Consolidation Characteristics Based on a Direct Analytical Solution of the Terzaghi Theory

Mohammed Shukri Al-Zoubi [1)]

[1)] Assistant Professor of Civil Engineering, Civil and Environmental Engineering Department, Faculty of Engineering, Mutah University, Jordan, malzoubi@mutah.edu.jo

ABSTRACT

A new method is proposed for evaluating both the coefficient of consolidation c_v and end of primary settlement δ_p based on a direct analytical solution of the Terzaghi theory. In this study, the c_v value is shown to be inversely proportional to the δ_p value. The proposed method utilizes both the early and later stages of consolidation (i.e., the entire range of consolidation) for the evaluation of both parameters. The proposed method requires four consolidation data points; two points for back-calculating the initial compression and two points for extrapolating the δ_p value. Results of oedometer tests on three clayey soils show that the c_v and δ_p values of the proposed method are quite comparable to those of the Casagrande method but generally lower than those of the Taylor method.

KEYWORDS: Terzaghi theory, Taylor, Casagrande, Coefficient of consolidation, End of primary settlement.

INTRODUCTION

The computation of settlement and rate of settlement requires the determination of the coefficient of consolidation (c_v) and end of primary settlement (EOP δ_p). Numerous methods have been developed based on the Terzaghi theory for evaluating both the coefficient of consolidation and end of primary settlement (e.g., Taylor, 1948; Casagrande and Fadum, 1940; Scott, 1961; Cour, 1971; Parkin, 1978; Sivaram and Swamee, 1977; Sridharan and Rao, 1981; Parkin and Lun, 1984; Sridharan et al., 1987; Robinson and Allam, 1996; Robinson, 1997 and 1999; Mesri et al., 1999a; Feng and Lee, 2001; Al-Zoubi, 2004a and 2004b; Singh, 2007).

The Casagrande method (the logarithm of time method; Casagrande and Fadum, 1940) determines the coefficient of consolidation at 50% consolidation; this method requires the determination of the initial and final compressions corresponding to 0 and 100% consolidation, respectively. The determination of the 100% consolidation is achieved by utilizing the similarity in the shape of the theoretical and experimental curves without the direct use of the theory. The Casagrande method yields EOP settlement that is almost identical to those obtained from pore water pressure measurements (Mesri, 1999b; Robinson, 1999). On the other hand, the Taylor method (the square root of time method; Taylor, 1948) determines the c_v value at 90% consolidation and requires the determination of the initial compression that corresponds to 0% consolidation. The determination of the 90% consolidation is obtained by the direct use of the Terzaghi theory where the ratio of the secant slopes at 50% to that at 90% consolidation is assumed constant and the same for both the observed and theoretical

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compression – square of time relationships as will be shown later in this paper. Both the Casagrande and Taylor methods utilize the same theoretical basis for evaluating the initial dial gauge reading that corresponds to 0% consolidation (Al-Zoubi, 2004a), but these two methods differ in the way the end of primary consolidation is identified. The Taylor method generally yields lower δ_p values and higher c_v values as compared to the Casagrande method.

In general, different values for the coefficient of consolidation and/or the end of primary consolidation have been obtained using the various existing methods developed based on the Terzaghi theory that assumes constant coefficient of consolidation. These differences in δ_p and c_v values obtained from these methods for a particular pressure increment may be attributed to one or more of the following factors: (a) variations in c_v that may increase, decrease or remain constant during a pressure increment (Al-Zoubi, 2004a and b), (b) resistance of a clay structure to compression (Mesri et al., 1994), (c) recompression-compression effects due to spanning preconsolidation pressure σ'_p (Mesri et al., 1994), (d) duration of pressure increment including secondary compression (Murakami, 1977); long duration of pressure increments may produce recompression-compression effects similar to those of preloading (Mesri et al., 1994), (e) procedure adopted to obtain δ_p (the range of primary consolidation or part of this range or at least a point within this range must be matched with the Terzaghi theory to be able to estimate the coefficient of consolidation) and (f) the existing methods may involve additional assumptions to those of the Terzaghi theory.

