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The Inclusion of Warping in Free Vibration of Structures

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ABSTRACT

Thin walled cellular structures are widely used in many civil, mechanical and aerospace engineering applications. These applications have increased with the economic necessity of providing high strength with low weight and cost. Structural analysis of thin walled girder is usually performed by the beam theory. In the present study, the effect of cross-sectional warping on the dynamic behavior of box girder deck is investigated using discrete element approach in idealizing the structure and incorporating the warping as a seventh degree of freedom in a space frame element. Shear deformation due to uniform torsion in addition to transverse shear deformation are taken into account in the problem formulation. The analysis is performed using the computer programs DNG6 and DNG7.It can be seen that the transverse shear contributes considerably to lowering the natural frequencies of the flexural vibration modes, and the inclusion of warping considerably increased the natural frequencies of the torsional-dominant vibration modes.

KEYWORDS: Vibration, Warping, Thin wall.

INTRODUCTION

The first beam theory has been established by Bernoulli and Navier. After that, Vlasov's, Benscoter as well as Kollbrunner and Haidin's theories followed. In the following article, the mathematical derivation of the governing equations required for the formulation of both stiffness and mass matrices is presented.

Equation of Motion

The equation of motion of any damped system is given by Clough and Peneien (Clough and Peneien, 1995).

$$[M]\{\ddot{U}\}+[C]\{\dot{U}\}+[k]\{U\}=\{P(t)\} \qquad (1)$$

where $\{U\}$, $\{\dot{U}\}$ and $\{\ddot{U}\}$ are the time-dependant displacement, velocity and acceleration vectors, respectively, and $\{F(t)\}$ is the applied load vector.

The system is assumed to have classical damping. Thus, the damping matrix is of the form:

$$[C] = 2\xi_i\omega_i\,[M] \qquad (2)$$

in which ξ_i is the damping ratio corresponding to the mode (i).

ω_i**:** is the natural angular velocity (or circular frequency) of the system which vibrates at the mode shape (i).

The equation of motion for free vibration undamped system can be obtained by omitting the damping matrix and the load vector from Equation (1), such that:

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \qquad (3)$$

Also, the free vibration motion of the system is

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simple harmonic, which may be expressed as:

$$U(t) = \Phi sin\omega_i t \tag{4}$$

in which:

Φ: represents the mode shape of the system which does not change with time, but only the amplitude varies.

The acceleration vector in free vibration will be:

$$\ddot{U}(t) = -\omega_i^2 \Phi_i sin\omega_i t = -\omega_i^2 \{U(t)\} \tag{5}$$

Substituting equations (4) and (5) into equation (3) results in:

$$-\omega_i^2 [M]\Phi_i sin\omega_i t + [K]\Phi_i sin\omega_i t = \{0\} \tag{6}$$

Omitting the sine terms, this equation can be written as:

$$\Big[[K] - \omega_i^2 [M]\Big]\Phi_i = \{0\} \tag{7}$$

The nontrivial solution is possible only when

$$|[K] - \omega_i^2 [M]| = 0 \tag{8}$$

Equation (8) is called the frequency equation of the system. Expanding this equation will give an algebraic equation of an nth degree known as the characteristic equation of the system (Paz, 2006).

The n roots of this equation. $\left(\omega_1^2\ \omega_2^2 \ldots\ldots\ldots\ldots\omega_n^2\right)$ represent the square of the circular frequencies of n modes of vibration (eigen values) which are possible in the system, while the corresponding n eigen vectors of this eigen problem represent the n mode shapes of vibration of the system. It is known that the amplitude of these mode shapes is arbitrary, it is customary to normalize them. The most often used normalizing procedure in computer programs for structural vibration analysis involves adjusting each mode shape.

$$\{\Phi_i\}^T [M]\{\Phi_i\} = I \tag{9}$$

The mode shapes normalized in this fashion are said to be orthonormal relative to the mass matrix (Clough and Peneien, 1995) or M-orthonormalized. Then:

$$\{\Phi_i\}^T [K]\{\Phi_i\} = [\lambda] \tag{10}$$

$$\{\Phi_i\}^T [M]\{\Phi_i\} = [I] \tag{11}$$

where $[\Phi_i]$ is a modal matrix whose columns are M-orthonormalized mode shapes.

$[\lambda]$ is a diagonal matrix which stores the square of the circular frequencies, and

$[I]$ is an identity matrix.

So, it is important to realize that all solution methods are iterative in nature, because basically solving the eigen value problem:

$$K\Phi = \lambda\Phi M \tag{12}$$

is equivalent to calculating the roots of the polynomial $P(\lambda)$, which has an order equal to the order of K and M.

