Application of Shock Capturing Method for Free Surface Flow Simulation

Kh. Korichi [1] ***and A. Hazzab*** [1]

[1] Laboratory of Modeling and Methods of Calculation, University Dr Moulay Tahar of Saïda, BP 138 Ennasr Saïda 20002, ALGERIA. E-Mail: kh.korichi@gmail.com

ABSTRACT

The present study is a contribution to free surface flow simulation by numerical resolution of Saint-Venant's equations, which form a nonlinear hyperbolic system. For reasons of stability and accuracy, we apply finite volume method, based on Riemann problem's resolution using shock capturing schemes. Application tests for steady and unsteady flows confirm the capacity of these schemes to maintain stability and precision.

KEYWORDS: Free surface flow simulation, Shallow water equations, Saint-Venant, Finite volume, Riemann problem, Godunov's scheme, Limiters.

INTRODUCTION

The natural phenomena such as floods and inundations, as well as the damages which produce make the subject of several researches. Chow (1959), Henderson (1966) and Cunge et al. (1980) explored the free surface flow studies on rivers and in particular on channels. But the great improvement returns to Barré de Saint-Venant (1871) who established, for the first time, his one-dimensional model which governs flows in shallow water. It was successfully applied to simulate several phenomena such as; flood inundations (Ying et al., 2004), dambreaks (Fennema and Chaudhry, 1987) and tsunami waves (George, 2006).

Mathematically, such phenomena are governed by conservation laws which are represented by systems of non-linear hyperbolic PDEs. The nonlinearity in the Saint-Venant system presents shock and rarefaction waves. This theory and its approximation have been well detailed by Le Veque (Garcia-Navarro and Saviron, 1987), Lax-Wendroff (George, 2006) and Gallouet (George, 2004). Given the importance of these studies, several techniques have been developed in recent decades, using a variety of numerical methods. Among them, shock capturing finite volume schemes, based on Riemann problem solution, are developed in this modest work.

Fabre (2001) explores this study by applying the one-dimensional Saint-Venant model. However, Le Veque (1998) illustrates various possible situations for analytical solutions of Riemann problems and the numerical treatment of shock and rarefaction waves.

Thus Riemann's approach is explored by Roe (1981) using first-order Godunov's scheme (Godunov, 1959) and second order Lax's scheme (Lax, 1972) which is a scheme without slope limiters. Van Leer (1977) and Warming and Beam (1975) introduce these slope limiters to prove the accuracy of such schemes.

Based on Fennema's and Chaudhry's works (Fennema and Chaudhry, 1987), Liggett and Cunge (1975) and MacDonald et al., (1997), we present various tests of steady and unsteady flows. Dambreak problems on dry bottom and wet bottom are studied. Thereafter, we treat the source terms represented by a flow over a concave bed.

MATHEMATICAL MODEL

Free surface flows, as their name indicates, are runoff flowings under the influence of gravity. They are managed by a system of PDEs (shallow water equations) (cf. Figure (1)). This one-dimensional model rises from two principles of conservation; that of mass:

Accepted for Publication on 15/10/2010.

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0 \qquad (1)$$

and momentum:

$$\frac{\partial hu}{\partial}+\frac{\partial}{\partial x}\left(\frac{(hu)^2}{h}\right)+\frac{\partial}{\partial x}\left(g\,\frac{h^2}{2}\right)=-ghb(x)_x \qquad (2)$$

where: *hu* is the unit discharge; *h* is the average depth of water; *g* is the gravitational acceleration and *b(x)* is the bed slope.

One-dimensional vectorial form of this conservation law, with source term, is written as follows:

$$q_t + f(q)_x = \psi(x,t) \qquad (3)$$

where *q* is the conservative variables' vector; *f* (*q*) is the flux vector and ψ is the source term vector. Knowing that:

$$q=\begin{pmatrix} h \\ hu \end{pmatrix},\quad f(q)=\begin{pmatrix} hu \\ hu^2+\frac{1}{2}gh^2 \end{pmatrix} \quad \text{and} \quad \psi=\begin{pmatrix} 0 \\ -ghb(x)_x \end{pmatrix}$$

u is the average velocity,

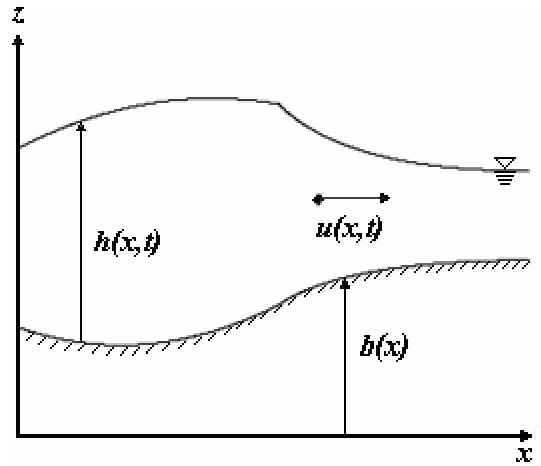

Figure 1: Different parameters in shallow water model

Shallow water equations represent a nonlinear hyperbolic system of PDEs, where the Jacobian matrix

$$\mathrm{J} = f'\left(q\right)=\begin{pmatrix} 0 & 1 \\ -u^2+gh & 2u \end{pmatrix} \qquad (4)$$

is diagonalisable with real and distinct eigenvalues

$\lambda^1=u-\sqrt{gh}$, $\lambda^2=u+\sqrt{gh}$.

RESOLUTION OF SAINT-VENANT EQUATIONS

A particular characteristic of hyperbolic systems is the existence of discontinuous solutions, such as shock waves. Several approaches are used to treat these cases. In the present study, we explore that based on Riemann problem's resolution.

Riemann Problem

Riemann problem represents a system of PDEs with special initial conditions which are constant on both sides of discontinuity, (cf. Figure (2)). Here, discontinuity is localised in (x = 0).

