le t \le x,\ 0 \le r \le 1,\ k(x,t) = 1 \quad and \quad f(x,r) = (\underline{f}(x,r), \overline{f}(x,r)) \ \ i.e.$$

$$f(x,\alpha) = (x(r^2 + r), x(7-r)).$$

To solve Eq. (9) by homotopy perturbation method 1st we construct convex homotopy,

$$\begin{cases} H(\underline{v}, p, r) = \underline{v}(x,r) - (r^2 + r)x - p\int_0^x \underline{v}^2(t,r)dt = 0 \\ H(\overline{v}, p, r) = \overline{v}(x,r) - (7-r)x - p\int_0^x \overline{v}^2(t,r)dt = 0 \end{cases}. \quad (10)$$

Assume the solution of Eq. (10) can be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \quad (11)$$

Utilizing Eq. (11) in Eq. (10) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) = x(r^2 + r) \\ \overline{v}_0(x,r) = \overline{f}(x,\alpha) = x(7-r) \end{cases}, \quad (12)$$

$$p^1 : \begin{cases} \underline{v}_1(x,r) = \frac{1}{3}x^3(r^2 + r)^2 \\ \overline{v}_1(x,r) = \frac{1}{3}x^3(7-r)^2 \end{cases}, \quad (13)$$

$$p^2 : \begin{cases} \underline{v}_2(x,r) = \frac{2}{15}x^5(r^2 + r)^3 \\ \overline{v}_2(x,r) = \frac{2}{15}x^5(7-r)^3 \end{cases}, \quad (14)$$

$$p^3 : \begin{cases} \underline{v}_3(x,r) = \frac{17}{315}x^7(r^2 + r)^4 \\ \overline{v}_3(x,r) = \frac{17}{315}x^7(7-r)^4 \end{cases}, \quad (15)$$

and so on…

As we know the solution is given as

$$\begin{cases} \underline{u}(x,r) = \sum_{i=0}^{\infty} \underline{v}_i(x,r) \\ \overline{u}(x,r) = \sum_{i=0}^{\infty} \overline{v}_i(x,r) \end{cases}. \quad (16)$$

Thus by utilizing above iterative results the approximate solution is given as

$$\begin{cases} \underline{u}(x,r) = x(r^2 + r) + \frac{1}{3}x^3(r^2 + r)^2 + \frac{2}{15}x^5(r^2 + r)^3 + \frac{17}{315}x^7(r^2 + r)^4 + \ldots \\ \overline{u}(x,r) = x(7-r) + \frac{1}{3}x^3(7-r)^2 + \frac{2}{15}x^5(7-r)^3 + \frac{17}{315}x^7(7-r)^4 + \ldots \end{cases}. \quad (17)$$

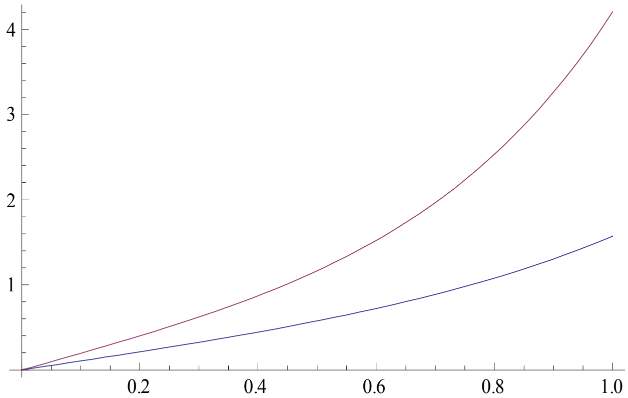

Fig. 1 Plot of Solution of Example 1, x ∈ [0, 1]

Example 4.2 Consider the non-linear fuzzy Volterra integral equation of 2nd kind

$$u(x,r) = f(x,r) + \int_0^x u^2(t,r)dt, \tag{18}$$

where

$$\lambda = 1,\ 0 \le x \le 1,\ 0 \le t \le x,\ 0 \le r \le 1,\ k(x,t) = 1 \quad and \quad f(x,r) = (\underline{f}(x,r), \overline{f}(x,r)) \ \ i.e.$$

$$f(x,r) = ((e^x - \frac{1}{2}(e^{2x} - 1))r, (e^x - \frac{1}{2}(e^{2x} - 1))(2 - r)).$$

