Development of Truss Linear Macro-Element

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ABSTRACT

Macro-elements are among the powerful means in reducing the number of equations to be solved in finite element analysis. In the proposed method, several finite truss elements will be transformed into a single element called the macro-element. This is done by equating the potential energy of the macro-element to the potential energy of the equivalent truss finite elements. If the order of the macro-element function corresponds to the order of the structural behavior that it models, an exact solution is achieved.

In this paper, a truss linear macro-element is developed. The developed macro-element was tested and the results were compared with the results of conventional finite element solutions and with closed form solutions. Excellent results were achieved with substantial reduction in the number of equations.

KEYWORDS: Truss finite element, Linear quadrilateral macro-element, Quadratic quadrilateral macro-element.

INTRODUCTION

The analysis of large structural systems using the conventional finite element method is impractical. This is because of the necessity to use a relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of degrees of freedom (d.o.f.) needed to successfully model large systems. One of these methods is to use macro-elements.

In this paper, a truss linear macro-element was developed.

Such macro-elements are based on the transformation of many structural truss elements into a single equivalent macro-element. This is done by preserving the same potential energies of the structure modeled by truss elements and the same structure modeled by macro-elements (Alani, 1983).

FORMULATION OF MACRO-ELEMENTS

In this paper, the formulation of truss linear macro-elements is developed (Alani, 2002). In this modeling, several basic truss elements are combined to form a macro-element (Alani, 2002).

The original structure that consists of many truss finite elements will be replaced by an equivalent model containing one or more macro-elements.

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The macro- elements are assembled, and the analysis is continued in a manner analogous to that used in the finite element method.

BASIC ASSUMPTIONS FOR MACRO-ELEMENT FORMULATION

The formulation is based on the following assumptions (Alani, 2006):

1. The potential and kinetic energies of the original finite elements and the equivalent macro-element models are equal.
2. All the elements that are composing the macro-element must be of the same type, such as: truss elements, beam elements, plane stress elements, plate bending elements…etc.
3. The order of the assumed displacement field of the macro-element is at least of the same order as that of the original finite elements.
4. The macro-element behavior follows the theory which controls the behavior of the structural elements that compose the macro-element.
5. The compatibility requirements for the macro-element are the same as those of the original finite elements.

NECESSARY STEPS NEEDED FOR DEVELOPMENT OF A MACRO-ELEMENT

The necessary steps for the development of a macro-element are as follows (Alani, 2010):

Step (1): Divide the original structure that consists of many finite elements into macro-elements.
Step (2): Select the order of the macro-element displacement function. This step depends on the order and number of the finite-elements composing the macro-element. Accuracy of the results depends greatly on this step.
Step (3): Set-up the stiffness matrices of the finite-elements forming the macro-element.
Step (4): Calculate the local coordinates (S, T) for the nodal points of the finite elements with respect to the macro-element nodes so as to formulate the transformation matrix [T] required in the next step.
Step (5): Formulate the transformation matrix [T], which relates the nodal degrees of freedom of the macro-element to the nodal degrees of freedom of the original structure modeled by finite elements.

The stiffness matrix of each finite element is multiplied by its corresponding transformation matrix to produce the participation of this element in establishing the macro-element stiffness matrix, as will be seen later.

The stiffness matrix of the macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent model as follows:

$$U_o = U_m \tag{1}$$

where:

U_o: The strain energy of the original structure modeled by many finite elements that constitute one macro-element.
U_m: The strain energy of the macro-element.

$$\frac{1}{2}\lfloor q_o \rfloor [SK_o]\{q_o\} = \frac{1}{2}\lfloor q_m \rfloor [K_m]\{q_m\} \tag{2}$$

where:

q_o: Displacement vector of the structure modeled by many finite elements that constitute one macro-element.

q_m: Displacement vector of one macro-element.

$[SK_o]$: The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element.

$[K_m]$: The stiffness matrix of the macro-element.

Let the displacement vector of the original structure (which constitutes one macro-element) $\{q_o\}$ be related to that of the macro-element, $\{q_m\}$ as:

$$\{q_o\} = [T]\ \{q_m\} \tag{3}$$

where [T] is the transformation matrix for the macro-element.

Substituting Eq. (3) into Eq. (2) gives:

$$\begin{aligned} \lfloor q_m \rfloor [T]^T [SK_o][T]\{q_m\} &= \lfloor q_m \rfloor [K_m]\{q_m\} \\ [T]^T [SK_o][T] &= [K_m] \end{aligned} \tag{4}$$

In this solution, matrix $[SK_o]$ is not needed, only $[K_o]$, the stiffness matrix of a finite element bounded by the macro-element is needed. To explain this, let

n : the number of finite elements comprising the macro-element.

$[T_e]$: the finite element transformation matrix.