If we neglect the source term, the Riemann problem is written as:

$q_t + f(q)_x = 0$ with;

$$q(x,0)=\begin{cases} q_l & \text{if } x<0 \\ q_r & \text{if } x>0 \end{cases} \qquad (5)$$

Therefore, the resolution of Riemann problem is to find an intermediate state q_m between the left state q_l and the right state q_r.

In Saint-Venant's system, we consider four configurations (Fabre, 2001), (cf. Figure (3)):

1. Two centred shock waves (from origin) which correspond to a collision of two waves.
2. Two centred rarefaction waves (from origin) which correspond to a separation of two waves.
3. One rarefaction wave propagates towards the left side and one shock wave propagates towards the right side which represents the configuration of a dambreak.
4. One shock wave propagates towards the left side and one rarefaction wave propagates towards the right side which represents a configuration of a hydraulic jump.

The first case (two shock waves) corresponds to the solutions of Saint-Venant's equations with initial conditions;

$$h(x,0),\quad u(x,0)=\begin{cases} u_l & \text{if } x<0 \\ -u_l & \text{if } x>0 \end{cases} \quad \text{with } u_l>0$$

The solution is represented by two shock waves emanating from the origin, which create a stationary intermediate state, (cf. Figure (4)). The determination of this intermediate state is calculated by applying the jump's condition of Rankine-Hygoniot:

$$s(q_*-q)=f(q_*)-f(q) \qquad (6)$$

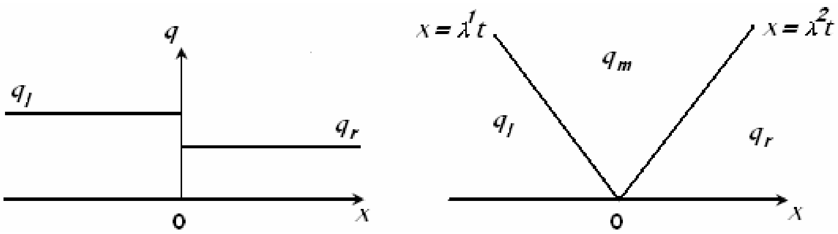

Figure 2: Illustration of Riemann's problem

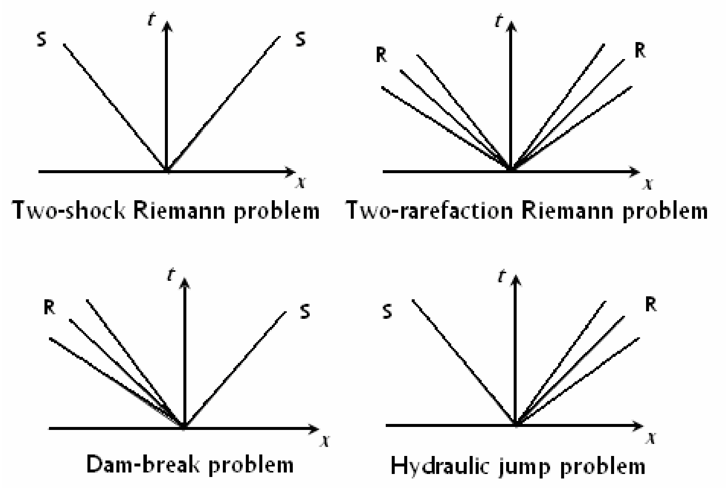

Figure 3: Four configurations of Riemann problem for Saint-Venant's equations

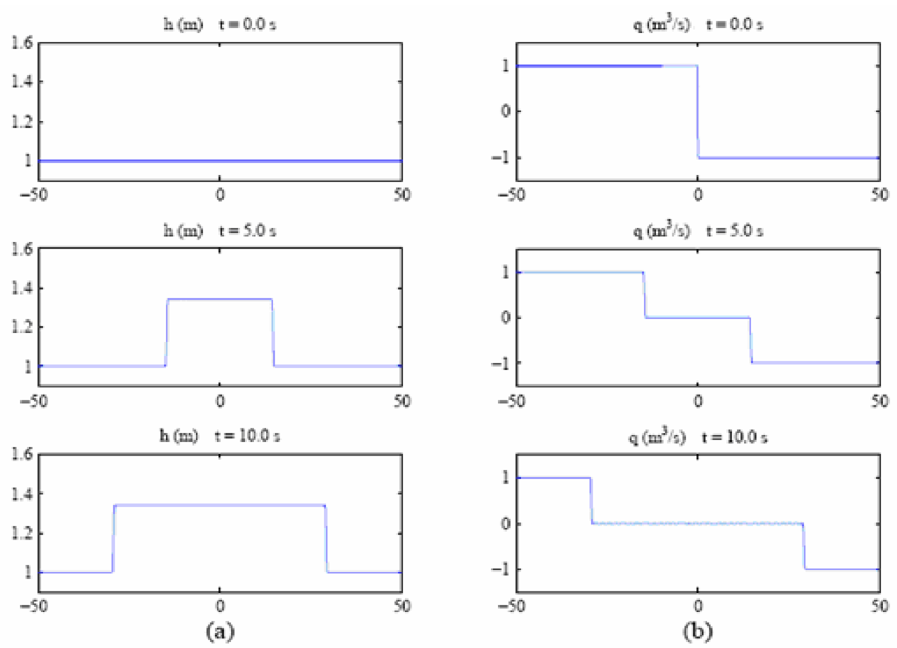

Figure 4: Riemann problem solutions for Saint-Venant equations with u_l = - u_r > 0 (a) depth, (b) discharge

This results in a system of two equations;

$$\begin{cases} s\left(h*-h\right)=h*u*-hu \\ s\left(h*u*-hu\right)=h*u_*^2-hu^2+\frac{1}{2}g\left(h_*^2-h^2\right) \end{cases} \qquad (7)$$

Graphically, this state represents the intersection of two Hugoniot's curves plotted from the left and right states, (cf. Figure (5)).

