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A New Method of Dimensional Analysis (Fluid Mechanics Applications)

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ABSTRACT

A new simple method of dimensional analysis is developed. The new method can handle any number of variables. This method needs only finding the inverse of the repeated variables' matrix. Since A^{-1} can be obtained numerically, this method can be easily programmed, and thus time is saved and the error that may occur at any step of hand calculations is avoided. Additionally, the combinations are used for the first time in the dimensional analysis subject. The procedure of the new method and the use of combinations in the dimensional analysis are illustrated through four different examples, which represent all the possible problems in the dimensional analysis.

KEYWORDS: Repeated variables, Unrepeated variables, Matrix inverse, Combinations.

INTRODUCTION

It is possible to obtain analytical solutions for some problems in fluid mechanics. However, flow problems are usually complex, and it is difficult to obtain analytical solutions for them, and, therefore, they must be solved experimentally. Since experimental studies are usually quite expensive, it is necessary to reduce the required number of experiments to the minimum. This can be achieved using dimensional analysis (Potter et al., 1997; Potter, 2009).

The dimensional analysis is a mathematical method to organize any field or laboratory study of the relationship between variable physical quantities. By dimensional analysis, these physical quantities are organized into dimensionless quantities. The number of these dimensionless groups is less than the number of the variables involved in the physical phenomenon. Thus, the required number of experiments is reduced to the minimum.

The dimensional analysis alone does not give the complete solution to the problem, but it is considered as an effective tool in modeling problems that do not have analytical solution and must be solved experimentally (Massey and Ward-Smith, 2006).

The use of dimensional analysis is not confined to fluid problems, but it extends to solve structural problems, as we will see in Example 3.

The dimensionless groups obtained from dimensional analysis are used in the similarity studies and hydraulic models. Another benefit of these dimensionless groups is that they can be used by any measurement system since these groups are dimensionless.

From this brief introduction, it seems that the dimensional analysis subject deserves deep investigation in order to present a simple mathematical model.

Literature Review

Many researchers, such as (Rayleigh, 1892; Novak

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and Cabelka, 1981; Kothandaraman and Rudramoorthy, 2007; Buckingham, 1915; Massey and Ward-Smith, 2006; Potter, 2009; Hunsaker and Rightmire, 1947; Streeter et al., 1998; Langharr, 1951; Barr, 1971) presented their mathematical models. All these methods need hand calculations. As a result, errors may occur at any step of calculations as well as these methods are time consuming, especially when the number of variables is large. Moreover, some of these methods, in the researcher's opinion, are considered complex.

In 2000, Al-Ghazaily and Al-Suhaily presented their method of dimensional analysis (Al-Ghazaily and Al-Suhaili, 2000). This method was easily programmed. Therefore, their method can be used with a significant saving of time and avoiding hand calculations. In order to introduce their method, a general example was taken and solved by all the methods mentioned before with a brief explanation of each method.

The method of Al-Ghazaily and Al-Suhaily can be compared with Langharr's method. In Langharr's method, the solution matrix is obtained from the dimensional matrix by using an analytical method. In the method of Al-Ghazaily and Al-Suhaily, the solution matrix is obtained from the dimensional matrix by using a simple method known as the row operations' method which can be programmed. Thus, Al-Ghazaily and Al-Suhaily presented a program for their method.

Objectives of the Present Research

i. Introducing a new method of dimensional analysis;
ii. Using the combinations in dimensional analysis.

The New Method of Dimensional Analysis

All variables used in fluid mechanics can be expressed in terms of three basic dimensions — the length dimension L, the time dimension T and either the force dimension F or the mass dimension M (Albertson et al., 1960). For heat transfer problems, the temperature dimension θ is added as a basic dimension. The new method can handle any number of basic dimensions. Therefore, 4 basic dimensions are selected. The system of dimensions MLTθ is used. The system of dimensions FLTθ can also be used.

The dimensional matrix of the repeated variables can be written as:

	R_1	R_2	R_3	R_4
M	a_{11}	a_{12}	a_{13}	a_{14}
L	a_{21}	a_{22}	a_{23}	a_{24}
T	a_{31}	a_{32}	a_{33}	a_{34}
θ	a_{41}	a_{42}	a_{43}	a_{44}

The number of repeated variables equals the number of basic dimensions. Therefore, the dimensional matrix of the repeated variables is of the order (4x4). The 4 repeated variables (R_1, R_2, R_3 and R_4) are assumed not to form a dimensionless group.