Every time $[T_e]$ carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite elements under consideration. The transformation for each finite element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of $[K_m]$, and:

$$\sum_{e=1}^{n} [T_e]^T [K_o][T_e] = [K_m] \tag{5}$$

The transformation matrix [T] is simply the evaluation of the shape functions of the macro-element at the nodes of the finite elements. This evaluation is based on local coordinates for the nodal points of the finite elements with respect to the macro-element nodes.

To form a general transformation matrix T_i corresponding to an arbitrary nodal point I of the original structure within a certain macro-element, consider the notation NKI which means that the shape function K of node I of this macro-element is evaluated at point I using its local coordinates within the macro-element, then the transformation matrix will depend on the macro-element type as will be seen later.

Step (6): Construction of the macro-element nodal load vector

The external loadings are applied at nodes of the finite element model.

However, these nodes may not necessarily coincide with the macro-element nodes. It is required to calculate the equivalent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed into an equivalent macro-element structural load vector by equating the external work done on the original structure modeled by finite elements and that of the macro-element model as following:

$$W_o = W_m \qquad (6)$$

where:

W_o: the external work done on the original structure that constitutes one macro-element.

W_m: the external work done on the macro-element.

$$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\} \qquad (7)$$

where:

$\{F_o\}$: The assembled nodal load vector of the finite elements constituting one macro-element.

$\{F_m\}$: The equivalent nodal load vector of the macro-element.

Substituting Eq. (3) into Eq. (7) gives

$$\lfloor q_m \rfloor [T]^T \{F_o\} = \lfloor q_m \rfloor \{F_m\}$$

$$[T]^T \{F_o\} = \{F_m\} \qquad (8)$$

where [T] is the same transformation matrix used in deriving $[K_m]$.

Step (7): Assemble all the macro-element stiffness matrices into a structural stiffness matrix and also construct the macro-element structural load vector.

Step (8): Apply the boundary conditions which will be at the macro-element nodes. Other boundary conditions corresponding to the eliminated nodes of the finite- elements of the original structure will be ignored.

Step (9): Solve for the equivalent model nodal displacements in a straight forward manner.

Step (10): Using the results obtained in step (9), the displacements at any point inside the macro-element may be calculated making use of the macro-element shape function.

Step (11): After the structure is analyzed for nodal displacements, the stresses at selected points in each macro-element may be obtained in the usual manner.

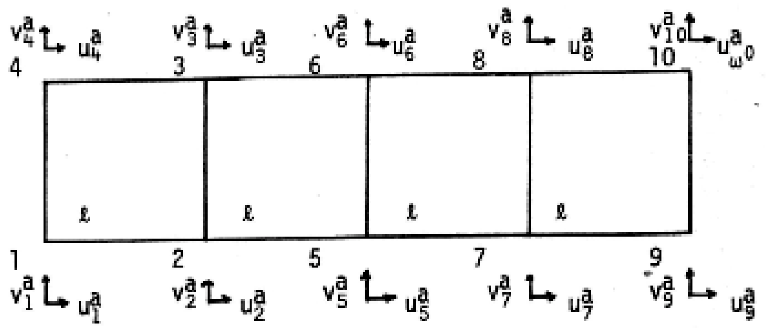

Figure 1: Case a

FORMULATION OF TRUSS LINEAR MACRO-ELEMENTS

Truss problems in two dimensions are plane stress problems in general. In order to demonstrate the formulation, let a simple problem, be considered. The procedure will then be generalized to any problem. In this problem, the structure consists of repeated cells called the repeated elements. Each repeated cell is composed of many truss elements. It is not necessary to have the structure consist of repeated elements as will be shown later. Consider the structure shown in Fig. 1 and denote this system as case **a**.

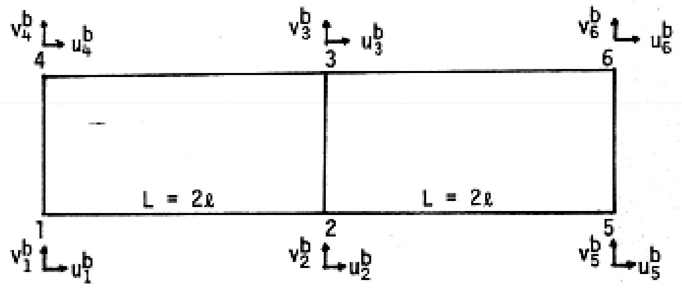

Figure 2: Case b

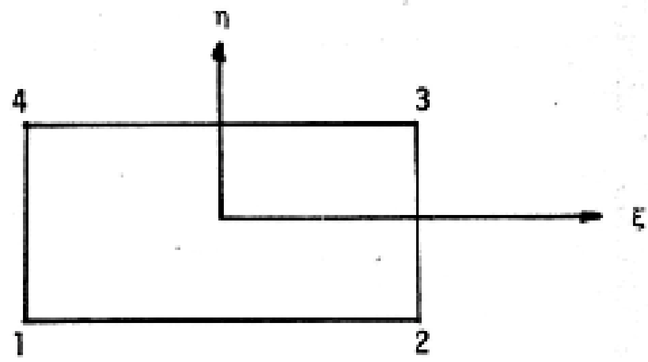

Figure 3: Four-node linear element

This structure will be modeled by two repeated macro-elements as shown in Fig. 2. Denote this system as case **b**.