To solve Eq. (18) by homotopy perturbation method 1st we construct convex homotopy,

$$\begin{cases} H(\underline{v}, p, r) = \underline{v}(x,r) - (e^x - \frac{1}{2}(e^{2x} - 1))r - p\int_0^x \underline{v}^2(t,r)dt = 0 \\ H(\overline{v}, p, r) = \overline{v}(x,r) - (e^x - \frac{1}{2}(e^{2x} - 1))(2 - r) - p\int_0^x \overline{v}^2(t,r)dt = 0 \end{cases}, \tag{19}$$

Assume the solution of Eq. (19) can be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \tag{20}$$

Utilizing Eq. (20) in Eq. (19) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) = (e^x - \frac{1}{2}(e^{2x} - 1))r \\ \overline{v}_0(x,r) = \overline{f}(x,r) = (e^x - \frac{1}{2}(e^{2x} - 1))(2 - r) \end{cases}, \tag{21}$$

$$p^1 : \begin{cases} \underline{v}_1(x,r) = (-\frac{47}{48} + \frac{1}{4}x + e^x + \frac{1}{4}e^{2x} - \frac{1}{3}e^{3x} + \frac{1}{16}e^{4x})r^2 \\ \overline{v}_1(x,r) = (-\frac{47}{48} + \frac{1}{4}x + e^x + \frac{1}{4}e^{2x} - \frac{1}{3}e^{3x} + \frac{1}{16}e^{4x})(2 - r)^2 \end{cases}, \tag{22}$$

$$p^2 : \begin{cases} \underline{v}_2(x,r) = (\frac{551}{2880} - \frac{47}{48}x + \frac{1}{8}x^2 - \frac{35}{24}e^x + \frac{161}{96}e^{2x} - \frac{5}{18}e^{3x} + \frac{41}{192}e^{4x} + \frac{11}{120}e^{5x} - \frac{1}{96}e^{6x} + \frac{1}{2}xe^x - \frac{1}{8}xe^{2x})r^3 \\ \overline{v}_2(x,r) = (\frac{551}{2880} - \frac{47}{48}x + \frac{1}{8}x^2 - \frac{35}{24}e^x + \frac{161}{96}e^{2x} - \frac{5}{18}e^{3x} + \frac{41}{192}e^{4x} + \frac{11}{120}e^{5x} - \frac{1}{96}e^{6x} + \frac{1}{2}xe^x - \frac{1}{8}xe^{2x})(2 - r)^3 \end{cases},$$

$\vdots$,

(23)
As we know the solution is given as

$$\begin{cases} \underline{u}(x,r) = \sum_{i=0}^{\infty} \underline{v}_i(x,r) \\ \overline{u}(x,r) = \sum_{i=0}^{\infty} \overline{v}_i(x,r) \end{cases}. \tag{24}$$

Thus by utilizing above iterative results the approximate solution is given as

$$\begin{cases} \underline{u}(x,r) = e^x r - \frac{1}{2}(e^{2x}-1)r - \frac{47}{48}r^2 + \frac{1}{4}xr^2 + e^x r^2 + \frac{1}{4}e^{2x}r^2 - \frac{1}{3}e^{3x}r^2 + \frac{1}{16}e^{4x}r^2 + \dots \\ \overline{u}(x,r) = e^x(2-r) - \frac{1}{2}(e^{2x}-1)(2-r) - \frac{47}{48}(2-r)^2 + \frac{1}{4}x(2-r)^2 + e^x(2-r)^2 + \frac{1}{4}e^{2x}(2-r)^2. \\ \quad -\frac{1}{3}e^{3x}(2-r)^2 + \frac{1}{16}e^{4x}(2-r)^2 + \dots \end{cases} \tag{25}$$

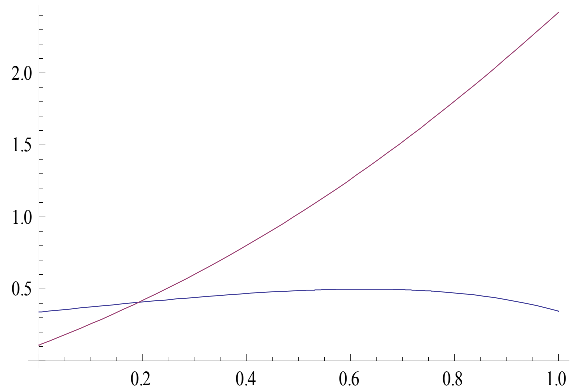

Fig. 2 Plot of Solution of Example 2, x∈ [0, 1]