In this paper, a new method is proposed in order to improve the estimation of the end of primary settlement (δ_p) and the coefficient of consolidation (c_v). The proposed method is compared to the Taylor and Casagrande methods utilizing results of oedometer tests on three clayey soils. The basic properties of these three soils are given in Table 1. As can be seen from Table 1, the soils utilized in the present study cover a relatively wide range of liquid limit and plasticity characteristics; the liquid limit for these soils ranges from 29% to 108% and the plasticity index ranges from 12% to 66%.

THE PROPOSED METHOD

The actual theoretical one-dimensional consolidation relationship between average degree of consolidation U and the time factor T obtained from the Terzaghi theory may, depending on the range of U, be given by the following two expressions (Terzaghi, 1943; Olson, 1986):

For $U \le 52.6\,\%$

$$U = \sqrt{\frac{4}{\pi}}\sqrt{T} \qquad (1)$$

For $U \ge 52.6\,\%$

$$Ln(1-U) = Ln\frac{8}{\pi^2} - \frac{\pi^2}{4}T \qquad (2)$$

In the Terzaghi theory, the consolidation time t is defined in terms of time factor T, maximum drainage path H_m and coefficient of consolidation c_v as follows:

$$t = \frac{T\,H_m^2}{c_v} \qquad (3)$$

On the other hand, the settlement δ_t may be expressed in terms of the average degree of consolidation U and EOP settlement δ_p by the following expression:

$$\delta_t = U\ \delta_p \qquad (4)$$

where $\delta_p = d_p - d_o$; d_p is the dial reading at the end of primary consolidation and δ_t is the settlement at time t during consolidation and is equal to $d_t - d_o$; d_t is the dial reading at time t and d_o is the dial reading corresponding to 0% consolidation, which may be given as follows (e.g., Al-Zoubi, 2004a):

$$d_o = \frac{d_{t2} - d_{t1}\sqrt{t_2/t_1}}{1-\sqrt{t_2/t_1}} \qquad (5)$$

where d_{t1} and d_{t2} are the dial gauge readings at time t_1 and time t_2, respectively, and are selected such that these two points are on the initial linear portion of the $d_t - \sqrt{t}$ curve. This is the same basis utilized by the Casagrande and Taylor methods since the three methods utilize the same equation (Eq. 1) for obtaining the initial compression d_o. Hence, the Taylor and Casagrande methods are similarly affected by the factors that influence the initial portion of the consolidation curve.

Table 1: The basic properties of the three soils utilized in the present study.

Soil	**Particle size**			**Liquid Limit (%)**	**Plastic Limit (%)**	**Specific Gravity**
	Sand (%)	**Silt (%)**	**Clay (%)**			
Azraq Green Clay (AGC)	8	23	69	108	42	2.76
Chicago Blue Clay (CBC)*	4	64	32	29	17	2.73
Madaba Clay (MDC)	14	41	45	55	25	2.78

* These basic properties for the Chicago Blue Clay were obtained by the Author; whereas the consolidation data were obtained from Taylor (1948).

Table 2: Results of the proposed method using consolidation data obtained from Taylor (1948), page 248.

Time (min)	**0**	**0.25**	**1**	**2.25**	**4**	**6.25**	**9**	**12.25**	**16**	**average**	**COV(*)** (%)
Dial Reading ($\times 10^{-4}$ in) 1 in = 25.4 mm	**1500**	**1451**	**1408**	**1354**	**1304**	**1248**	**1197**	**1143**	**1093**		
m (mm/min$^{-1/2}$) (between any two consecutive points)	-----	-----	-----	0.274	0.254	0.284	0.259	0.274	-----	0.269	4.55
d_0 (25.4 x 10^{-4} mm)	-----	-----	-----	1516	1504	1528	1503	1521	-----	1514	0.72
Time (min)	**20.25**	**25**	**30.25**	**36**	**42.25**	**60**	**100**	**200**	**400**	**1440**	
Dial Reading ($\times 10^{-4}$ in) 1 in = 25.4 mm	**1043**	**999**	**956**	**922**	**892**	**830**	**765**	**722**	**693**	**642**	
settlement δ_{ti}	1.201	1.313	1.422	1.509	1.585	1.742	1.908	2.017	2.090	2.220	
EOP δ_{pi}	1.674	1.717	1.780	1.791	1.806	1.864	1.911	2.018	2.092	2.220	
Coefficient of consolidation c_v / H_m^2 (10^{-3} min^{-1})	21.1	20.1	18.7	18.4	18.1	17.0	16.2	-----	-----	-----	