A single macro-element is isolated as shown in Fig. 3. The order of the element function to be selected depends largely on the configuration of the original structure. It is then evident that a quadrilateral element with quadratic interpolation functions exactly models the structure.

For linear-sided quadrilateral elements, as shown in Fig. 4, the shape functions are (Cook, 1981):

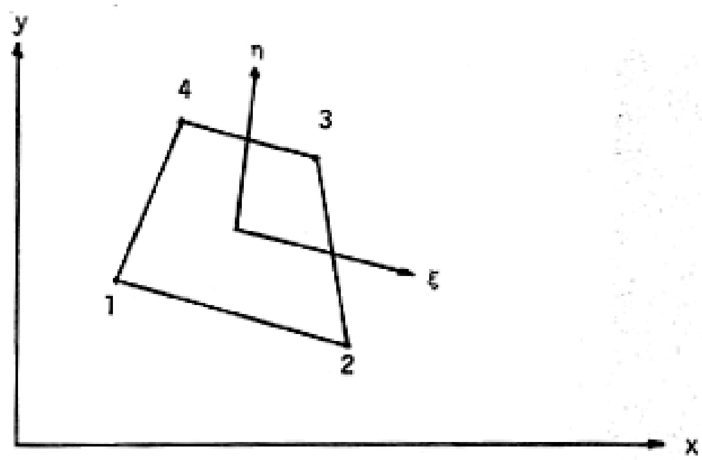

Figure 4: Linear-sided quadrilateral element

$$\left.\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\} \qquad (9)$$

Let the deformed configuration of the original structure be represented by a linear function. A linear quadrilateral macro-element can exactly model this structure. The displacement functions are:

$$\left.\begin{aligned} u(\xi,\eta) &= N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 \\ v(\xi,\eta) &= N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 \end{aligned}\right\} \qquad (10)$$

Formulate the transformation matrix, which is step 5, and then return to steps 3 and 4. The displacement relations between cases **a** and **b** are:

$$\left.\begin{aligned} \{q_o\} &= [T]\{q_m\} \\ \{q_o\}^T &= \left[u_1^a v_1^a u_2^a v_2^a u_3^a v_3^a u_4^a v_4^a u_5^a v_5^a u_6^a v_6^a\right] \\ \{q_m\}^T &= \left[u_1^b v_1^b u_2^b v_2^b u_3^b v_3^b u_4^b v_4^b\right] \end{aligned}\right\} \qquad (11)$$

But

$$\left.\begin{aligned} &u_1^a = u_1^b, v_1^a = v_1^b, u_4^a = u_4^b, v_4^a = v_4^b \\ &u_5^a = u_2^b, v_5^a = v_2^b, u_6^a = u_3^b \text{ and } v_6^a = v_3^b \end{aligned}\right\} \qquad (12)$$

The aim now is to find the relations between $u_2^a, v_2^a, u_3^a, v_3^a$ and the displacements of the macro-element, which is case **b**.

If Eqn. 10 is evaluated at $\xi = 0$ and n = -1, the required relations with u_2^a and v_2^a are achieved. If this equation is evaluated at $\xi = 0$ and n = +1, the required relations with u_3^a and u_3^a are formed.

$$u_2^a = u_{(0,-1)}^b = [1/4\,(1-0)\,(1+1)]u_1^b + [1/4\,(1+0)\,(1+1)]u_2^b + [1/4\,(1+0)\,(1-1)]u_3^b + [1/4\,(1-0)\,(1-1)]u_4^b$$

$$\left.\begin{aligned} &u_2^a = \tfrac{1}{2}u_1^b + \tfrac{1}{2}u_2^b \\ &\text{In the same way} \\ &v_2^a = v_{(0-1)}^b = \tfrac{1}{2}v_1^b + \tfrac{1}{2}v_2^b \end{aligned}\right\} \qquad (13)$$

$$\begin{aligned} u_3^a = u_{(0,1)}^b &= [1/4\,(1-0)\,(1-1)]u_1^b + [1/4\,(1+0)\,(1-1)]u_2^b + \\ &[1/4\,(1+0)\,(1+1)]u_3^b + [1/4\,(1-0)\,(1+1)]u_4^b \qquad (14) \\ u_3^a &= 1/2\,u_3^b + 1/2\,u_4^b \end{aligned}$$

Using the same method

$v_3^a = 1/2\, v_3^b + 1/2\, v_4^b$

$u_5^a = u_2^b, v_5^a = v_2^b, u_6^a = u_3^b, v_6^a = v_3^b$

Eqns. 12, 13 and 14 suffice to construct the required transformation matrix. The above relations are summarized below.