Example 4.3 Consider the non-linear fuzzy Fredholm integral equation of 2nd kind

$$u(x,r) = f(x,r) + \int_0^1 u^2(t,r)dt, \tag{26}$$

where

$$0 \le x,t \le 1,\ 0 \le r \le 1,\ \ k(x,t) = 1\ \ and\ \ f(x,r) = (\underline{f}(x,r), \overline{f}(x,r))\ \ i.e.$$
$$f(x,r) = (r,(2-r)).$$

To solve Eq. (26) by homotopy perturbation method 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,r) = \underline{v}(x,r) - r - p\int_0^1 \underline{v}^2(t,r)dt = 0 \\ H(\overline{v},p,r) = \overline{v}(x,r) - (2-r) - p\int_0^1 \overline{v}^2(t,r)dt = 0. \end{cases} \tag{27}$$

Assume the solution of Eq. (27) can be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \tag{28}$$

Utilizing Eq. (28) in Eq. (27) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) = r \\ \overline{v}_0(x,r) = \overline{f}(x,r) = (2-r) \end{cases}, \tag{29}$$

$$p^1:\begin{cases}\underline{v}_1(x,r)=r^2\\ \overline{v}_1(x,r)=(2-r)^2\end{cases}, \tag{30}$$

$$p^2:\begin{cases}\underline{v}_2(x,r)=2r^3\\ \overline{v}_2(x,r)=2(2-r)^3\end{cases}, \tag{31}$$

$$p^3:\begin{cases}\underline{v}_3(x,r)=5r^4\\ \overline{v}_3(x,r)=5(2-r)^4\end{cases}, \tag{32}$$

$\vdots$,

As we know the solution is given as

$$\begin{cases}\underline{u}(x,r)=\sum_{i=0}^{\infty}\underline{v}_i(x,r)\\ \overline{u}(x,r)=\sum_{i=0}^{\infty}\overline{v}_i(x,r).\end{cases} \tag{33}$$

Thus by utilizing above iterative results the approximate solution is given as

$$\begin{cases}\underline{u}(x,r)=r+r^2+2r^3+5r^4+\dots\\ \overline{u}(x,r)=(2-r)+(2-r)^2+2(2-r)^3+5(2-r)^4+\dots\end{cases}. \tag{34}$$

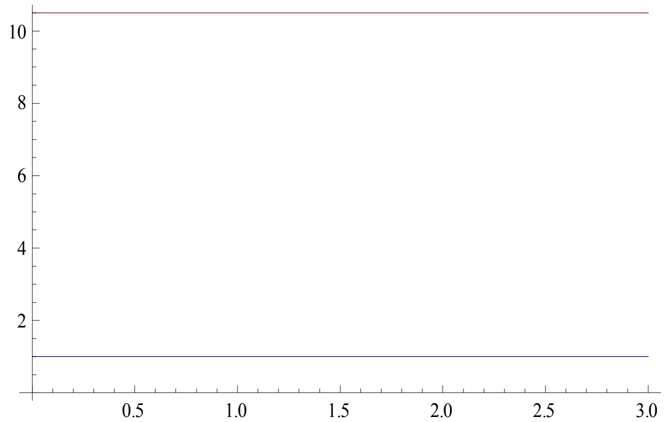

Fig. 3 Plot of solution of Example 3, x∈ [0, 3]