(*) **COV** is the coefficient of variation.

Table 3: Comparison of δ_p and c_v values of the Proposed, Taylor and Casagrande methods using the consolidation data of Table 2.

Method	EOP settlement δ_p (mm)	c_v / H_m^2 (x 10^{-3} min^{-1})
Taylor	1.846	17.4
Casagrande	1.927	15.9
Proposed (this study)	1.921	16.0

However, these methods differ in the way by which the primary consolidation range (or EOP δ_p) is obtained as shown later.

Based on Eqs. 1, 3 and 4, the coefficient of consolidation may be given by the following expression (Al-Zoubi, 2004a):

$$c_v = \frac{\pi}{4}\left(\frac{m H_m}{\delta_p}\right)^2 \tag{6}$$

where m is the slope of the initial linear portion of the observed $\delta_t - \sqrt{t}$ curve that may be computed as follows:

$$m = \frac{\delta_{t2} - \delta_{t1}}{\sqrt{t_2} - \sqrt{t_1}} = \frac{d_{t2} - d_{t1}}{\sqrt{t_2} - \sqrt{t_1}} \tag{7}$$

Equation 6 shows that the c_v value is dependent on both the value of the slope m as well as that of the end of primary settlement δ_p. Equation 6 shows also that the coefficient of consolidation can not be obtained from only the initial portion because Eq. 6 involves three unknown values (i.e., d_0, d_p and m; where $\delta_p = d_p - d_0$). Therefore, the value of d_p must be determined from the later stages of consolidation (theoretically, from the range of $U \geq 52.6\,\%$) while both d_0 and m can be obtained from the initial portion of the $\delta_t - \sqrt{t}$ curve. At least one additional data point (d_{ti}, t_i) must be selected from the consolidation data for estimating the end of primary settlement δ_p in addition to the two data points (d_{t1}, t_1) and (d_{t2}, t_2) required for obtaining the initial compression d_0 using Eq. 5 and the slope m of the initial linear portion of the observed $\delta_t - \sqrt{t}$ curve using Eq. 7.

A theoretical expression for estimating the EOP settlement δ_p may be obtained by combining Eqs. 2 through 6 as follows:

$$f(\delta_p, \delta_{ti}, t_i) = Ln(\delta_p - \delta_{ti}) - Ln\delta_p + 1.938\frac{m^2}{\delta_p^2}t_i = 0 \tag{8}$$

where $\delta_{ti} = d_{ti} - d_0$ is the settlement at time t_i and $\delta_p = d_p - d_0$.

In order to solve Eq. 8 for δ_p, three data points {i.e., (d_{t1}, t_1), (d_{t2}, t_2) and (d_{ti}, t_i)} must be selected from the consolidation data. The first two data points (d_{t1}, t_1) and (d_{t2}, t_2) are required for obtaining the initial compression d_0 and the slope m as described above. The third data point (d_{ti}, t_i) can be taken at any time beyond the initial linear portion (i.e., the subscript i refers to any data point in the range of $U \geq 52.6$ %).

The solution of Eq. 8 using the selected three data points requires iterations for obtaining the EOP settlement δ_p (and then obtaining the coefficient of consolidation c_v using Eq. 6). However, this solution can be obtained graphically or numerically by using any method for finding the roots of an equation. The © Microsoft Excel Solver was, however, utilized in this study for solving Eq. 8.