$u_1^a = u_1^b, v_1^a = v_1^b, u_2^a = 1/2\, u_1^b + 1/2\, u_2^b$

$v_2^a = \frac{1}{2} v_1^b + \frac{1}{2} v_2^b, u_3^a = 1/2\, u_3^b + 1/2\, u_4^b$

$v_3^a = \frac{1}{2} v_3^b + \frac{1}{2} v_4^b, u_4^a = u_4^b, v_4^a = v_4^b$

$u_5^a = u_2^b, v_5^a = v_2^b, u_6^a = u_3^b, v_6^a = v_3^b$

Arranging in matrix form,

$$\underset{\substack{\{q_0\} \\ 12\times1}}{\begin{Bmatrix} u_1^a \\ v_1^a \\ u_2^a \\ v_2^a \\ u_3^a \\ v_3^a \\ u_4^a \\ v_4^a \\ u_5^a \\ v_5^a \\ u_6^a \\ v_6^a \end{Bmatrix}} = \underset{\substack{[T] \\ 12\times8}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \underset{\substack{\{q_m\} \\ 8\times1}}{\begin{Bmatrix} u_1^b \\ v_1^b \\ u_2^b \\ v_2^b \\ u_3^b \\ v_3^b \\ u_4^b \\ v_4^b \end{Bmatrix}} \qquad (15)$$

Notice that the transformation matrix $[T]$ is simply the evaluation of the shape functions of the macro-element at the nodes of the original structure. For example v1, this degree of freedom is acting at node 1 of system **a** with its local coordinate (-1, -1) according to the macro-element. Then

$$v_1^a = v_{(-1,-1)}^b = N_1 v_1^b + N_2 v_2^b + N_3 v_3^b + N_4 v_4^b =$$
$$[1/4\,(1+1)\,(1+1)]v_1^b + [1/4\,(1-1)\,(1+1)]v_2^b +$$
$$[1/4\,(1-1)\,(1-1)]v_3^b + [1/4\,(1+1)\,(1-1)]v_4^b$$
$$v_1^a = v_1^b$$

The formulation of the transformation matrix can be generalized as shown in Eqn. 16

$$\begin{Bmatrix} u_1^a \\ v_1^a \\ u_2^a \\ v_2^a \\ . \\ . \\ u_{n-1}^a \\ v_{n-1}^a \\ u_n^a \\ v_n^a \end{Bmatrix} = \begin{bmatrix} N_1|_1^a & 0 & N_2|_1^a & 0 & N_3|_1^a & 0 & N_4|_1^a & 0 \\ 0 & N_1|_1^a & 0 & N_2|_1^a & 0 & N_3|_1^a & 0 & N_4|_1^a \\ N_1|_2^a & 0 & N_2|_2^a & 0 & N_3|_2^a & 0 & N_4|_2^a & 0 \\ 0 & N_1|_2^a & 0 & N_2|_2^a & 0 & N_3|_2^a & 0 & N_4|_2^a \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ N_1|_{n-1}^a & 0 & N_2|_{n-1}^a & 0 & N_3|_{n-1}^a & 0 & N_4|_{n-1}^a & 0 \\ 0 & N_1|_{n-1}^a & 0 & N_2|_{n-1}^a & 0 & N_3|_{n-1}^a & 0 & N_4|_{n-1}^a \\ N_1|_n^a & 0 & N_2|_n^a & 0 & N_3|_n^a & 0 & N_4|_n^a & 0 \\ 0 & N_1|_n^a & 0 & N_2|_n^a & 0 & N_3|_n^a & 0 & N_4|_n^a \end{bmatrix} \begin{Bmatrix} u_1^b \\ v_1^b \\ u_2^b \\ v_2^b \\ u_3^b \\ v_3^b \\ u_4^b \\ v_4^b \end{Bmatrix} \quad \text{….(16)}$$

where

$N_i\Big|_j^a$ is the shape function i evaluated at node j of case **a**.

The assumptions are based on the notion that the structure is a truss. The formulation of the stiffness matrix of the truss element required in step 3 will be carried out. The displacement function of a truss element is linear. This is because a truss element has two nodes with one degree of freedom per node. See Fig. 5.