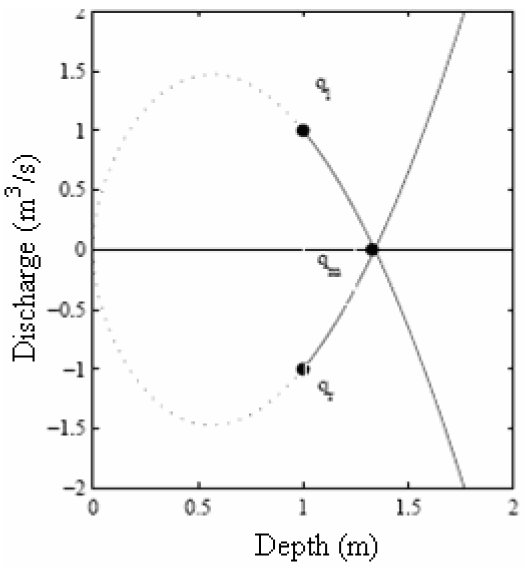

Figure 5: Construction of two-shock's solution for Riemann problem of Saint-Venant equations

In the second case (two rarefaction waves), the problem is posed with the following initial conditions;

$$h\left(x,0\right), \quad u\left(x,0\right)=\begin{cases} u_l & \text{if } x<0 \\ -u_l & \text{if } x>0 \end{cases} \text{ with } u_l<0$$

This solution is schematized as (cf. Figure (6)).

This solution is calculated by applying the Riemann invariants relation such as;

$$w^1=u+2\sqrt{gh}$$
$$w^2=u-2\sqrt{gh} \qquad (8)$$

Intermediate sate is given by:

$$u_m=u_l+2\left(\sqrt{gh_l}-\sqrt{gh_m}\right)$$
$$u_m=u_l-2\left(\sqrt{gh_r}-\sqrt{gh_m}\right) \qquad (9)$$

Graphically, the relation $q=f(h)$ is called the integrals curve and the intermediate state q_m represents the intersection of the two integral curves, (cf. Figure (7)).

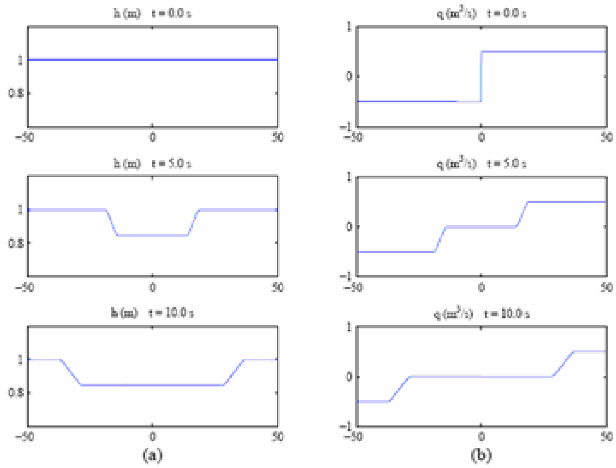

Figure 6: Riemann problem solutions for Saint-Venant equations with $u_l = - u_r < 0$ (a) depth, (b) discharge

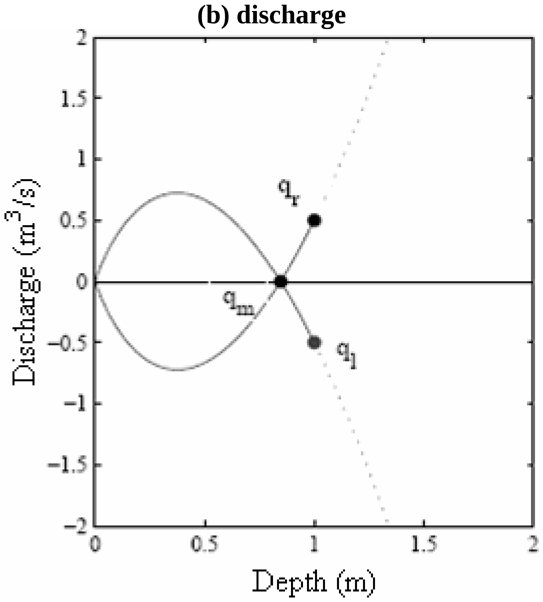

Figure 7: Construction of two-rarefaction waves solution for Riemann's problem of Saint-Venant equations

For the general case (rarefaction and shock waves) which illustrates a dambreak problem with the following initial conditions:

$$h\left(x,0\right)=\begin{cases} h_l & \text{if } x<0 \\ h_r & \text{if } x>0 \end{cases} \text{ with } \left(u,0\right)=0,$$

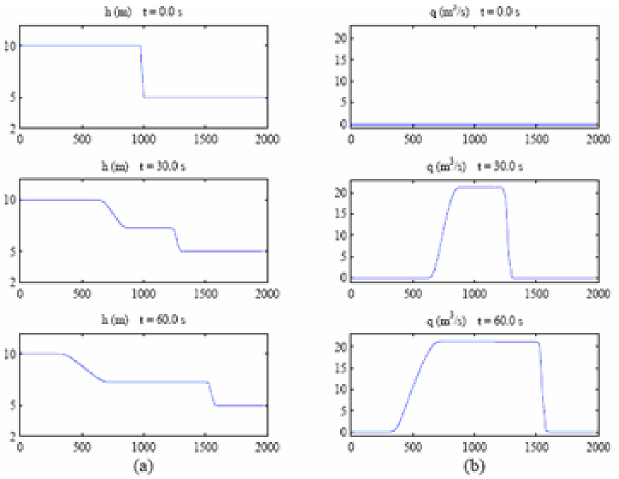

Figure 8: Riemann problem's solutions for a dam break (a) depth, (b) discharge

a shock wave propagates towards the right side and a rarefaction wave propagates towards the left side, (cf. Figure (8)), therefore the solution is obtained by applying Rankine-Hugoniot's relation for the shock wave and Riemann invariant's relation for the rarefaction wave, (cf. Figure (9)).