Example 4.4 Consider the non-linear fuzzy Fredholm integral equation of 1st kind

$$f(x,r)=\int_0^1 e^{x-2t}u^2(t,r)dt, \tag{35}$$

where

$$0\le x,t\le 1,\ 0\le r\le 1,\ k(x,t)=e^{x-2t}\quad and\quad f(x,r)=(\underline{f}(x,r),\overline{f}(x,r))\ \ i.e.$$

$$f(x,r)=(e^x r, e^x(3-r)).$$

We first set

$$\begin{cases}w(x,r)=u^2(x,r)\\ u(x,r)=\sqrt{w(x,r)}\end{cases}, \tag{36}$$

To carry out Eq. (35) into

$$f(x,r)=\int_0^1 e^{x-2t}w(t,r)dt, \tag{37}$$

Which is equivalent to

$$w(x,r)=f(x,r)-\int_0^1 (e^{x-2t}w(t,r)-w(x,r))dt. \tag{38}$$

Now to solve Eq. (38) by homotopy perturbation method 1st we define homotopy as follows,

$$\begin{cases} H(\underline{w}, p, r) = \underline{w}(x,r) - e^{x}r + p\int_0^1 (e^{x-2t}\,\underline{v}(t,r) - \underline{v}(x,r))dt = 0 \\ H(\overline{w}, p, r) = \overline{w}(x,r) - e^{x}(3-r) + p\int_0^1 (e^{x-2t}\,\overline{v}(t,r) - \overline{v}(x,r))dt = 0 \end{cases}. \quad (39)$$

Assume the solution of Eq. (39) can be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \quad (40)$$

Utilizing Eq. (40) in Eq. (39) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) = e^{x}r \\ \overline{v}_0(x,r) = \overline{f}(x,r) = e^{x}(3-r) \end{cases}, \quad (41)$$

$$p^1 : \begin{cases} \underline{v}_1(x,r) = e^{x-1}r \\ \overline{v}_1(x,r) = e^{x-1}(3-r) \end{cases}, \quad (42)$$

$$p^2 : \begin{cases} \underline{v}_2(x,r) = e^{x-2}r \\ \overline{v}_2(x,r) = e^{x-2}(3-r) \end{cases}, \quad (43)$$

$$p^3 : \begin{cases} \underline{v}_3(x,r) = e^{x-3}r \\ \overline{v}_3(x,r) = e^{x-3}(3-r) \end{cases}, \quad (44)$$

$\vdots$

As we know the solution is given as

$$\begin{cases} \underline{w}(x,r) = \sum_{i=0}^{\infty} \underline{v}_i(x,r) \\ \overline{w}(x,r) = \sum_{i=0}^{\infty} \overline{v}_i(x,r). \end{cases} \quad (45)$$

Thus by utilizing the above iterative results the series form solution is given as

$$\begin{cases} \underline{w}(x,r) = e^{x}r(1 + e^{-1} + e^{-2} + e^{-3} + \ldots) \\ \overline{w}(x,r) = e^{x}(3-r)(1 + e^{-1} + e^{-2} + e^{-3} + \ldots) \end{cases}. \quad (46)$$

And the exact form solution is given as

$$\begin{cases} \underline{w}(x,r) = r\left(\frac{e^{x+1}}{e-1}\right) \\ \overline{w}(x,r) = (3-r)\left(\frac{e^{x+1}}{e-1}\right) \end{cases}. \quad (47)$$

Now by doing back substitution from Eq. (36) the exact solution is given as

$$\begin{cases} \underline{u}(x,r) = \sqrt{\underline{w}(x,r)} = \sqrt{r(\frac{e^{x+1}}{e-1})} \\ \overline{u}(x,r) = \sqrt{\overline{w}(x,r)} = \sqrt{(3-r)(\frac{e^{x+1}}{e-1})} \end{cases}. \quad (48)$$

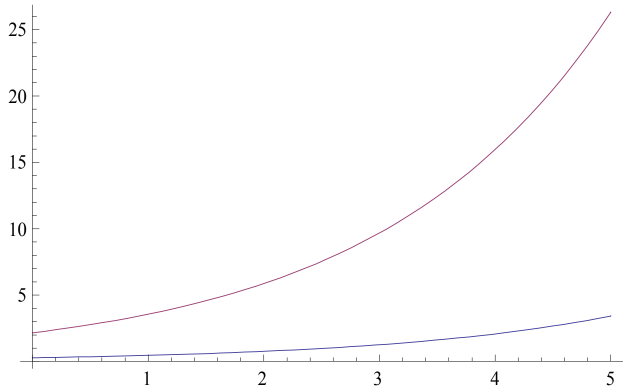

Fig. 4 Plot of Solution of Example 4, x∈ [0, 5]