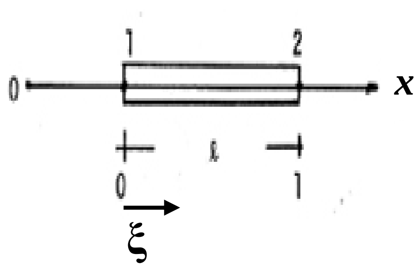

Figure 5: Truss element in local

The shape functions in dimensionless coordinates are:

$N_1 = 1 - \xi$

$N_2 = \xi$

The displacement function is:

$u = N_1 u_1 + N_2 u_2$

$$u = u(\xi) \text{ and } \xi = \xi(x) \qquad \text{……………….(17-a)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\,\frac{\partial \xi}{\partial x}$$

But

$$\frac{x - x_1}{\ell} = \xi \qquad \text{……………….(17-b)}$$

$$\partial x = \ell d\xi,\ \frac{\partial \xi}{\partial x} = \frac{1}{\ell} \qquad \text{……………….(17-c)}$$

$$\varepsilon = \frac{\partial u}{\partial x} = \frac{1}{\ell}\,\frac{\partial u}{\partial \xi} \qquad \text{……………….(17-d)}$$

$\sigma = E\varepsilon$

Then, the total potential energy is:

$$\Pi = 1/2 \int_{vol} \varepsilon^T E\, \varepsilon dv - \sum_1^m F_i u \quad i = 1, 2\text{…..}m \qquad \text{……………….(17-e)}$$

where m is the number of concentrated loads at the nodes of the truss structure. But

$$\varepsilon = \frac{1}{\ell}\quad \frac{\partial u}{\partial \xi} = \frac{1}{\ell}\left\{ \left(\frac{\partial N_1}{\partial \xi} u_1 + \frac{\partial N_2}{\partial \xi} u_2\right) \right\} \qquad \text{……………….(17-f)}$$

$$= \frac{1}{\ell}\begin{bmatrix} -1 & 1 \end{bmatrix} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{u} \qquad \text{……………….(17-g)}$$

$$\therefore \prod = \tfrac{1}{2}\{u\}^T \left[\int_{vol} \frac{1}{\ell} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{E}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \right] dv\, \{u\} - \sum_1^m \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} F_i \qquad \text{……………….(17-h)}$$

If the cross-section of the element is assured constant all over the length, then

$$\Pi = 1/2 \frac{AE}{\ell} \{u\}^T \left[\int_0^1 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} (\ell d\xi) \right] \{u\} - \{u\}^T \sum_1^m [N]^T \quad F_i \qquad \text{…….(17-i)}$$

Minimize the total potential energy with respect to $\{u\}^T$

$$\frac{\partial \Pi}{\partial \{u\}^T} = 0 \qquad \text{………………(17-j)}$$

$$\therefore \frac{AE}{\ell} = \int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi \, \{u\} \sum_1^m \, [N]^T \, F_i \qquad \text{………………(17-k)}$$

$$\underbrace{\frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{K_e} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\sum_1^m [N]^T F_i}_{F_e} \qquad \text{………………(17-l)}$$

where

$[K_e]$ is the element stiffness matrix in local coordinates.
$\{F_e\}$ is the load vector applied at the truss nodes. To transform the element stiffness matrix to global coordinates, a coordinate transformation matrix is needed. In two-dimensional problems, this has the form:

$$[\lambda] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad \text{………………(17-m)}$$

where θ is the angle between the local axis and global axis.

$$[K_0] = [\lambda]^T \, [K_e][\lambda] \qquad \text{………………(17-n)}$$

$$[K_0] = \frac{AE}{\ell} \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad \text{………………(17-o)}$$

$$[K_0] = \frac{AE}{\ell} \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \qquad \text{……………(17-p)}$$

Now, the formulation of the macro-element is:

$$\left. \begin{array}{l} [\mathrm{K_m}] = \sum_1^{e} [\mathrm{T_e}]^T \, [\mathrm{K_0}][\mathrm{T_e}] \\ \text{The load vector in global is:} \\ \{\mathrm{Fm}\} = \sum_1^{n} [\mathrm{N}]^T \begin{Bmatrix} \mathrm{F_{ix}} \\ \mathrm{F_{iy}} \end{Bmatrix} \end{array} \right\} \qquad \text{………………(18)}$$

where F_{ix}, F_{iy} are the components of the load F_i in the x and y global directions. Construction of the transformation matrix T is based on the evaluation of the shape functions at specific points inside the macro-element. These points usually are the nodes of the original structure.

It is then clear that a local coordinate for each of those nodes is required. There are two methods to do this. The first is the closed form solution and the second is an iterative scheme. Although the closed form solution of the local coordinates is readily formulated for linear quadrilateral elements, it is more complex to formulate for higher order elements.