The dimensional matrix of the unrepeated variables can be written as:

	U_1	U_2	...	U_n
M	b_{11}	b_{12}	…	b_{1n}
L	b_{21}	b_{22}	...	b_{2n}
T	b_{31}	b_{32}	…	b_{3n}
θ	b_{41}	b_{42}	…	b_{4n}

The dimensional matrix of the unrepeated variables is of the order (4xn), since the number of unrepeated variables is not restricted and belongs to the physical problem.

The number of dimensionless groups equals the number of the unrepeated variables.

The first dimensionless group takes the form:

$$\frac{U_1}{R_1^{x_{11}} R_2^{x_{21}} R_3^{x_{31}} R_4^{x_{41}}} = M^0 \, L^0 \, T^0 \, \theta^0 \, .$$

Writing the dimensions of the repeated and unrepeated variables results in:

$$\frac{\left(M^{b_{11}}\,L^{b_{21}}\,T^{b_{31}}\,\theta^{b_{41}}\right)^{1}}{(M^{a_{11}}\,L^{a_{21}}\,T^{a_{31}}\,\theta^{a_{41}})^{x_{11}}\,(M^{a_{12}}\,L^{a_{22}}\,T^{a_{32}}\,\theta^{a_{42}})^{x_{21}}(M^{a_{13}}\,L^{a_{23}}\,T^{a_{33}}\,\theta^{a_{43}})^{x_{31}}(M^{a_{14}}\,L^{a_{24}}\,T^{a_{34}}\,\theta^{a_{44}})^{x_{41}}}=M^{0}\,L^{0}\,T^{0}\,\theta^{0}.$$

This relation requires that:

$$\begin{aligned}
a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + a_{14}x_{41} &= b_{11}\\
a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + a_{24}x_{41} &= b_{21}\\
a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} + a_{34}x_{41} &= b_{31}\\
a_{41}x_{11} + a_{42}x_{21} + a_{43}x_{31} + a_{44}x_{41} &= b_{41}
\end{aligned}$$

These equations represent a system of linear algebraic equations in the unknowns x_{11}, x_{21}, x_{31} and x_{41}. In matrix form, this system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{Bmatrix} x_{11}\\ x_{21}\\ x_{31}\\ x_{41}\end{Bmatrix}=\begin{Bmatrix} b_{11}\\ b_{21}\\ b_{31}\\ b_{41}\end{Bmatrix}$$

The second dimensionless group takes the form:

$$\frac{U_2}{R_1^{x_{21}}R_2^{x_{22}}R_3^{x_{23}}\,R_4^{x_{24}}}=M^{0}\,L^{0}\,T^{0}\,\theta^{0}$$

Writing the dimensions of the repeated and unrepeated variables results in:

$$\frac{\left(M^{b_{12}}\,L^{b_{22}}\,T^{b_{32}}\,\theta^{b_{42}}\right)^{1}}{(M^{a_{11}}\,L^{a_{21}}\,T^{a_{31}}\,\theta^{a_{41}})^{x_{12}}\,(M^{a_{12}}\,L^{a_{22}}\,T^{a_{32}}\,\theta^{a_{42}})^{x_{22}}(M^{a_{13}}\,L^{a_{23}}\,T^{a_{33}}\,\theta^{a_{43}})^{x_{32}}(M^{a_{14}}\,L^{a_{24}}\,T^{a_{34}}\,\theta^{a_{44}})^{x_{42}}}=M^{0}\,L^{0}\,T^{0}\,\theta^{0}\quad.$$

This relation requires that:

$$\begin{aligned}
a_{11}x_{12} + a_{12}x_{22} + a_{13}x_{32} + a_{14}x_{42} &= b_{12}\\
a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} + a_{24}x_{42} &= b_{22}\\
a_{31}x_{12} + a_{32}x_{22} + a_{33}x_{32} + a_{34}x_{42} &= b_{32}\\
a_{41}x_{12} + a_{42}x_{22} + a_{43}x_{32} + a_{44}x_{42} &= b_{42}
\end{aligned}$$

These equations represent a system of linear algebraic equations in the unknowns x_{12}, x_{22}, x_{32} and x_{42}. In matrix form, this system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{Bmatrix} x_{12}\\ x_{22}\\ x_{32}\\ x_{42}\end{Bmatrix}=\begin{Bmatrix} b_{12}\\ b_{22}\\ b_{32}\\ b_{42}\end{Bmatrix}$$