The treatment of the Riemann problem is made more flexible thanks to specific numerical methods named shock capturing methods which we will present hereafter.

Finite Volume Method (FVM)

Among all the numerical techniques, the finite volume method is well adapted to treat conservation laws, hyperbolic and nonlinear problems such as Saint-Venant's equations by applying the Riemann solver. This method is based on the discretization of the integral form, by subdividing the domain in a number of finite volumes (cells) (Le Veque, 2004). In each cell, the integral relations are applied locally and the exact conservation in each cell is realized, (cf. Figure (10)).

Each cell is defined by: $C_i = \left(x_{i-1/2}, x_{i+1/2}\right)$

As shown in Figure (10), the value Q_i^n represents the approximation of the average value on the interval (i) at time t_n, we write:

$$Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)dx \equiv \frac{1}{\Delta x} \int_{C_i} q(x,t_n)dx \qquad (10)$$

The integration of equation (10) in time gives the general form of finite volume method applied to the conservation laws;

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) \qquad (11)$$

where $F_{i-1/2}^n$ is an approximation of the flux at $x = x_{i-1/2}$, so:

$$F_{i-1/2}^n \approx \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f\left(q\left(x_{i-1/2}, t\right)\right)dt \qquad (12)$$

First Order Godunov's Scheme

Classes of methods that provide a stable and consistent approximation to the numerical fluxes (12) are the Godunov-type methods. The original Godunov method (Godunov, 1959) is a first order upwind-type scheme, where Riemann problems are solved at each grid cell interface before each time step to determine the numerical flux for that time step. This approach is based on three following stages:

- Approximation of the solution by a piecewise constant function on each interval (cf. Figure (11));
- Solving in each interface $xi + 1/2$ Riemann problem;
- Evolving the solution by writing:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right]$$

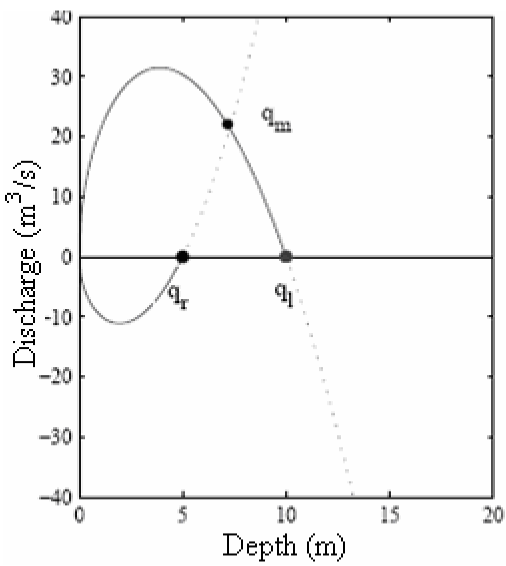

Figure 9: Construction of rarefaction-shock waves solution for Riemann's problem in dambreak problem

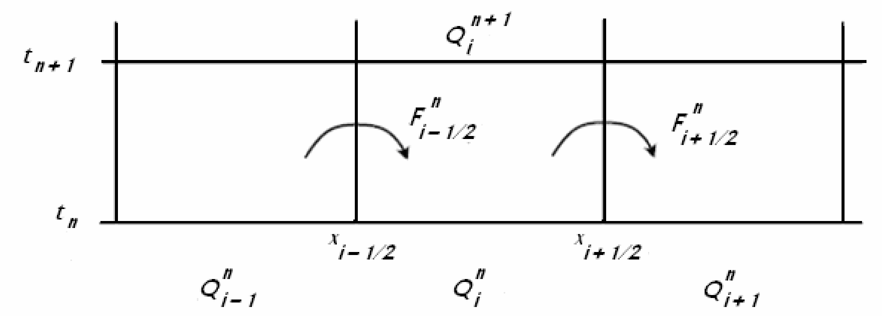

Figure: 10: Principle of the finite volume method

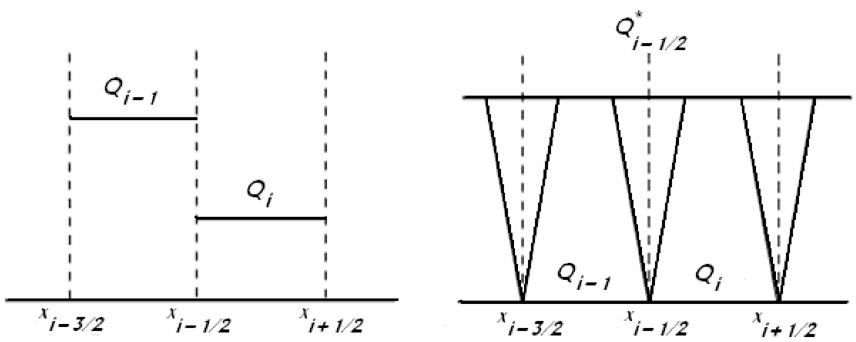

Figure 11: Principle of Godunov's scheme

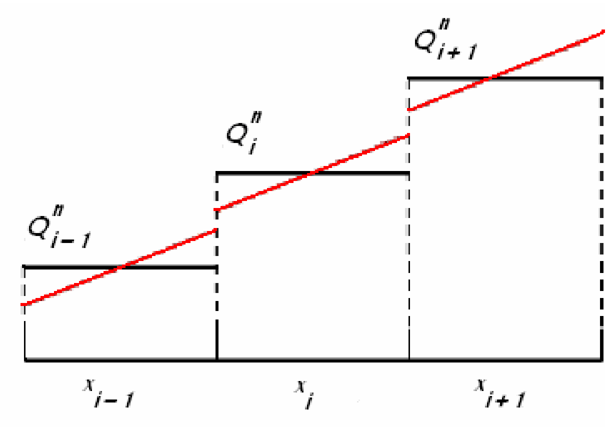

Figure 12: Slope's variable in each cell

High Order Schemes

Godunov's scheme is conservative. It respects the entropy condition, but it is a first order scheme and requires an iterative resolution in each time step. For more accuracy, Van Leer proposed adding a second order term (Van Leer, 1977), whereas Lax Wendroff (1960) and Roe (1981) developed other Riemann solvers (Roe, 1981).