CLOSED FORM SOLUTION FOR LINEAR QUADRILATERAL MACRO-ELEMENTS

Let a_i, b_i be the values of the global coordinates of node i as shown in Fig. 6. The relation between the local coordinates (ξ, n) and the global coordinates (X, Y) of any point inside the macro-element is:

$$\left.\begin{aligned} x &= N_1 a_1 + N_2 a_2 + N_3 a_3 + N_4 a_4 \\ y &= N_1 b_1 + N_2 b_2 + N_3 b_3 + N_4 b_4 \end{aligned}\right\} \quad \text{..................(19)}$$

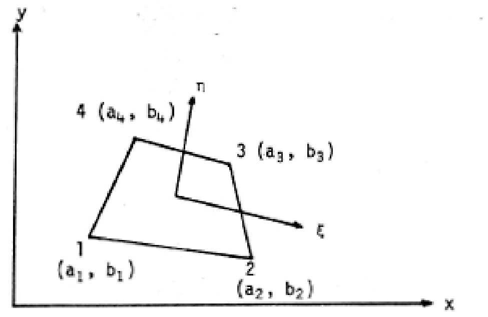

Figure 6: Linear quadrilateral macro-element

where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are known coordinate values. For linear quadrilateral elements, the shape functions are:

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

Substituting the above equations into Eqn. 19, one obtains:

$$\left.\begin{aligned} x &= \tfrac{1}{4}a_1(1-\xi)(1-\eta) + \tfrac{1}{4}a_2(1+\xi)(1-\eta) + \tfrac{1}{4}a_3(1+\xi)(1+\eta) + \tfrac{1}{4}a_4(1-\xi)(1+\eta) \\ y &= \tfrac{1}{4}b_1(1-\xi)(1-\eta) + \tfrac{1}{4}b_2(1+\xi)(1-\eta) + \tfrac{1}{4}b_3(1+\xi)(1+\eta) + \tfrac{1}{4}b_4(1-\xi)(1+\eta) \end{aligned}\right\} \quad \text{........(20)}$$

x and y are the global coordinates of the point for which the local coordinates ξ and η are required. Let first point of Eqn. 20 be simplified and solved for ξ.

$$4x=\left[a_1(1-\eta)+a_4(1+\eta)\right]+\left[a_2(1-\eta)+a_3(1+\eta)\right]+\xi\left[a_2(1-\eta)+a_3(1+\eta)-a_1(1-\eta)-a_4(1+\eta)\right]$$

$$\xi=\frac{4x-\left[(1-\eta)(a_1+a_2)+(1+\eta)(a_3+a_4)\right]}{(1-\eta)(a_2-a_1)+(1+\eta)(a_3-a_4)} \qquad \text{…….(21-a)}$$

Also from the second part of Eqn. (20):

$$\xi=\frac{4y-\left[(1-\eta)(b_1+b_2)+(1+\eta)(b_3+b_4)\right]}{(1-\eta)(b_2-b_1)+(1+\eta)(b_3+b_4)} \qquad \text{…….(21-b)}$$

The two equations in (21-a) and (21-b) are equal.

$$\frac{4x-\left[(1-\eta)(a_1+a_2)+(1+\eta)(a_3+a_4)\right]}{(1-\eta)(a_2-a_1)+(1+\eta)(a_3-a_4)}=\frac{4y-\left[(1-\eta)(b_1+b_2)+(1+\eta)(b_3+b_4)\right]}{(1-\eta)(b_2-b_1)+(1+\eta)(b_3+b_4)}$$

Simplifying

$$4y-\left[\underbrace{(b_1+b_2+b_3+b_4)}_{D_1}+\underbrace{(-b_1-b_2+b_3+b_4)}_{D_2}\eta\right]\left[\underbrace{(a_2-a_1+a_3-a_4)}_{D_3}+\underbrace{(a_1-a_2+a_3-a_4)}_{D_4}\eta\right]=$$

$$4x-\left[\underbrace{(a_1+a_2+a_3+a_4)}_{C_1}+\underbrace{(-a_1-a_2+a_3+a_4)}_{C_2}\eta\right]\left[\underbrace{(b_2-b_1+b_3-b_4)}_{C_3}+\underbrace{(b_1-b_2+b_3-b_4)}_{C_4}\eta\right]$$

$$(4y-D_1-D_2\eta)(D_3+D_4\eta)=(4x-C_1-C_2\eta)(C_3+C_4\eta)$$

where: $4y-D_1$ is constant.
Expanding and collecting the terms with the same power of η

$$\underbrace{(C_2C_4-D_2D_4)}_{A}\eta^2+\underbrace{(C_2C_3-4xC_4+C_1C_4-D_2D_3+4yD_4-D_1D_4)}_{B}\eta+\underbrace{(4yD_3-D_1D_3-4xC_3+C_1C_3)}_{C}=0$$

$$A\eta^2+B\eta+C=0 \qquad \text{……………….(22)}$$

$$\eta=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

Two roots exist for η. To select the correct root, one of them is substituted in Eqn. (21-a) and solved for ξ. Do the values of ξ and η satisfy Eqn. (20)? If not, the second root of η is solved for ξ.