The system of equations for the n dimensionless group takes the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{Bmatrix} x_{1n}\\ x_{2n}\\ x_{3n}\\ x_{4n}\end{Bmatrix}=\begin{Bmatrix} b_{1n}\\ b_{2n}\\ b_{3n}\\ b_{4n}\end{Bmatrix}$$

Combining the individual matrices in one matrix gives the final system of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n}\\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n}\\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n}\\ x_{41} & x_{42} & x_{43} & \cdots & x_{4n}\end{bmatrix}=\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n}\\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n}\\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n}\\ b_{41} & b_{42} & b_{43} & \cdots & b_{4n}\end{bmatrix}$$

or $AX = B$

Multiplying by A^{-1} results in $A^{-1}AX = A^{-1}B$

and the solution is $\boxed{\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}}$.

As we see, the new method can handle any number of unrepeated variables since the matrices A and X are conformable (the number of columns in A (= 4) = the number of rows in X (= 4), and therefore matrix B is of the order (4xn)).

In summary, the new method only needs finding the inverse of the repeated variables' matrix. The matrix inverse can be obtained using analytical or numerical methods, which can be found in any elementary text book of mathematics. Since the matrix inverse can be obtained numerically, the new method can be easily programmed, and thus time is saved and the error that may occur at any step of hand calculations is avoided.

The Use of Combinations in Dimensional Analysis

In literature, several methods are used to select the repeated variables:

i. Variables with uncomplicated dimensional formulae are chosen as repeated variables (Massey and Ward-Smith, 2006).

ii. In fluid mechanics, the variables in a physical phenomenon are classified as geometric characteristics, fluid properties and flow characteristics (Albertson et al., 1960). One variable is chosen from each group as a repeated variable.

iii. In fluid mechanics, the variables in a physical phenomenon are classified as geometric terms, kinematic terms and dynamic terms (Streeter et al., 1998). One variable is chosen from each group as a repeated variable. The dependent quantity should not be selected as a repeated variable.

Consider the following problem (Streeter et al., 1998, Example 5.3, page 231).

The losses Δh/L per unit length of pipe in turbulent flow through a smooth pipe depend upon velocity V, diameter D, gravity g, dynamic viscosity μ and density ρ. With dimensional analysis, determine the general form of the equation.

The variables and their dimensions are written as follows:

[Δh/L]	[V]	[D]	[ρ]	[μ]	[g]
dimensionless	LT^{-1}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-2}

These variables, according to Albertson et al. (1960), can be classified as follows:

geometric characteristics	flow characteristics	fluid properties
D	V and g	ρ and μ

According to Streeter et al. (1998), the variables can be classified as follows:

geometric terms	kinematic terms	dynamic terms
D	V and g	ρ and μ

Streeter et al. (1998) chose only one group of repeated variables V, D and ρ.

In studying a new physical phenomenon, the researcher needs to find all the dimensionless groups. This requires taking all the repeated variables' groups. For the knowledge of the researcher, all text books and researches, including the modern text books such as Potter (2009), consider only one or two groups of repeated variables.

In the present research, the combinations are used to find all the groups of repeated variables, and thus all the dimensionless groups will be obtained. The idea of using the combinations in the dimensional analysis is considered for the first time.

Since the sequence of the repeated variables is insignificant, the number of repeated variables' groups for (r) repeated variables and (k) variables can be obtained using the combination:

$$C_r^k = \frac{k!}{r!\ (k-r)!}.$$

For the problem considered, r =3 and k = 5. The number of repeated variables' groups is:

$$C_3^5 = \frac{5!}{3!\ (5-3)!} = 10,$$

and the repeated variables' groups are:

V Dρ, VDμ, VDg, Vρμ, Vρg, Vμg, Dρμ, Dρg, Dμg and ρμg.

Since the dimensionless group is obtained by dividing the unrepeated variables' group by the repeated variables' group, each repeated variables' group forming a dimensionless group should be excluded.

By applying the new method, it is easy to discover whether the repeated variables' group forms a dimensionless group. When a repeated variables' group forms a dimensionless group, the determinant of the dimensional matrix of the repeated variables will equal zero. That is, the dimensional matrix of the repeated variables is singular.