Roe's Solver. It is the most known and used solver. It is an approximate Riemann solver by replacing the exact Jacobian of Saint-Venant's system in each interval by a constant Jacobian. Roe gives this Jacobian as follows:

$$\hat{A}_{i-1/2} = \begin{bmatrix} 0 & 1 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} \end{bmatrix} \tag{13}$$

The depth and the average velocity are given by:

$$\bar{h} = \frac{1}{2}\left(h_{i-1} + h_i\right) \text{ and } \hat{u} = \frac{\sqrt{h_{i-1}}u_{i-1} + \sqrt{h_i}u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}} \tag{14}$$

Flux Limiters: The first order schemes give stable but less accurate solutions. However, the second-order schemes give accurate solutions but oscillate at discontinuities. The idea is, thus, to combine the advantages of these two schemes by applying flux limiters. The purpose of this idea is to improve the behavior of the high order schemes. This technique appeared at the beginning of the Seventies in *the hybrid scheme* of Harten and Zwas (1972). More generally, we can combine any formula of first order flux $F_L(Q_{i-1},Q_i)$ (such as upwind flux) by high order flux $F_H(Q_{i-1},Q_i)$ (such as Lax-Wendroff) to obtain a flux limiter scheme, such as:

$$F_{i-1/2}^{n}=F_L(Q_{i-1},Q_i)+\Phi_{i-1/2}^{n}\left[F_H(Q_{i-1},Q_i)-F_L(Q_{i-1},Q_i)\right]$$

- If $\Phi_{i-1/2}^{n}=0$, it is a first order scheme;
- If $\Phi_{i-1/2}^{n}=1$, it is a high order scheme.

So, the limiter represents a function $\Phi(\theta)$ in which θ is the variable's slope in each cell.

Slope Limiters: These are based on piecewise linear reconstitution (Le Veque, 2004), where the variable represents a piecewise linear function having a slope in each cell, (cf. Figure (12)), and we write:

$$\tilde{q}_i^n(x,t)=Q_i^n+(x-x_i)\,\theta_i^n$$

According to the evaluation of these slopes's functions, we set various schemes that are:

Schemes	***Minmod***	***Superbee***	***Van Leer***	***Monotonized Centered MC***
Slope's variable	$\phi(\theta)=\min\text{mod}(1,\theta)$	$\max\left(0,\min(1,2\theta),\min(2,\theta)\right)$	$\phi(\theta)=\dfrac{\theta+\lvert\theta\rvert}{1+\lvert\theta\rvert}$	$\max\left(0,(\min(1,\theta)/2,2,2\theta)\right)$

Source term Treatment

When the effects of the source term are important (influence of the friction and bottom topography), we have to introduce the source term vector. We talk so about balance laws and we write;

$$q_t+f(q)_x=\psi(q,x,t)$$

where:

$$\psi(q,x,t)=\begin{pmatrix}0\\-ghb_x\end{pmatrix}$$

Fractional Step Method: Le Veque (1998) explores two principal numerical methods to treat these balance laws. These are the quasi-steady method and the fractional step method which splits the balance law into two equations;

- Homogeneous conservation law $q_t+f(q)_x=0$;
- Ordinary differential equation ODE $dq/dt=\psi(q,x)$.

The solution is obtained by alternating between the solutions of these two equations in each time step.

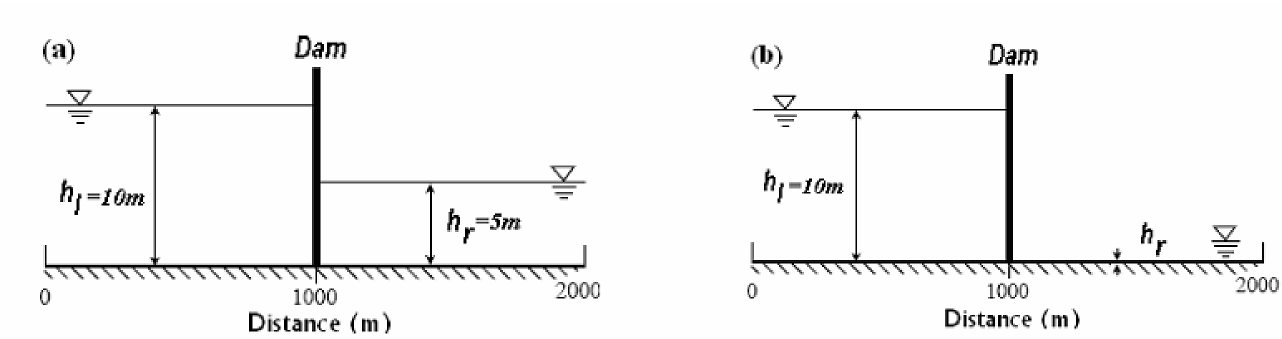

Figure 13: Initial conditions for dambreak test (a) on wet bottom (b) on dry bottom