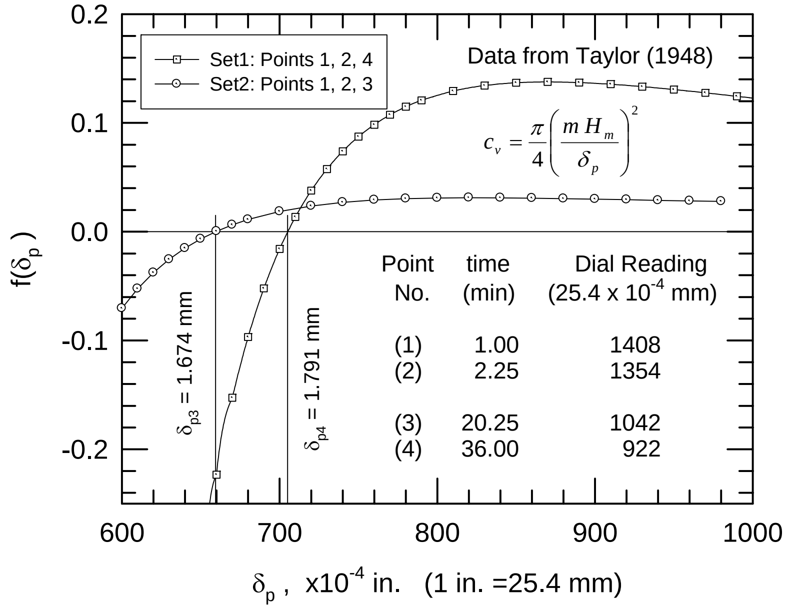

Figure (1): Graphical solution of Eq. 8 using two sets of selected data points.

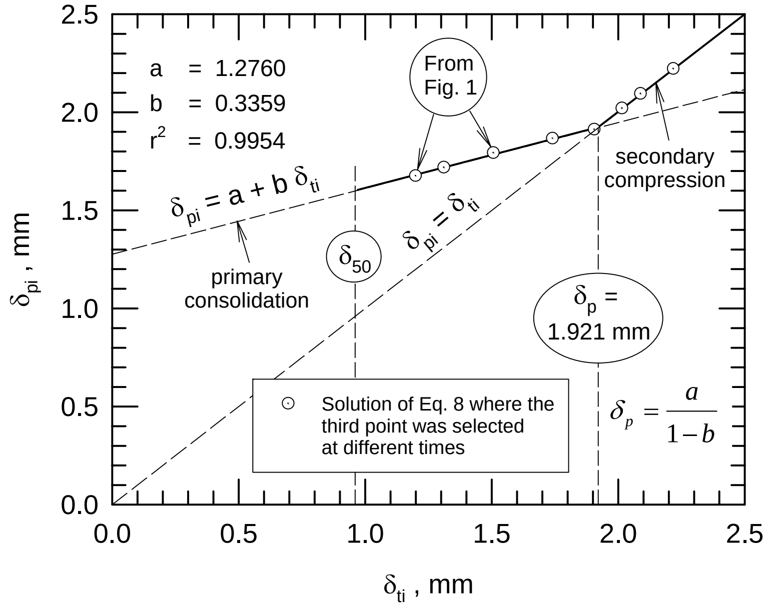

Figure (2): Estimates of EOP settlement δ_{pi} obtained from the analytical solution using Eq. 8 a function of δ_{ti}.