As it is seen, by the use of combinations there is no need to classify the variables as geometric characteristics, fluid properties and flow characteristics or as geometric terms, kinematic terms and dynamic terms. Additionally, we find all the groups of repeated variables, and thus all the dimensionless groups will be obtained.

The Procedure of the New Method Including the Use of Combinations in Dimensional Analysis

Step 1

1. Write the dimensions of all variables.

If there is a dimensionless variable, such as an angle or a ratio between two lengths, it is considered as a dimensionless group.

2. Determine the number of repeated variables.

The number of repeated variables (r) = the number of basic dimensions repeated more than once.

3. Determine the number of dimensionless groups.

The number of dimensionless groups (m) = the number of unrepeated variables.

4. Determine the number of repeated variables' groups.

If the number of variables (after excluding the dimensionless variables and the dependent variable) is k and the number of repeated variables is r, then the number of repeated variables' groups is found using the combination:

$$C_r^k = \frac{k!}{r!\ (k-r)!}\,.$$

Step 2

1. For each repeated variables' group write:

I. The dimensional matrix of the repeated variables (**A**).

II. The dimensional matrix of the unrepeated variables (**B**).

2. Find A^{-1}.

If the determinant of A, $|A| = 0$, then A is a singular matrix. This means that the repeated variables form a dimensionless group. This repeated variables' group should be excluded.

3. The solution is $X = A^{-1}B$.

4. Find the dimensionless groups, then write the functional relationship of the dimensionless groups: $\emptyset\,(N_1, N_2, \ldots, N_m) = 0$.

If, for example, N_2 contains the dependent variable, then write $N_2 = f\,(N_1, N_3, ..., N_m)$

The procedure of the new method is best illustrated by several examples.

Example 1 (Streeter et al., 1998, Example 5.2, page 230)

A V-notch weir is a vertical plate with a notch of angle Φ cut into the top of it and placed across an open channel. The liquid in the channel is backed up and forced to flow through the notch. The discharge Q is a function of the elevation H of upstream liquid surface above the bottom of the notch. In addition, the discharge depends upon gravity and upon the velocity of approach V_0 to the weir. Determine the form of the discharge equation.

Solution

The variables and their dimensions are written as follows:

[Q]	[H]	[g]	[V_0]	[Φ]
L^3T^{-1}	L	LT^{-2}	LT^{-1}	dimensionless

Φ is dimensionless, thus it is considered as a dimensionless group. The number of basic dimensions repeated more than once is 2. Therefore, the number of repeated variables r = 2. The number of unrepeated variables is 2. Therefore, the number of dimensionless groups m = 2.

The number of variables (after excluding Φ (the dimensionless variable) and the dependent variable Q) is 3, then the number of repeated variables' groups is $C_2^3 = 3$, and the repeated variables' groups are: Hg, HV_0 and gV_0.

Case 1: Hg as repeated variables

The dimensional matrix of the repeated variables (**A**)

	H	g
L	1	1
T	0	-2

The dimensional matrix of the unrepeated variables (**B**)

	Q	V_0
L	3	1
T	-1	-1

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

The dimensionless groups are:

$$N_1 = \frac{Q}{H^{x_{11}} g^{x_{21}}} \qquad and \qquad N_2 = \frac{V_0}{H^{x_{12}} g^{x_{22}}}$$

The solution is $X = A^{-1}B$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore, the dimensionless groups are: $N_1 = \frac{Q}{H^{5/2} g^{1/2}}$, $N_2 = \frac{V_0}{H^{1/2} g^{1/2}}$ $\quad and \quad N_3 = \emptyset$

f (N_1, N_2, N_3) = 0. Since N_1 contains the dependent variable (Q), then $Q = H^{\frac{5}{2}} g^{\frac{1}{2}} f_1\left(\frac{V_0}{\sqrt{gH}}, \emptyset\right)$.

Case 2: HV_0 as repeated variables

The dimensional matrix of the repeated variables (**A**)

	H	V_0
L	1	1
T	0	-1

The dimensional matrix of the unrepeated variables (**B**)

	g	Q
L	1	3
T	-2	-1

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

The dimensionless groups are: $N_1 = \frac{g}{H^{x_{11}} V_0^{x_{21}}}$ *and* $N_2 = \frac{Q}{H^{x_{12}} V_0^{x_{22}}}$.