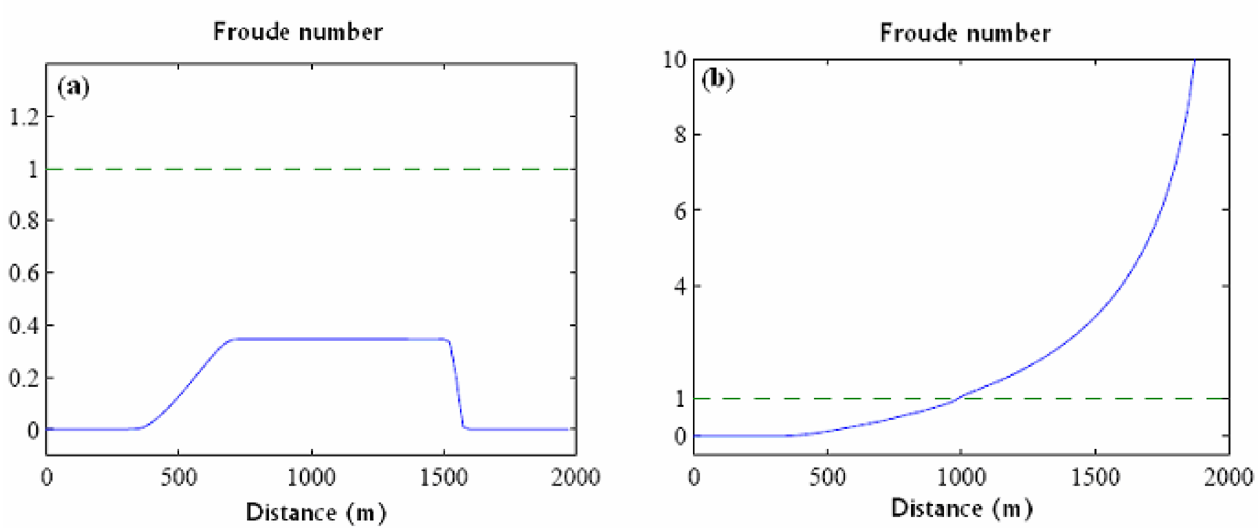

Figure 14: Froude's number analysis for dambreak test (a) on wet bottom (subcritical flow) (b) on dry bottom (supercritical flow)

NUMERICAL TESTS

The essential goal of these tests is to compare the various schemes, first order Godunov scheme and second-order slope limiter schemes. The comparative study involves the examination of steady flows (dambeak) (Ying et al., 2004). Thereafter, we treat the source term which represents a flow over concave bed (Van Leer, 1977; Liggett and Cunge, 1975). We use a finite volume code "Clawpack" (Internet) (Conservation Law Package) by introducing the appropriate changes.

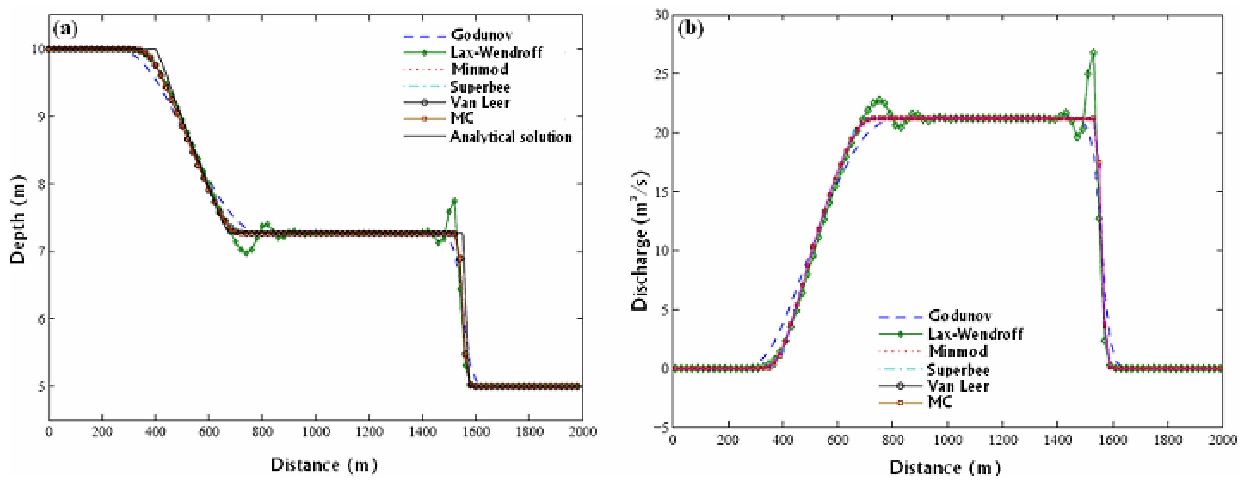

Figure 15: Comparison of different schemes – dambreak on a wet bottom (a) depth, (b) discharge

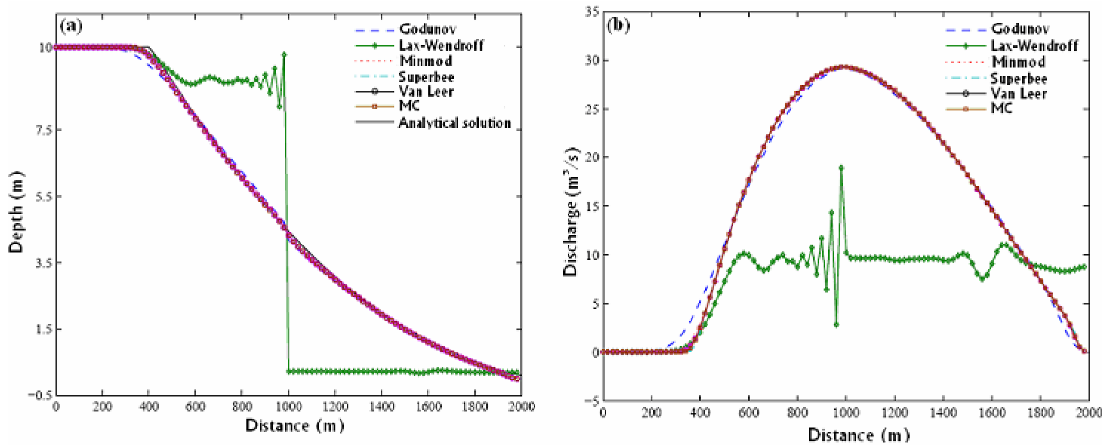

Figure 16: Comparison of different schemes – dambreak on a dry bottom (a) depth, (b) discharge