Solution Methods for Eigen Problems

It is important to realize that all solution methods are iterative in nature, because basically solving the eigen value problem $K\Phi = \lambda M\Phi$ is equivalent to calculating the roots of the polynomial $P(\lambda)$, which has an order equal to the order of K and M. The general groups of solution methods (Chardrupatla and Belugundu, 2002) are:

Vector Iteration Methods

Various vector iteration methods are based on properties of the Rayleigh quotient. For the generalized eingen value problem of Equation (12), the Rayleigh quotient $Q(V)$ is given by:

$$Q(V) = \frac{V^T KV}{V^T MV}; \tag{13}$$

where V is an arbitrary vector which defines a mode shape. A fundamental property of Rayleigh quotient is that it lies between the smallest and the largest eigen values; that is:

$$\lambda_1 \leq Q(V) \geq \lambda_n . \tag{14}$$

Power iteration, inverse iteration and subspace iteration methods are based on this property.

Power iteration leads to the evaluation of the largest eigen value, the inverse iteration can be used in evaluating the lowest eigen value, and subspace iteration technique is suitable for large scale problems.

Transformation Methods

The basic approach is to transform the matrices to a simpler form and then determine the eigen values and eigen vectors. The major methods in this category are the generalized Jacobi method and the QR method.

In the QR method, the matrices are first reduced to a tridiagonal form using Householder matrices (Bathe and Wilson, 1976).

The generalized Jacobi method uses the transformation to simultaneously diagonalize the stiffness and mass matrices. This method needs the full matrix locations and is quite efficient for calculating all eigen values and eigen vectors for small problems.

Polynomial Iteration Methods

The polynomial iteration techniques which operate on the fact that:

$P(\lambda_i) = 0$ is given by:

$$P(\lambda) = det(K - \lambda M) . \tag{15}$$

An advantage of the polynomial iteration method is that each eigen value is determined independently and cumulative errors do not occur. However, there is no guarantee which value of λ has been calculated without recourse to other methods; e.g., Sturm sequence checks.

The Strum sequence property of the characteristic polynomial is as follows:

$$P(\lambda) = det(K - \lambda M)$$

and

$$P^{(r)}(\lambda^{(r)}) = det(K^{(r)} - \lambda^{(r)} M^{(r)}) \tag{16}$$

$r = 1, \ldots\ldots\ldots n-1$

where $P^{(r)}(\lambda^{(r)})$ is the characteristic polynomial of the r'th associated constraint problem corresponding to $K\phi = \lambda M\phi$.

Considering the effectiveness of the solution procedures, none of these methods is always to be most efficient, but the solution technique to be used should be selected according to the specific problem to be solved.

The procedure for these solutions of large eigen problems was very expensive for a long time, therefore approximate solution techniques will be used.

Subspace iteration method is one of these approximate methods used for a the cases considered in this study.

Subspace Iteration Method

The subspace iteration method is particularly suited for the calculation of a few eigen values and eigen vectors of large finite element systems.

The basic objective in this method is to solve for the smallest P eigen values and the corresponding eigen vectors satisfying the formula (Bathe and Wilson, 1976; Bathe and Wilson, 1972).

$$[K]\{\Phi\} = [M]\{\Phi\}[\lambda] \tag{17}$$

in which Φ is a matrix storing the p eigen vectors and λ is the corresponding vector of eigen values.

$[K]$ is the stiffness matrix.

$[M]$ is the mass matrix.

The subspace iterations are performed as follows:

$$[K][X_{k+1}] = [M][X_K] \tag{18}$$

- For K=1, 2… iterate from subspace E_k to subspace E_{k+1}.

- Calculate the projections of the matrices.

$$[K_{k+1}] = [X_{k+1}]^T[K][X_{k+1}] \qquad (19)$$

$$[M_{k+1}] = [X_{k+1}]^T[K][X_{k+1}] \qquad (20)$$

- solve for the eigen system of the projected matrices.

$$[K_{k+1}][Q_{k+1}] = [M_{k+1}][Q_{k+1}][\lambda_{k+1}] \qquad (21)$$

where $[Q_{k+1}]$ is the matrix of eigen vectors of the subspace and $[\lambda_{k+1}]$ is the diagonal matrix of the corresponding eigen values.

- calculate an improved approximation to eigen vectors:

$$X_{k+1} = \bar{X}_{k+1}Q_{k+1} \qquad (22)$$

Then, provided that the vectors in [$\mathbf{X}_1$] are not orthogonal to one of the required eigen vectors, the i[th] diagonal entry in $[\lambda_{k+1}]$ converges to λ_i or ω_i^2 and the i[th] vector in $[X_{k+1}]$ converges to Φ_i.

An important aspect is the convergence of the method. Assuming that in the iteration the vectors in X_{k+1} are of order in such a way that the i[th] diagonal element in λ_{k+1} is larger than the (i-1) [st] element, I=2..., P, than the i[th] column in X_{k+1} converges linearly to Φ_i and the convergence rate is λ_i/λ_{p+1}.

Although this is an asymptotic convergence rate, it indicates that the smallest eigen values converge faster. Also, faster convergence can be obtained by using q iteration vectors, with q > p.