Those roots yield the correct x and y values. These equations are readily programmed to find the local coordinates of any point inside the macro-element if the global coordinates are known.

Now, $[K_e]$ and $[T]$ are known. Extract from the transformation matrix $[T]$ that part corresponding to the degrees of freedom of $[K_e]$. Let this part be called $[T_e]$, then

$$[K_m]=\sum_{e=1}^{n}[T_e]^T[K_e][T_e]$$

The external loadings are applied at the truss structure nodes, although they may not coincide with the macro-element nodes. Thus, a subroutine to calculate the equivalent consistent nodal load vector for the macro-element is required. This is done for each concentrated load by the evaluation of the equation shown below.

$$\underset{16\times 1}{\{F_m\}i}=\underset{16\times 2}{[N]^T}\,\underset{2\times 1}{Fi\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}} \qquad \text{……………………} \ (23)$$

where

$$N=\begin{bmatrix}N1 & 0 & N2 & 0 & N3 & 0 \ldots N8 & 0\\ 0 & N1 & 0 & N2 & 0 & N3 & 0 & N8\end{bmatrix}$$

where F_i is the force at node i; the macro-element nodal load vector from F_i only. Summing up the effect of all loadings yields:

$$\{F_m\}=\sum_{i=1}^{n}[N]^T F_i\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix} \qquad \text{……………………} \ (24)$$

Here, the shape functions are evaluated at the local coordinate values of the point of application of F_i. Next, the total stiffness and load vector of the structure composed of macro-elements is formulated. This is done by the conventional assembling subroutine used in finite element methods. The boundary conditions are applied.

The linear system is solved using Gauss alimentation of any solving subroutine. The solution is for the macro-elements nodal values.

APPLICATIONS

Problem No.1: A cantilever truss consists of five truss members with eight degress of freedom and EA are the same for all members. Three degress of freedom are constrained as shown in Fig.7. This truss problem is modeled by a linear quadrilateral element with eight degrees of freedom, as shown in Fig. 8. The results are tabulated in Table 1.

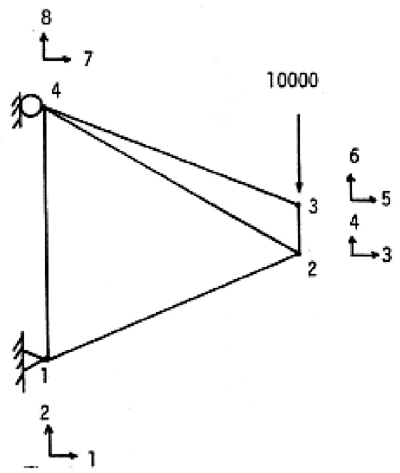

Figure 7: Cantilever truss in two dimensions: problem no. 1

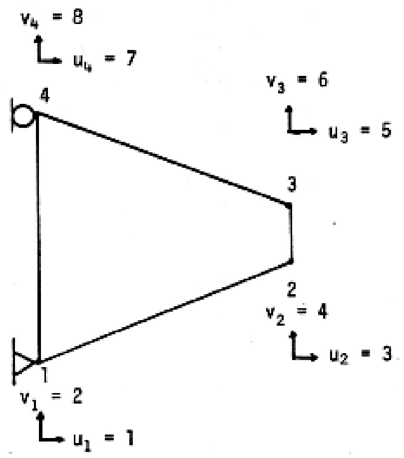

Figure 8: Macro-element method modeling of truss problem no. 1 in two dimensions

Problem No.2: A cantilever truss consists of nine truss members with a total of 12 degress of freedom. All members are of the same AE. Three degrees of freedom are constrained as shown in Fig.9.

This problem was modeled by a linear quadrilateral element as shown in Fig.10. The results are shown in Table 2.

Problem No.3: The same as problem No. 2, but using a quadratic quadrilateral isoparametric macro-element with eight nodes as shown in Fig.11. The results are tabulated in Table 3.

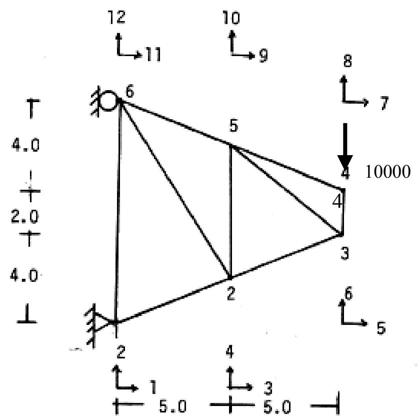

Figure 9: Cantilever truss in two dimensions: problems no. 2 and no. 3

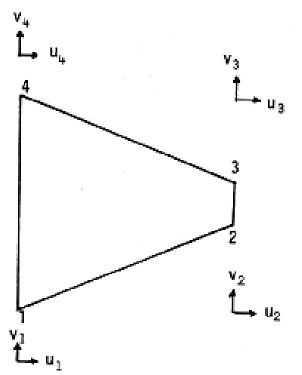

Figure 10: Macro-element method modeling of truss problem no. 2 in two dimensions