Dam Break Test

The first test is an examination of a dam break in a flat and horizontal channel (no friction effect), in two situations; on wet bottom and on dry bottom. We use a time step Δt=0.1 during 60s. The length of the domain is x=2000m. The breakpoint is located at x=1000m. The initial upstream depth is h_l = 10 m, while the depth in the downstream is initially h_r=5 m (on wet bottom), and h_r=0.0001m (on dry bottom). Initial unit discharge is null, $hu_l = hu_r = 0$ m^2/s (cf. Figure (13)).

Froude's number analysis indicates that for the flow on a wet bottom, the flow regime is subcritical everywhere, while on a dry bottom, the flow regime is transcritical, since it passes from a subcritical to a torrential regime, (cf. Figure (14)).

After time t = 60s, we observe that Godunov's scheme diffuses near discontinuities, while Lax-Wendroff's scheme oscillates at shock and rarefaction waves, (cf. Figure (15)). Though, in the case of a dambreak on a dry bottom, the Lax-Wendroff's scheme fails completely, (cf. Figure (16)).

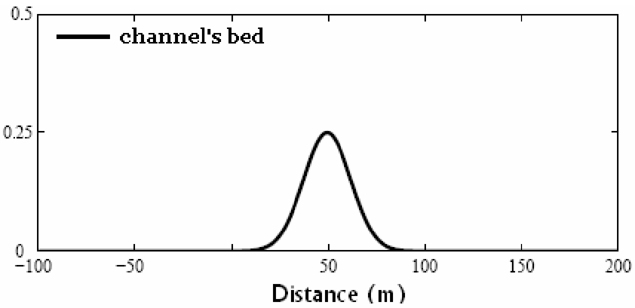

Figure 17: Bottom of the channel

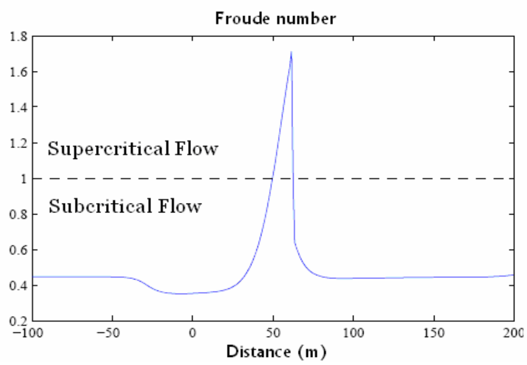

Figure 18: Evolution of Froude's number

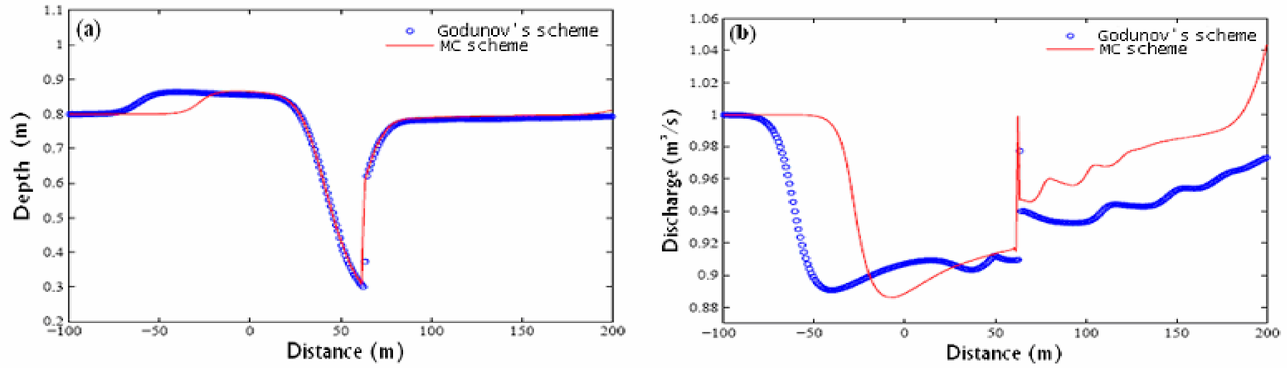

Figure 19: Comparison between Godunov and MC schemes (a) depth, (b) discharge

Flow over Concave Bed Test

In this case, we neglect the friction effect. The bed (cf. Figure (17)) is determined by a function $b(x)$ as;

$$b(x)=\begin{cases}0.25\ \exp\left(-33.75(x/100-0.5)^2\right) & \text{if}\quad 0<x<100\\ 0 & \text{otherwise}\end{cases}\qquad(15)$$

Initial depth and discharge are respectively h=0.8m, hu=1m^2/s. The time of simulation is t = 60s and the length of horizontal channel is L=200m which is splitted into Δx=1m.

At the end of the simulation, we observe that a hydraulic jump was formed at the downstream of the bump.

Froude number's analysis shows that the flow mode is transcritical, since it moves from a subcritical to a torrential mode, (cf. Figure (18)).

Using fractional step method and comparing Godunov's and Monotonized Centered schemes, we note that the Godunov's scheme presents a numerical diffusion. However, discharge solution makes an artificial numerical jump (cf. Figure (19)).