In the implementation, q= min (2p, p+8) has been used (Bathe and Wilson, 1976).

Considering the convergence rate, it should be noted that multiple eigen values do not decrease the rate of convergence, λ_{q+1}/λ_q .

The convergence is measured as follows:

$$\frac{\left|\lambda_i^{(k+1)} - \lambda_i^{(k)}\right|}{\lambda_i^{(k+1)}} \le tol = 10^{-6} . \qquad (23)$$

The Stiffness and Mass Matrices

The formulation of the stiffness and mass matrices is based on approximate displacement functions.

The structure is modelled by a three-dimensional beam-column element with warping deflection and bimoment inertia as an additional degree-of-freedom (7 degrees of freedom per node).

Warping Displacement and Stress

The warping displacement is done by:

$$\varepsilon x = (\overline{\omega}_o - \overline{\omega})\frac{d^2\theta}{dx^2} ; \qquad (24)$$

and the warping stress is done by:

$$\sigma x = E(\overline{\omega}_o - \overline{\omega})\frac{d^2\theta}{dx^2} . \qquad (25)$$

Torsion-Bending Element Stiffness

The element stiffness matrix corresponding to the torsional-warping degree-of-freedom is derived by assuming the displacement field $\theta(x)$ corresponding to the mentioned degree-of-freedom.

$$\theta(x) = H(x)\ \upsilon_p \qquad (26)$$

υ_p = local displacement vector.

Using cubic Hermitean polynomials or “beam functions” (Weaver and Johnston, 1990), then:

$$H^T(x) = \begin{Bmatrix} -x + \frac{2x^2}{L} - \frac{x^3}{L^2} \\ \frac{x^2}{L} - \frac{x^3}{L^2} \\ 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \end{Bmatrix} \qquad (27)$$

According to the method of virtual work, the stiffness coefficients are derived from the equation:

$$k_p = \int_0^L [a_S(x)]^T\ k_S\ a_S(x)\,dx \cdot \qquad (28)$$

Therefore, the stiffness matrix will take the form:

$$k_p = \int_0^L \left\{ \frac{d^2H}{dx^2} \; \frac{dH}{dx} \right\} \begin{bmatrix} E\Gamma & 0 \\ 0 & GJ \end{bmatrix} \left\{ \begin{matrix} \dfrac{d^2H}{dx^2} \\ \dfrac{dH}{dx} \end{matrix} \right\} dx = K_{pB} + K_{pS} \qquad (29)$$

$$Kp = \frac{2E\Gamma}{L} \begin{bmatrix} 2 & 1 & -\frac{3}{L} & \frac{3}{L} \\ & 2 & -\frac{3}{L} & \frac{3}{L} \\ & & \frac{6}{L^2} & -\frac{6}{L^2} \\ sym. & & & \frac{6}{L^2} \end{bmatrix} +$$

$$\frac{GJL}{30} \begin{bmatrix} 4 & -1 & -\frac{3}{L} & \frac{3}{L} \\ & 4 & -\frac{3}{L} & \frac{3}{L} \\ & & \frac{36}{L^2} & -\frac{36}{L^2} \\ sym. & & & \frac{36}{L^2} \end{bmatrix} \qquad (30)$$

Mass Formulation

The consistent mass matrix can be calculated from (Paz, 2006).

$$[M] = \int_0^L [H(x)]^T m [H(x)] dx \qquad (31)$$

$$M_p = \frac{\bar{\rho} A r_g^2 L^3}{420} \begin{bmatrix} 4 & -3 & -\frac{22}{L} & -\frac{13}{L} \\ & 4 & \frac{13}{L} & \frac{22}{L} \\ & & \frac{156}{L^2} & \frac{54}{L^2} \\ & & & \frac{156}{L^2} \end{bmatrix} \qquad (32)$$

$\bar{\rho}$: denotes the mass density;
A: cross-sectional area; and
r_g: the radius of gyration of the cross-section.

Computer Programming

For the purpose of this study, two computer programs (DNG6 and DNG7) are used in this study. These programs are coded in FORTRAN99 language. The programs are depending on the same prepared data file.

The Computer Program DNG6

The program will formulate the stiffness and mass matrices for each member using a six degree freedom system.

The Computer Program DNG7

The program will formulate the stiffness and mass matrices for each member using a seven degree freedom system, the effect of warping is included as a seventh degree of freedom. The transverse shear deformation is also included.

Validation of the Proposed D.E. Idealization Methods

The proposed discrete elements (D.E.) idealization methods with six and seven degrees of freedom per node are validated by comparing the natural frequencies of certain examples obtained by this method against the results obtained by using MSC/NASTRAN program.

Example 01: The free vibration of a rectangular beam shown in Figure (1) is considered. The analysis is performed using the exact solution in Chopra (2007) and the computer programs (DNG6 and DNG7) and MSC/NASTRAN as shown in Table (1.1). The results obtained by all the above methods are tabulated in Table (1).