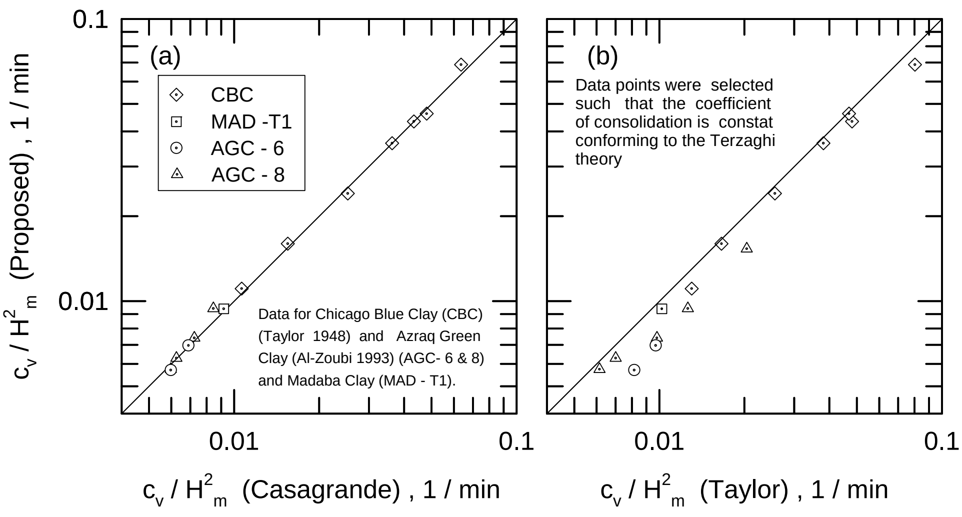

Figure (3): Comparison of the values of the coefficient of consolidation using the proposed, Taylor and Casagrande methods.

Figure 1 shows the graphical presentation of Eq. 8 for two sets of consolidation data that are listed in the figure. The first set is represented by the data points 1, 2 and 3; while the second set is represented by the data points 1, 2 and 4. The complete set of data as obtained from Taylor (1948) is listed in Table 2. Figure 1 shows that the δ_p values that make $f(\delta_p)=0$ are equal to 1.674 and 1.791 mm for first and second sets, respectively; these values were obtained using ©Microsoft Excel Solver. Based on these results, it can be seen that the δ_p value depends on the selected third point (d_{ti}, t_i). The solution of Eq. 8 was also repeated using other different data points (δ_{ti}, t_i) in order to examine the effect of the selection of the third point (δ_{ti}, t_i) on the estimated δ_p value and to assess the relationship between the estimated δ_p value and the selected δ_{ti} value. These estimated δ_{pi} values are listed in Table 2 and are also plotted against the δ_{ti} value in Fig. 2. The subscript i is added to δ_p because of the dependence of δ_p on the δ_{ti} value.

Table 2 and Fig. 2 confirm that the estimated δ_{pi} value depends on the selected third point (δ_{ti}, t_i); a similar trend was reported by Sivaram and Swamee (1977). This dependence of the estimated δ_{pi} value on the selected δ_{ti} value can be attributed to the fitting of the observed time-compression curve in which the actual time to EOP consolidation has a definite value (i.e., t_p) to the Terzaghi theory in which the theoretical time to EOP consolidation is infinity.

Figure 2 interestingly shows that the estimated δ_{pi} value increases linearly with the increase of δ_{ti}. This linear relationship between δ_{pi} and δ_{ti} in the primary consolidation range can be expressed as follows:

$$\delta_{pi} = a + b\delta_{ti} \quad (9)$$

where a and b are the intercept and slope of this linear relationship, respectively.

On the other hand, Table 2 and Fig. 2 show that as the time-compression curve goes into in the secondary compression range the obtained δ_{pi} value becomes practically equal to the assumed δ_{ti} value. This relationship in the secondary compression range (represented by the 45^o line in Fig. 2) can be given by the following expression:

$$\delta_{pi} = \delta_{ti}. \quad (10)$$

Based on the above, it is suggested to obtain the EOP settlement from the point of intersection between the two straight lines that represent the primary consolidation range (Eq. 9) and secondary compression range (Eq. 10). Hence, the EOP settlement δ_p for a given pressure increment may be obtained from the following expression:

$$\delta_p = \frac{a}{1-b} \qquad (11)$$

Equation 11 shows that the EOP settlement δ_p can be obtained from the linear relationship between δ_{pi} and δ_{ti} in the primary consolidation range by extrapolation without the need to continue the test into the secondary compression range as demonstrated in Fig. 2, because δ_p is only a function of a and b that can be obtained from the primary consolidation range. This extrapolation requires at least two data points in the range $U \geq 526\%$ to obtain the EOP settlement.