CONCLUSION

In conclusion, we resume that Saint-Venant's equations represent a nonlinear hyperbolic conservation law. Such category has discontinuous solutions. Analytical resolution passes by the treatment of Riemann's problem which represents the system of PDEs with constant initial conditions on both sides of this discontinuity. That operation is difficult. This imposes the use of numerical techniques. In our case, the choice of finite volume method is dictated by the conservative nature of Saint-Venant's equations.

In this study, we explored first order Godunov's scheme, second-order Lax-Wendroff scheme without limiters and second order slope limiters' schemes such as: Minmod, Superbee, Van Leer and Monotonized Centered schemes. From numerical tests and analysis of transitory and stationary flows, we note that slope limiters' schemes are most advantageous. They yield more accurate and stable solutions. Finally, we envisage introducing friction effects to examine real cases.

Acknowledgements

Clawpack software is written by Randall J. Le Veque, University of Washington. But we made modifications for the applications quoted on the top.

Notations

A	-	*Jacobian matrix;*
$b(x)$	-	*Bottom function;*
C_I	-	*Elementary cell;*
F	-	*Flux vector terms through interfaces;*
Fr	-	*Froude's number;*
$f(q)$	-	*Flux function;*
g	m/s^2	*Gravitational acceleration;*
h	m	*Depth of water;*
hu	m^2/ s	*Unit discharge;*
Q	-	*Conservative variables vector;*
q_t	-	*Partial differential in time;*
q_x	-	*Partial differential in space;*
Q_i^n	-	*Approximation of the dependent variable;*
t	s	*Time;*
u	m/s	*Average velocity;*
x	-	*Longitudinal co-ordinate;*
ψ	-	*Source terms vector;*
$\varnothing$	-	*Flux limiter function.*

REFERENCES

Audusse, E. 2004. Modélisation hyperbolique et analyse numérique pour les écoulements en eaux peu profondes. (PhD Thesis), Universite Paris VI Pierre et Marie Curie - Laboratoire Jacques-Louis Lions.

Chow, V. T. 1959. Open-channel hydraulics. McGraw-Hill Book Company, New York, N. Y., 680 pages.

Crossley, A. J. 1999. Accurate and efficient numerical solution for Saint Venant equations of open channel. (PhD Thesis), University of Nottingham.

Cunge, J., Holly, F. and Verwey, A. 1980. Practical aspects of computational river hydraulics. Pitman Publishing, Ltd., 420 pages.

Fabre, J. 2001. Ondes. ENSEEIHT, 155 pages.

Fennema, R. J. and Chaudhry, M. H. 1987. Simulation of one-dimensional dam-break flows. *International Journal of Hydraulic Research*, 25 (1): 41-51.

Gallouet, T. Approximation des systemes de lois de conservations hyperboliques. Rapport technique.

Garcia-Navarro, P. and Saviron, J. M. 1987. Numerical simulation of unsteady flow at open channel junctions. *Journal of Hydraulic Research*, 30 (5), 73: 595-609.

George, D. L. 2006. Finite volume methods and adaptive refinement for tsunami propagation and inundation. (PhD Thesis), University of Washington.

George, D. L. 2004. Numerical approximation of the nonlinear shallow water equations with topography and dry beds: a Godunov-type scheme. (Master Thesis), University of Washington.

Godunov, S. K. 1959. A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics. *Mat. Sb.,* 47: 271-306.

Harten, A. and Zwas, G. 1972. Self-adjusting hybrid schemes for shock computations, *J. Comput.Phys.,* 568.

Henderson, F. M. 1966. Open-channel flow. MacMillan Publishing Co., NewYork, N.Y, 522 pages.

http://www.amath.washington.edu/~claw

Korichi, Kh. 2006. Application de la méthode des volumes finis pour la simulation des écoulements à surface libre. (Master Thesis), C.U. de Mascara, 200 pages.

Lax, P. D. and Wendroff, B. 1960. Systems of conservation laws. *Comm. Pure Appl. Math.,* 13: 217-237.

Lax, P. D. 1972. Hyperbolic systems of conservation laws and the mathematical theory of shockwaves. SIAM Regional Conference Series in Applied Mathematics.

Le Veque, R. J. 1998. Balancing source terms and flux gradients in high resolution Godunov methods, the quasi-steady wave propagation algorithm. AMS.

Le Veque, R. J. 2004. Finite volume method for hyperbolic problems. Cambridge University Press, 580 pages.

Le Veque, R. J. 1990. Numerical methods for conservation laws.

Liggett, J. A. and Cunge, A. 1975. Numerical methods of solution of the unsteady flow equations in open channels. Unsteady Flow in Open Channels, K. Mahmood and V. Yevjevich, Eds.,Water Resources Publications, Fort Collins, Colorado.

Macdonald, I. 1996. Analysis and computaion of steady open channel flow. Thèse (PhD), University of Reading, Department of Mathematics.

Macdonald, I., Baines, M. J., Nicols, N. K. and Samuels, P. G. 1997. Analytic benchmark solutions for open channel flows. *Journal of Hydraulic Engineering*.

Roe, P. L. 1981. Approximate Riemann solvers, parameter vectors and difference schemes. *J. Comput. Phys.,* 43: 357-372.

Saint-Venant, A. J. C. 1871. Théorie du mouvement non-permanant des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. Compte-Rendu à l'académie des Sciences de Paris, (73): 147-154.

Van Leer, B. 1977. Towards the ultimate conservative difference scheme, IV. A new approach to numerical convection. *J. Comput. Phys*., 276-299.

Warming, R.F. and Beam, R.M. 1975. Upwind second-order difference schemes and applications in unsteady aerodynamic flows. In: Proc. AIAA 2nd Computational Fluid Dynamics Conf., Hartford, Conn.

Ying, X., Sam, S., Wang, Y., Abdul, A. and Khan, A. 2004. Numerical simulation of flood inundation due to dam and levee breach. National Center for Computational Hydroscience and Engineering, The University of Mississippi.