Sectional Properties (kn, m)

A= 0.25
$I_Y = I_Z = 5.20833$ E-3
J = 8.7875 E-3
$I_W = h^3 b^3 / 144$
$R_g = 0.1874833$,
$\rho = 2.4$ kg /m^3, E = 2E7, L= 6 m
$\omega_n = (1.875)^2 \sqrt[2]{\frac{EI}{L^4 \bar{m}}} = 40.69$ rad/sec

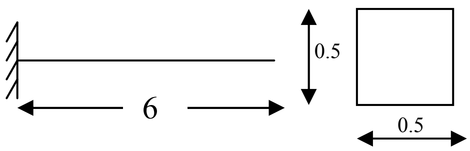

Figure 1: Rectangular beam with fixed free end condition

Example 02:

The free vibration of a single box thin-wall section shown in Figure (2) will be analyzed to verify the performance of the proposed thin wall beam column element in the free vibration response. The analysis is performed using (DNG6 and DNG7) programs and MSC/NASTRAN package. The results are presented in Table (1.2). The results obtained by all the above methods are given in Table (2).

Sectional Properties (kn, m)

L = 500 mm, b = 25 mm, h =50 mm,
t = 1mm, E = 200 GN/m^2,
G = 76.9 GN/m^2, Mt = 100 N.m

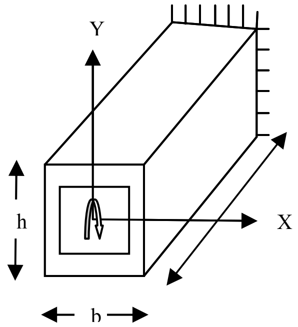

Figure 2: Cantilever box beam subjected to a twisting moment

RESULTS AND DISCUSSION

Free vibration analysis with 6 elements, warping and shear included « DNG6 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.64520D+01	0.40539D+02
2	0.64524D+01	0.40542D+02
3	0.39604D+02	0.24884D+03
4	0.39604D+02	0.24884D+03
5	0.79538D+02	0.49975D+03
6	0.10774D+03	0.67697D+03
7	0.10774D+03	0.67697D+03
8	0.12063D+03	0.75791D+03
9	0.20399D+03	0.12817D+04
10	0.20399D+03	0.12817D+04

The calculated eigen vectors 'node number7'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.000D+00	- 0.105D+01	0.000D+00	0.240D+00	0.000D+00
0.000D+00	0.105D+01	0.000D+00	0.000D+00	0.000D+00	0.240D+00
0.000D+00	0.104D+01	0.129D−07	0.000D+00	- 0.101D−07	0.815D+00
0.000D+00	-0.129D−07	0.104D+01	0.000D+00	- 0.815D+00	- 0.101D−07
0.215D−10	0.000D−00	0.000D+00	0.365D+01	0.000D+00	0.000D+00
- 0.157D−10	- 0.104D+01	0.348D−08	0.000D+00	- 0.431D−08	- 0.128D+01
0.157D−10	- 0.348D−08	- 0.104D+01	0.000D+00	0.128D+01	- 0.431D−08
0.749D+00	- 0.105D−06	0.105D−06	0.625D−05	- 0.373D−06	- 0.373D−06
- 0.128D−10	- 0.104D+01	- 0.352D−08	0.000D+00	0.587D−08	- 0.173D+01
- 0.128D−10	- 0.352D−08	0.104D+01	- 0.103D−11	- 0.173D+01	- 0.587D−08

Table (1.1a). Example N° 1: free vibration analysis of a rectangular beam with a fixed- free end condition using DNG6

Free vibration analysis with 6 elements, warping and shear not included « DNG6 »
The calculated natural eigen values

No.	Hz	rad/sec
1	0.64768D+01	0.40695D+02
2	0.64768D+01	0.40695D+02
3	0.40599D+02	0.25509D+03
4	0.40599D+02	0.25509D+03
5	0.79538D+02	0.49975D+03
6	0.11386D+03	0.71539D+03
7	0.11386D+03	0.71539D+03
8	0.12063D+03	0.75791D+03
9	0.22414D+03	0.14083D+04
10	0.22414D+03	0.14083D+04

The calculated eigen vectors 'node number7'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.745D+00	0.745D+00	0.000D+00	- 0.171D+00	0.171D+00
0.000D+00	0.745D+00	- 0.745D+00	0.000D+00	0.171D+00	0.171D+00
0.000D+00	0.746D+00	0.745D+00	0.000D+00	- 0.594D+00	0.595D+00
0.000D+00	- 0.745D+00	0.746D+00	0.000D+00	- 0.595D+00	- 0.594D+00
0.000D+00	0.000D+00	0.000D+00	0.400D+01	0.000D+00	0.000D+00
0.000D+00	- 0.738D+00	- 0.758D+00	0.000D+00	- 0.991D+00	- 0.966D+00
0.000D+00	- 0.758D+00	- 0.738D+00	0.221D−11	0.966D+00	- 0.991D+00
- 0.750D+00	0.000D+00	0.000D+00	0.355D−11	0.000D+00	0.000D+00
0.000D+00	0.329D+00	- 0.101D+01	0.000D+00	0.186D+01	0.605D+00
0.000D+00	- 0.101D+01	- 0.329D+00	0.000D+00	0.605D+00	- 0.186D+01