Example 4.5 Consider the non-linear singular fuzzy integral equation of 2nd kind

$$u(x,r) = f(x,r) + \int_0^x \frac{u^2(t,r)}{\sqrt{x-t}} dt, \quad (49)$$

where

$$\lambda = 1,\ 0 \le x \le 1,\ 0 \le t < x, 0 \le r \le 1,\ k(x,t) = \frac{1}{\sqrt{x-t}} \quad and \quad f(x,r) = (\underline{f}(x,r), \overline{f}(x,r)) \ \ i.e.$$

$$f(x,r) = ((xr - \frac{16}{15}x^{\frac{5}{2}}r^2), (x(3-r) - \frac{16}{15}x^{\frac{5}{2}}(3-r)^2)).$$

To solve Eq. (49) by homotopy perturbation method 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,r) = \underline{v}(x,r) - (xr - \frac{16}{15}x^{\frac{5}{2}}r^2) - p\int_0^x \frac{\underline{v}^2(t,r)}{\sqrt{x-t}} dt = 0 \\ H(\overline{v},p,r) = \overline{v}(x,r) - (x(3-r) - \frac{16}{15}x^{\frac{5}{2}}(3-r)^2) - p\int_0^x \frac{\overline{v}^2(t,r)}{\sqrt{x-t}} dt = 0 \end{cases}. \quad (50)$$

Assume the solution of Eq. (50) can be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases}. \quad (51)$$

Utilizing Eq. (51) in Eq. (50) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \underline{f}(x,r) = xr - \frac{16}{15}x^{\frac{5}{2}}r^2 \\ \overline{v}_0(x,r) = \overline{f}(x,r) = x(3-r) - \frac{16}{15}x^{\frac{5}{2}}(3-r)^2 \end{cases}, \quad (52)$$

$$p^1:\begin{cases}\underline{v}_1(x,r)=\frac{16}{15}x^{\frac{5}{2}}r^2-\frac{7}{12}\pi x^4r^3+\frac{131072}{155925}x^{\frac{11}{2}}r^4\\ \overline{v}_1(x,r)=\frac{16}{15}x^{\frac{5}{2}}(3-r)^2-\frac{7}{12}\pi x^4(3-r)^3+\frac{131072}{155925}x^{\frac{11}{2}}(3-r)^4\end{cases}, \quad (53)$$

$\vdots$,

As we know the solution is given as

$$\begin{cases}\underline{u}(x,r)=\sum_{i=0}^{\infty}\underline{v}_i(x,r)\\ \overline{u}(x,r)=\sum_{i=0}^{\infty}\overline{v}_i(x,r)\end{cases}. \quad (54)$$

We observe the noise term phenomena appears between coefficients of p^0 and p^1 thus by utilizing above iterative results and cancelling terms the exact solution is given as

$$\begin{cases}\underline{u}(x,r)=xr\\ \overline{u}(x,r)=x(3-r)\end{cases}. \quad (55)$$

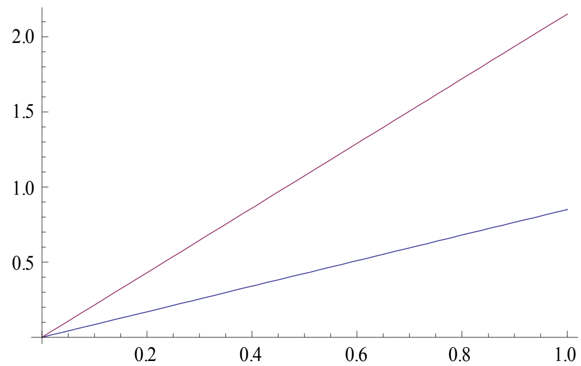

Fig. 5 Plot of Solution of Example 5, x∈ [0, 1]

Example 4.6 Consider the non-linear Able's fuzzy integral equation of 1st kind

$$f(x,r)=\int_0^x\frac{u^3(t,r)}{\sqrt{x-t}}dt, \quad (56)$$

where

$$\lambda=1,\ 0\le x\le 1,\ 0\le t<x, 0\le r\le 1,\ k(x,t)=\frac{1}{\sqrt{x-t}}\ and\ f(x,r)=(\underline{f}(x,r),\overline{f}(x,r))\ i.e.$$

$$f(x,r)=(\frac{32}{35}r^3x^{\frac{7}{2}},\frac{32}{35}(5-r)^3x^{\frac{7}{2}}).$$