Hence, the coefficient of consolidation using the proposed method requires at least four data points to be selected such that the first two points (theoretically, in the range $U \leq 526\%$) are utilized for back-calculating the initial compression d_0 and the second two points (theoretically, in the range $U \geq 526\%$) are utilized for extrapolating the EOP settlement. Results of oedometer tests on specimens of three clay soils are utilized for evaluating the proposed method.

For the consolidation data of Table 2 and Fig. 2, the EOP settlement δ_p obtained using the proposed method (δ_p = 1.921 mm) is quite similar to that of the Casagrande method (δ_p = 1.927 mm) but higher than that of the Taylor method (δ_p = 1.846 mm). The δ_p and c_v values of the proposed, Taylor and Casagrande methods are listed in Table 3, which shows that the c_v value of the proposed method is in good agreement with that of the Casagrande method but lower than that of the Taylor method. Figure 3, which depicts results of incremental oedometer tests conducted on three specimens of the three clayey soils, supports this observation. Figure 3(a) shows that the c_v values obtained from the proposed method are quite similar to those of the Casagrande method; whereas Fig. 3(b) shows that the c_v values obtained from the proposed method are generally lower than those of the Taylor method. It should be pointed out that Fig. 3 includes only the data points for which the coefficient of consolidation c_v was observed to be constant with consolidation pressure σ'_{vc} conforming to the Terzaghi theory.

Figures 3 (a) and (b) show also that the Casagrande method c_v values are generally lower than those obtained using the Taylor method. This observation is consistent with the reported trend for the Taylor and Casagrande methods in the geotechnical engineering literature (e.g., Lambe and Whitman, 1969; Hossain, 1995; Sridharan and Prakash, 1995; Robinson, 1999).

Based on the above (for an example, see Table 3), the similarity in the c_v values of the proposed and Casagrande methods is observed to be associated with similarity in the δ_p values. Also, the discrepancy in the c_v values of the proposed and Taylor methods is associated with discrepancy in the δ_p values. In other words, when these methods predicted very similar ranges for the primary consolidation (that corresponds to the Terzaghi theory) or similar δ_p values, the c_v values estimated from these methods were observed to be similar particularly for the cases in which the coefficient of consolidation was constant conforming to the Terzaghi theory. On the other hand, when these methods predicted different values for the EOP δ_p, the c_v values estimated from these methods were observed to be different. Equation 6, which explicitly relates the c_v value to the EOP δ_p value, supports these observations. This observation emphasizes that the identification of the initial and final compressions (and thus δ_p) are of primary importance for a realistic determination of the coefficient of consolidation (Olson, 1986; Robinson, 1999).

SUMMARY AND CONCLUSIONS

In the present study, a new method is developed for evaluating the coefficient of consolidation and end of primary settlement based on a direct solution of the Terzaghi theory. This new method determines the coefficient of consolidation utilizing the entire range of consolidation (i.e., the proposed method utilizes both the early and later stages of consolidation). The proposed method requires four data points; two data points are required in the early stages of consolidation ($U \leq 52.6\,\%$) for back-calculating the initial compression and two data points in the later stages of consolidation ($U \geq 52.6\,\%$) for extrapolating the end of primary settlement.

Results of oedometer tests on three clayey soils, which cover a relatively wide range of liquid limit and plasticity characteristics (the liquid limit for these three soils ranges from 29% to 108% and the plasticity index ranges from 12% to 66%), show that the c_v and δ_p values of the proposed method are quite similar to those of the Casagrande method. These results also show that the c_v values of the proposed method are generally lower than those of the Taylor method the δ_p values of the proposed method are generally higher than those of the Taylor method.

The present study confirms that the identification of the experimental range of primary consolidation that corresponds to the Terzaghi theory is of primary importance for a realistic determination of the coefficient of consolidation using the Terzaghi theory. Also, it is observed that the differences in the estimates of c_v values using the available methods are primarily due to the differences in δ_p values and not necessarily due to the effects of the initial and secondary compressions as usually stated in the literature.

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