Table (1.1b). Example N°1: free vibration analysis of a rectangular beam with a fixed- free end condition using DNG6

Free vibration analysis with 6 elements, transverse shear included « DNG7 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.64520D+01	0.40539D+02
2	0.64524D+01	0.40542D+02
3	0.39604D+02	0.24884D+03
4	0.39604D+02	0.24884D+03
5	0.82094D+02	0.51581D+03
6	0.10774D+03	0.67697D+03
7	0.10774D+03	0.67697D+03
8	0.12063D+03	0.75791D+03
9	0.20399D+03	0.12817D+04
10	0.20399D+03	0.12817D+04

The calculated eigen vectors 'node number7'

Deflection x	Deflection y	Deflection z	Rotation x	Warping u	Rotation y	Rotation z
0.000D+00	0.000D+00	- 0.105D+01	0.000D+00	0.000D+00	0.240D+00	0.000D+00
0.000D+00	0.105D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.240D+00
0.000D+00	0.104D+01	0.895D−08	0.000D+00	0.000D+00	- 0.671D−08	0.815D+00
0.000D+00	0.895D−08	- 0.104D+01	0.000D+00	0.000D+00	0.815D+00	0.671D−08
0.271D−10	0.000D−00	0.000D+00	0.405D+01	-0.424D−01	0.000D+00	0.000D+00
- 0.157D−10	- 0.104D+01	0.348D−08	0.000D+00	0.000D+00	- 0.431D−08	- 0.128D+01
0.157D−10	- 0.348D−08	- 0.104D+01	0.000D+00	0.000D+00	0.128D+01	- 0.431D−08
0.749D+00	- 0.105D−06	0.105D−06	-0.500D−05	0.531D−05	- 0.373D−06	- 0.373D−06
- 0.128D−10	- 0.104D+01	- 0.352D−08	0.000D+00	0.000D+00	0.586D−08	- 0.173D+01
- 0.128D−10	- 0.352D−08	0.104D+01	0.000D+00	0.000D+00	- 0.173D+01	- 0.586D−08

Table (1.1c). Example N°1: free vibration analysis of a rectangular beam with a fixed- free end condition using DNG7

Free vibration analysis with 6 elements, transverse shear not included « DNG7 »

The calculated natural eigenvalues

No.	Hz	rad/sec
1	0.64768D+01	0.40695D+02
2	0.64768D+01	0.40695D+02
3	0.40599D+02	0.25509D+03
4	0.40599D+02	0.25509D+03
5	0.82094D+02	0.51581D+03
6	0.11386D+03	0.71539D+03
7	0.11386D+03	0.71539D+03
8	0.12063D+03	0.75791D+03
9	0.22414D+03	0.14083D+04
10	0.22414D+03	0.14083D+04

The calculated eigen vectors 'node number7'

Deflection x	Deflection y	Deflection z	Rotation x	Warping u	Rotation y	Rotation z
0.000D+00	0.745D+00	0.745D+00	0.000D+00	0.000D+00	- 0.171D+00	0.171D+00
0.000D+00	0.745D+00	- 0.745D+00	0.000D+00	0.000D+00	0.171D+00	0.171D+00
0.000D+00	0.152D+00	0.104 D+01	0.000D+00	0.000D+00	- 0.832D+00	0.121D+00
0.000D+00	0.104D+01	- 0.152D+00	0.000D+00	0.000D+00	0.121D+00	0.832D+00
0.000D+00	0.000D+00	0.000D+00	0.405D+01	- 0.424D−01	0.000D+00	0.000D+00
0.000D+00	- 0.748D+00	- 0.747D+00	0.000D+00	0.000D+00	0.978D+00	- 0.979D+00
0.000D+00	0.747D+00	- 0.748D+00	0.000D+00	0.000D+00	0.979D+00	0.978D+00
- 0.750D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	- 0.107D+01	0.344D−02	0.000D+00	0.000D+00	- 0.633D−02	- 0.196D+01
0.000D+00	- 0.344D−02	- 0.107D+01	0.000D+00	0.000D+00	0.196D+01	- 0.633D−02

Table (1.1d). Example N°1: free vibration analysis of a rectangular beam with a fixed- free end condition using DNG7

Table 1. Natural frequencies (Hz) of a rectangular beam with a fixed free end condition (example N°1)