Consider the transformation

$$\begin{aligned}w(x,r)&=u^3(x,r)\\ u(x,r)&=\sqrt[3]{w(x,r)}.\end{aligned} \quad (57)$$

Carries Eq. (57) into

$$f(x,r)=\int_0^x\frac{w(t,r)}{\sqrt{x-t}}dt.$$

Equivalent to

$$f(x,r)=\int_0^x\frac{w(x,r)}{\sqrt{x-t}}dt+\int_0^x\frac{w(t,r)-w(x,r)}{\sqrt{x-t}}dt$$

$$w(x,r) = \frac{f(x,r)}{2\sqrt{x}} - \frac{1}{2\sqrt{x}} \int_0^x \frac{w(t,r) - w(x,r)}{\sqrt{x-t}} dt \,. \tag{58}$$

To solve Eq. (58) by HPM first we construct convex homotopy as follows,

$$\begin{cases} H(\underline{v}, p, r) = \underline{v}(x,r) - \frac{16}{35} r^3 x^3 + \frac{p}{2\sqrt{x}} \int_0^x \frac{\underline{v}(t,r) - \underline{v}(x,r)}{\sqrt{x-t}} dt = 0 \\ H(\overline{v}, p, r) = \overline{v}(x,r) - \frac{16}{35} (5-r)^3 x^3 + \frac{p}{2\sqrt{x}} \int_0^x \frac{\overline{v}(t,r) - \overline{v}(x,r)}{\sqrt{x-t}} dt = 0 \end{cases} . \tag{59}$$

Assume the solution of Eq. (59) an be written as power series in p

$$\begin{cases} \underline{v}(x,r) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,r) \\ \overline{v}(x,r) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,r) \end{cases} . \tag{60}$$

Now by putting Eq. (60) in Eq. (59) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,r) = \frac{16}{35} r^3 x^3 \\ \overline{v}_0(x,r) = \frac{16}{35} (5-r)^3 x^3 \end{cases} , \tag{61}$$

$$p^1 : \begin{cases} \underline{v}_1(x,r) = \frac{16}{35} r^3 (1 - \frac{16}{35}) x^3 \\ \overline{v}_1(x,r) = \frac{16}{35} (5-r)^3 (1 - \frac{16}{35}) x^3 \end{cases} , \tag{62}$$

$$p^2 : \begin{cases} \underline{v}_2(x,r) = \frac{16}{35} r^3 (1 - \frac{16}{35})^2 x^3 \\ \overline{v}_2(x,r) = \frac{16}{35} (5-r)^3 (1 - \frac{16}{35})^2 x^3 \end{cases} \tag{63}$$

$\vdots$,

As we know the solution is given as

$$\begin{cases} \underline{w}(x,r) = \sum_{i=0}^{\infty} \underline{v}_i(x,r) \\ \overline{w}(x,r) = \sum_{i=0}^{\infty} \overline{v}_i(x,r). \end{cases} \tag{64}$$

Thus by utilizing the above iterative results the series form solution is given as

$$\begin{cases} \underline{w}(x,r) = \frac{16}{35} r^3 x^3 + \frac{16}{35} r^3 (1 - \frac{16}{35}) x^3 + \frac{16}{35} r^3 (1 - \frac{16}{35})^2 x^3 + \ldots \\ \overline{w}(x,r) = \frac{16}{35} (5-r)^3 x^3 + \frac{16}{35} (5-r)^3 (1 - \frac{16}{35}) x^3 + \frac{16}{35} (5-r)^3 (1 - \frac{16}{35})^2 x^3 + \ldots \end{cases} . \tag{65}$$

And the exact form solution is given as

$$\begin{cases} \underline{w}(x,r) = r^3 x^3 \\ \overline{w}(x,r) = (5-r)^3 x^3 \end{cases}. \tag{66}$$

Now by doing back substitution from Eq. (57) the exact solution is given as

$$\begin{cases} \underline{u}(x,r) = \sqrt[3]{\underline{w}(x,r)} = rx \\ \overline{u}(x,r) = \sqrt[3]{\overline{w}(x,r)} = (5-r)x \end{cases}. \tag{67}$$