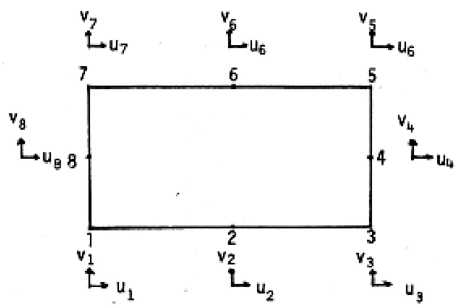

Figure 11: Macro-element method modeling of truss problem no. 3 in two dimensions

Table 1: Displacement results of truss problem no.1 in two dimensions

Displacements	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	2.879×10^{-5}	2.879×10^{-5}	0.0
4	-3.195×10^{-3}	-3.195×10^{-3}	0.0
5	-1.118×10^{-3}	-1.118×10^{-3}	0.0
6	-3.395×10^{-3}	-3.395×10^{-3}	0.0
7	0.0	0.0	0.0
8	-6.00×10^{-4}	-6.00×10^{-4}	0.0

Table 2: Displacement results of truss problem no. 2 in two dimensions

Displacements	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	-1.958×10^{-4}	-5.082×10^{-5}	74.0
4	-1.072×10^{-3}	-1.211×10^{-3}	12.96
5	-4.865×10^{-5}	-1.016×10^{-4}	52.10
6	-2.741×10^{-3}	-2.421×10^{-3}	11.67
7	-4.160×10^{-4}	-3.589×10^{-4}	13.72
8	-2.941×10^{-3}	-2.588×10^{-3}	12.00
9	2.517×10^{-4}	-1.794×10^{-4}	-
10	-1.272×10^{-3}	-1.582×10^{-3}	20.00
11	0.0	0.0	0.0
12	-6.000×10^{-4}	-5.770×10^{-4}	3.8

Table 3: Displacement results of truss problem no. 3 in two dimensions

Displacements	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	-1.958×10^{-4}	-1.958×10^{-4}	0.0
4	-1.072×10^{-3}	-1.072×10^{-3}	0.0
5	-4.865×10^{-5}	-4.865×10^{-5}	0.0
6	-2.741×10^{-3}	-2.741×10^{-3}	0.0
7	-4.160×10^{-4}	-4.160×10^{-4}	0.0
8	-2.941×10^{-3}	-2.941×10^{-3}	0.0
9	2.517×10^{-4}	2.517×10^{-4}	0.0
10	-1.272×10^{-3}	-1.272×10^{-3}	0.0
11	0.0	0.0	0.0
12	-6.000×10^{-4}	-6.000×10^{-4}	0.0

DISCUSSION OF THE RESULTS

The solved problems showed that using the macro-elements in the analysis reduced the number of equations to be solved. When the order of the displacement function of the macro-element is at least the same order as the displacement function of the original truss structure, excellent results are achieved with good amount of reduction in d.o.f.

In example-1, the number of nodes in the truss structure is four nodes, modeled by one linear macro-element which has four nodes with linear shape functions. This means that the shape functions of the macro-element are of the same order of the truss displacement and can exactly describe the displacement of the truss structure, and this is the case of Table-1, where there are no errors between the two solutions.

In example-2, the number of nodes in the truss structure is six nodes, modeled by one linear macro-element which has only four nodes with linear shape functions. This means that the shape functions of the macro-element can not exactly describe the displacement of the truss structure, and this is the case of Table-2, where there are large errors between the two solutions.

Example-2 was remodeled by two macro-elements of truss linear macro-element, and the answers of displacements were exactly the same answers of Table-3; i.e. no errors.

In Example-3, the number of nodes in the truss structure is six nodes, modeled by one quadratic quadrilateral isoperimetric macro-element with eight nodes. This means that the shape functions of the macro-element can exactly describe the displacement of the truss structure, and this is the case of Table-3, where there are no errors between the two solutions.

CONCLUSION

New modeling of a truss linear macro-element based on truss type finite elements was developed.

The solved examples demonstrated that using macro-elements in the analysis largely reduced the total number of d.o.f. required to model a certain structure. This, in turn, reduced the total number of equations to be solved.

Reduction in the total number of equations reduced computer time and memory space for storage.

At the same time, macro-elements provided accurate results. In addition, finite elements of different sizes and material properties can easily be used inside the macro-elements if required in the analysis. This developed macro-element theory was applied to different kinds of structural elements like beams, thin plates and thick plates, and good results were achieved in accuracy and time of execution. This theory can be applied to any kind of structure as long as the basic assumptions for macro-element formulation are satisfied.

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