Predominant mode	**Exact**	**DNG6 (6D.O.F.) without (warping and shear)**		**DNG6 (6 D.O.F.) with (warping and shear)**		**DNG7 (7D.O.F.) with shear**		**MSC/NASTRAN**	
ULY1	6.465	6elem.	6.476	6elem.	6.452	6elem.	6.452	12elem.	6.4559
ULZ1	6.465	6elem.	6.476	6elem.	6.452	6elem.	6.452	12elem.	6.4559
ULY2	40.58	6elem.	40.599	6elem.	39.604	6elem.	39.604	12elem.	40.144
ULZ2	40.58	6elem.	40.599	6elem.	39.604	6elem.	39.604	12elem.	40.144
ULY3	113.6	6elem.	113.86	6elem.	107.74	6elem.	107.74	12elem.	111.61
ULZ3	113.6	6elem.	113.86	6elem.	107.74	6elem.	107.74	12elem.	111.61
CTW		6elem.	79.538 Mode 5	6elem.	79.538	6elem.	82.09	12elem.	347.28 Mode 10
AXIAL		6elem.	120.63	6elem.	120.63	6elem.	120.63	12elem.	120.195

ULY1: 1st uncoupled lateral mode in Y- direction.
ULZ1: 1st uncoupled lateral mode in Z- direction.
CTW: Coupled torsional warping mode.
UAXIAL: Uncoupled axial mode.

Free vibration analysis with 15 elements, warping and transverse shear included« DNG6 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.39148D+01	0.24597D+02
2	0.65826D+01	0.41360D+02
3	0.23186D+02	0.14568D+03
4	0.38119D+02	0.23951D+03
5	0.60077D+02	0.37747D+03
6	0.70341D+02	0.44196D+03
7	0.96276D+02	0.60492D+03
8	0.10726D+03	0.67394D+03
9	0.16105D+03	0.10119D+04
10	0.16776D+03	0.10541D+04

The calculated eigen vectors 'node number16'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.000D+00	0.260D+01	0.000D+00	- 0.705D−02	0.000D+00
0.000D+00	0.260D+01	0.000D+00	0.000D+00	0.000D+00	0.699D−02
0.000D+00	- 0.127D−09	0.253D+01	0.000D+00	- 0.230D−01	0.000D+00
0.000D+00	0.250D+01	0.000D+00	0.000D+00	0.000D+00	0.233D−01
0.000D+00	0.000D+00	0.245D+01	0.000D+00	- 0.337D−01	0.000D+00
0.392D−01	0.225D−11	0.292D−11	0.111D+00	0.000D+01	0.000D+00
0.000D+00	0.241D+01	0.000D+00	0.000D+00	0.000D+00	0.315D−01
0.000D+00	0.000D+00	- 0.238D+01	0.000D+00	0.418D−01	0.000D+01
0.000D+00	0.000D+00	0.233D+01	0.000D+00	- 0.474D−01	0.000D+00
0.000D+00	0.233D+01	0.000D+00	0.000D+00	0.000D+00	0.397D−01

Table (1.2a). Example N° 2: free vibration analysis of a cantilever box beam using DNG6

Free vibration analysis with 15 elements, warping included and transverse shear not included « DNG6 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.39694D+01	0.24941D+02
2	0.49677D+02	0.31213D+03
3	0.80101D+02	0.50329D+03
4	0.14958D+03	0.93982D+03
5	0.24118D+03	0.15154D+04
6	0.25112D+03	0.15778D+04
7	0.35541D+03	0.22331D+04
8	0.40491D+03	0.25441D+04
9	0.46357D+03	0.29127D+04
10	0.57307D+03	0.36007D+04

The calculated eigen vectors 'node number16'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.550D−01	0.261D+01	0.000D+00	- 0.720D−02	0.152D−03
0.000D+00	0.000D+00	0.000D+00	0.784D−01	0.000D+00	0.000D+00
0.185D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	- 0.790D−01	0.000D+00	0.000D+00
- 0.186D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.802D−01	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.819D−01	0.000D+00	0.000D+00
- 0.189D+01	0.130D−10	- 0.177D−11	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.844D−01	0.000D+00	0.000D+00
- 0.193D+01	-0.151D−10	0.236D−11	0.000D+00	0.000D+00	0.000D+00

Table (1.2b). Example N° 2: free vibration analysis of a cantilever box beam using DNG6

Free vibration analysis with 15 elements, warping not included and transverse shear included « DNG6 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.39148D+01	0.24597D+02
2	0.65826D+01	0.41360D+02
3	0.23186D+02	0.14568D+03
4	0.38119D+02	0.23951D+03
5	0.49676D+02	0.31213D+03
6	0.60077D+02	0.37747D+03
7	0.96276D+02	0.60492D+03
8	0.10726D+03	0.67394D+03
9	0.16105D+03	0.10119D+04
10	0.16776D+03	0.10541D+04