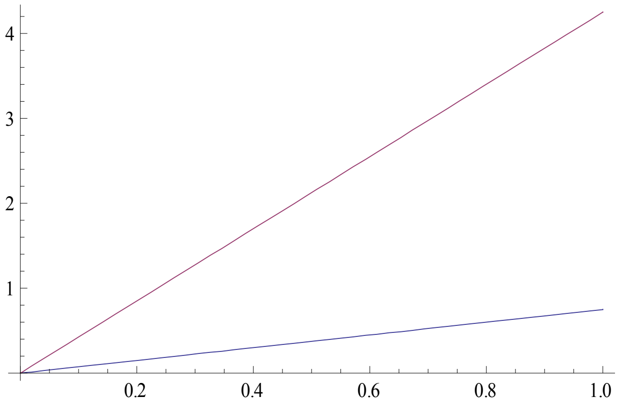

Fig. 6 Plot of Solution of example 6, x∈ [0, 1]

5. Conclusion

In this paper, some nonlinear fuzzy integral equations were handled by the homotopy perturbation method. The HPM has been shown to solve easily, accurately and affectingly a wide range of fuzzy non-linear problems which converge rapidly. Obtained results show that this new technique is easy to implement and produces accurate results. A considerable advantage of the used technique is that the approximate solutions are found very easily. The method can also be extended to the system of nonlinear fuzzy integral equations of mixed type with variable coefficients, but some modifications are needed.

References

[1] F.H. Hadi, B. Reza and N.S. Hassan, Solving linear fuzzy integral equations of second kind by artificial neural networks, Journal of Alexandria Engineering, (2014)53, 249-257.
[2] M. Farshid, Y.K. Mohammad and P. Mahmoud, Solving linear and non-linear ables fuzzy integral equations by Homotopy Analysis method, Journal of Taibah University for Science, 9(2015), 104-115.
[3] M. Otadi and M. Mosleh, Numerical solutions of fuzzy non-linear integral equations of second kind. Iranian Journal of Fuzzy System, Vol.11, No.1, (2014), pp.135-145.
[4] M. Mosleh and M. Otadi, Solution of fuzzy Volterra integral equations in a Bernstien Polynomial Basis, Journal of Advances in Information Technology, Vol.4, No.3, 2013.
[5] M. Zeinal, S. Shahmord and K. Mirnia, Fuzzy Integro-Differential Equations: Discrete solution and Error Estimation, Iranian Journal of Fuzzy System, Vol.10, No.1, (2013) pp.107-112.
[6] M. Jahantigh, T. Alahviranbo and M. Otadi, Numerical solutions of fuzzy integral equations, Journal of Applied Mathematical Sciences, Vol.2, 2008, No.1, 33-46.
[7] L.A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning,” Inform. Sci., vol. 8, pp. 199-249, 1975.
[8] Zadeh, L. A. (1965) "Fuzzy sets", Information and Control, 8, 338–353.
[9] D. Dubois and H. Prade, “Operations on fuzzy numbers,” J. Systems Sci., vol. 9, pp. 613-626, 1978
[10] Dubois D, Prade H (1982) towards fuzzy differential calculus. Fuzzy Sets Syst 8:1–7
[11] Sugeno, M., Theory of fuzzy integrals and its applications, Doctoral. Thesis, Tokyo institute of Technology, 1974.
[12] M. Sugeno. Fuzzy measures and fuzzy integrals -a survey. In Gupta, Saridis, and Gaines, editors, Fuzzy Automata and Decision Processes, pages 89-102.1977.
[13] J. H. He, some asymptotic methods for strongly nonlinear equation, Int. J. Mod. Phys. 20 (20) 10 (2006), 1144-1199.
[14] He JH (2000) A coupling method of a homotopy technique and a perturbation technique for non-linear

problems. Int J Non-linear Mech 35(1):37–43
[15] He JH (2003) Homotopy perturbation method: a new nonlinear analytical technique. Appl Math Comput 135(1):73–79
[16] He JH (2004) Comparison of homotopy perturbation method and homotopy analysis method. Appl Math Comput156 (2):527–539.