The calculated eigen vectors 'node number16'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.000D+00	0.260D+01	0.000D+00	- 0.705D−02	0.000D+00
0.000D+00	0.260D+01	0.000D+00	0.000D+00	0.000D+00	0.699D−02
0.000D+00	0.000D+00	-0.253D+01	0.000D+00	0.230D−01	0.000D+00
0.000D+00	0.250D+01	0.000D+00	0.000D+00	0.000D+00	0.233D−01
0.243D−02	0.000D+00	0.000D+00	0.111D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.245D+01	0.000D+00	- 0.337D−01	0.000D+00
0.000D+00	0.241D+01	0.000D+00	0.000D+00	0.000D+00	0.315D−01
0.000D+00	0.000D+00	- 0.238D+01	0.000D+00	0.418D−01	0.000D+01
0.000D+00	0.000D+00	-0.233D+01	0.000D+00	0.474D−01	0.000D+00
0.000D+00	-0.233D+01	0.000D+00	0.000D+00	0.000D+00	-0.379D−01

Table (1.2c). Example N° 2: free vibration analysis of a cantilever box beam using DNG6

Free vibration analysis with 15 elements, warping and transverse shear not included « DNG6 »

The calculated natural frequencies

No.	Hz	rad/sec
1	0.39694D+01	0.24941D+02
2	0.49677D+02	0.31213D+03
3	0.80101D+02	0.50329D+03
4	0.14958D+03	0.93982D+03
5	0.24118D+03	0.15154D+04
6	0.25112D+03	0.15778D+04
7	0.35541D+03	0.22331D+04
8	0.40491D+03	0.25441D+04
9	0.46357D+03	0.29127D+04
10	0.57307D+03	0.36007D+04

The calculated eigen vectors 'node number16'

Deflection x	Deflection y	Deflection z	Rotation x	Rotation y	Rotation z
0.000D+00	0.550D−01	0.261D+01	0.000D+00	- 0.720D−02	0.152D−03
0.000D+00	0.000D+00	0.000D+00	0.784D−01	0.000D+00	0.000D+00
0.185D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	- 0.790D−01	0.000D+00	0.000D+00
- 0.186D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.802D−01	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.819D−01	0.000D+00	0.000D+00
- 0.189D+01	0.130D−10	- 0.177D−11	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.844D−01	0.000D+00	0.000D+00
- 0.193D+01	-0.151D−10	0.236D−11	0.000D+00	0.000D+00	0.000D+00

Table (1.2d). Example N° 2: free vibration analysis of a cantilever box beam using DNG6

Free vibration analysis with 15 elements, warping and transverse shear included « DNG7»

The calculated natural frequencies

No.	Hz	rad/sec
1	0.39023D+01	0.24519D+02
2	0.65616D+01	0.41228D+02
3	0.23112D+02	0.14521D+03
4	0.37997D+02	0.23874D+03
5	0.51370D+02	0.32276D+03
6	0.59885D+02	0.37627D+03
7	0.79846D+02	0.50169D+03
8	0.95969D+02	0.60299D+03
9	0.10692D+03	0.67179D+03
10	0.15582D+03	0.97905D+03

The calculated eigen vectors 'node number16'

Deflection x	**Deflection y**	**Deflection z**	**Rotation x**	**Warping u**	**Rotation y**	**Rotation z**
0.000D+00	0.000D+00	0.259D+01	0.000D+00	0.000D+00	-0.703D−02	0.000D+00
0.000D+00	0.259D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.966D−02
0.000D+00	0.000D+00	0.253D+01	0.000D+00	0.000D+00	-0.229D−01	0.000D+00
0.000D+00	0.249D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.222D−01
0.000D+00	0.000D+00	0.000D+00	0.113D+00	-0.207D−04	0.000D+00	0.000D+00
0.000D+00	0.000D+00	0.245D+01	0.000D+00	0.000D+00	-0.336D−01	0.000D+00
0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.184D+01	0.240D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.314D−01
0.000D+00	0.000D+00	-0.237D+01	0.400D−11	0.000D+00	0.416D−01	0.000D+00
0.000D+00	0.000D+00	0.605D−11	-0.115D+00	0.188D−03	0.377D−11	0.000D+00

Table (1.2e). Example N° 2: free vibration analysis of a cantilever box beam using DNG7

Free vibration analysis with 15 elements, warping and transverse shear not included « DNG7»

The calculated natural frequencies

No.	**Hz**	**rad/sec**
1	0.39678D+01	0.24931D+02
2	0.66682D+01	0.41898D+02
3	0.24775D+02	0.15567D+03
4	0.41867D+02	0.26306D+03
5	0.51534D+02	0.32380D+03
6	0.69378D+02	0.43591D+03
7	0.80101D+02	0.50329D+03
8	0.11724D+03	0.73663D+03
9	0.13597D+03	0.85434D+03
10	0.15632D+03	0.98218D+03

The calculated eigen vectors 'node number16'

Deflection x	**Deflection y**	**Deflection z**	**Rotation x**	**Warping u**	**Rotation y**	**Rotation z**
0.000D+00	0.000D+00	0.261D+01	0.000D+00	0.000D+00	-0.720D−02	0.000D+00
0.000D+00	0.261D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.720D−02
0.000D+00	0.000D+00	0.261D+01	0.000D+00	0.000D+00	-0.250D−01	0.000D+00
0.000D+00	0.261D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.250D−01
0.000D+00	0.000D+00	0.000D+00	0.113D+00	-0.208D−04	0.000D+00	0.000D+00
0.000D+00	0.000D+00	-0.262D+01	0.000D+00	0.000D+00	0.411D−01	0.000D+00
0.185D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.000D+00
0.000D+00	0.262D+01	0.000D+00	0.000D+00	0.000D+00	0.000D+00	0.412D−01
0.000D+00	0.000D+00	0.262D+01	0.000D+00	0.000D+00	-0.575D−01	0.000D+00
0.000D+00	0.000D+00	0.000D+00	-0.116D+00	0.188D−03	0.000D+00	0.000D+00

Table (1.2f). Example N° 2: free vibration analysis of a cantilever box beam using DNG7

Table 2. Natural frequencies (Hz) of a cantilever box beam (Example N° 2)

Conditions	6-D.O.F. (15elements)	7-D.O.F. (15elements)	MSC/ NASTRAN
With shear and warping			
ULZ1	3.9148	3.9023	3.8924270
ULY1	6.5826	6.5616	6.5597000
ULZ2	23.186	23.112	21.809105
ULY2	38.119	37.997	37.459114

Discussion

From the results obtained for example 01, Tables (1.1a), (1.1b), (1.1c) and (1.1d), we can say that:

- For all cases the 1st, 2nd and 3rd uncoupled lateral (flexural) Y and Z modes are within the same frequencies, since the section is square and hence it has equal rigidities or stiffnesses and masses in the Y and Z directions.
- The approximate results obtained using (MSC/ NASTRAN) package are reached by dividing the beam into twelve elements, as shown in Table 1, while excellent results are obtained by only six elements in the (DNG6 and DNG7) programs.

It means that the beam element displacement field used by MSC/NASTAN is not very accurately representing the behavior as the Hermitean cubic polynomials used in the proposed 6and 7 D.O.F. methods given in this study. It can be seen that a half of the number of elements used in MSC/NASTRAN is enough to give more accurate results by using the proposed 6 and 7 D.O.F. (DNG6 and DNG7) programs.

- From the results obtained using (DNG6 and DNG7) programs, the transverse shear contributes considerably to lowering the natural frequencies of the flexural vibration modes (translational in Yand Z directions).
- The inclusion of warping considerably increased the natural frequencies of the torsional-dominant vibration modes.
- The axial modes are identical, since the axial stiffnesses and masses are not changed.

From the results obtained for example 02, it can be seen that:

- Table (1.2a) shows that mode number 6 (60.34 Hz) is the 1st torsional mode when warping and shear deformations are included, but it becomes the 5th mode (49.67 Hz), as shown in Table (1.2c), when warping is neglected. It means that warping affects the results of not only magnitude of the natural frequency but also the torsional mode numbers.
- Table (1.2b) shows that mode number 1(3.969 Hz) is the 1st uncoupled lateral mode (in the Z-direction), when shear deformation is not included, and it becomes (3.914), shown in Table (1.2a), when shear deformation is included. Also, the 2nd uncoupled lateral y- mode (6.582 Hz), shown in Table (1.2a), disappears and the 2 nd mode becomes the torsional mode when shear deformation is neglected, Table (1.2b).
- From Table 2, and by comparing the results obtained by 6D.O.F. method with the results obtained by MSC/NASTRAN, warping and shear deformation must be considered in the analysis for accurate results.
- In the case of 7D.O.F., the inclusion of shear deformation alters only the magnitude of the flexural modes but does not alter the nature of distribution of modes; i.e., if the 2nd mode is the lateral mode it will remain the 2nd mode in all cases even when shear deformation is not included.

- From Table 2, and by comparing the results obtained by 7 D.O.F. methods with the results obtained by MSC/NASTRAN, shear deformation must be considered in the analysis for better results.

CONCLUSIONS

Free vibration formulation of tubular structure and the methods that are used for the solution of the equation of motion are considered in this paper.

The analysis is performed using the computer programs DNG6 and DNG7.

The first ten eigen values and the corresponding mode shapes were considered in free vibration response of certain examples as well as the comparison between 7 D.O.F. per node and 6 D.O.F. per node with the results obtained by MSC/NASTRAN.

It can be seen that the transverse shear contributes considerably to lowering the natural frequencies of the flexural vibration modes (translational in Y- and Z-directions), and the inclusion of warping considerably increased the natural frequencies of the torsional-dominant vibration modes.

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