The solution is $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

Therefore, the dimensionless groups are: $N_1 = \frac{g}{H^{-1} V_0^2}$, $N_2 = \frac{Q}{H^2 V_0}$ *and* $N_3 = \emptyset$

f (N_1, N_2, N_3) = 0. Since N_2 contains the dependent variable (Q), then $Q = H^2 V_0 f_2 \left(\frac{g}{H^{-1} V_0^2}, \emptyset \right)$.

Since any dimensionless group can be inverted or raised to any power without affecting its dimensionless status, then:

$$Q = H^2 V_0 f_2 \left(\frac{V_0}{\sqrt{gH}}, \emptyset \right).$$

Case 3: gV_0 as repeated variables

The dimensional matrix of the repeated variables (**A**)

	g	V_0
L	1	1
T	-2	-1

The dimensional matrix of the unrepeated variables (**B**)

	Q	H
L	3	1
T	-1	0

$\mathbf{A^{-1}} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

The dimensionless groups are:

$$N_1 = \frac{Q}{g^{x_{11}} V_0^{x_{21}}} \quad \textit{and} \quad N_2 = \frac{H}{g^{x_{12}} V_0^{x_{22}}}$$

The solution is $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

Therefore, the dimensionless groups are:

$$N_1 = \frac{Q}{g^{-2} V_0^5}, \quad N_2 = \frac{H}{g^{-1} V_0^2} \quad \textit{and} \quad N_3 = \emptyset$$

f (N_1, N_2, N_3) = 0. Since N_1 contains the dependent variable (Q), then:

$$Q = g^{-2} V_0^5 f_3 \left(\frac{V_0}{\sqrt{gH}}, \emptyset \right).$$

Summary

Repeated variables' group	Dimensionless groups			Discharge equation
Hg	$\frac{Q}{H^{5/2} g^{1/2}}$	$\frac{V_0}{\sqrt{gH}}$	$\emptyset$	$Q = H^{5/2} g^{1/2} f_1 \left(\frac{V_0}{\sqrt{gH}}, \emptyset \right)$
HV_0	$\frac{Q}{H^2 V_0}$	$\frac{V_0}{\sqrt{gH}}$	$\emptyset$	$Q = H^2 V_0 f_2 \left(\frac{V_0}{\sqrt{gH}}, \emptyset \right)$
gV_0	$\frac{Q}{g^{-2} V_0^5}$	$\frac{V_0}{\sqrt{gH}}$	$\emptyset$	$Q = g^{-2} V_0^5 f_3 \left(\frac{V_0}{\sqrt{gH}}, \emptyset \right)$

In general, the last two forms are not very useful because frequently V_0 may be neglected with V-notch weirs. Therefore, if the repeated variables' group, HV_0 or gV_0 is chosen, the forms obtained are not very useful. This example shows that all repeated variables' groups should be considered, and then the forms that are not useful may be neglected.

Example 2 (Potter et al., 1997, Example 6.2, page 233)

The rise of liquid in a capillary tube is to be studied. It is anticipated that the rise h will depend on surface tension σ, tube diameter d, liquid density ρ, gravity g and angle β of attachment between the liquid and tube. Write the function form of the dimensionless variables.

Solution

The variables and their dimensions are written as follows:

[h]	[σ]	[d]	[ρ]	[g]	[β]
L	MT^{-2}	L	ML^{-3}	LT^{-2}	dimensionless

β is dimensionless, thus it is considered as a dimensionless group. The number of basic dimensions repeated more than once is 3. Therefore, the number of repeated variables r = 3. The number of unrepeated variables is 2. Therefore, the number of dimensionless groups m = 2.

The number of variables (after excluding β (the dimensionless variable) and the dependent variable h) is 4, then the number of repeated variables' groups is $C_3^4 = 4$, and the repeated variables' groups are: ρdσ, ρdg, ρgσ and σdg.

Case 1: ρdσ as repeated variables

The dimensional matrix of the repeated variables (**A**)

	ρ	d	σ
M	1	0	1
L	-3	1	0
T	0	0	-2

The dimensional matrix of the unrepeated variables (**B**)

	h	g
M	0	0
L	1	1
T	0	-2

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 3 & 1 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

The dimensionless groups are:

$$N_1 = \frac{h}{\rho^{x_{11}} d^{x_{21}} \sigma^{x_{31}}} \qquad and \qquad N_2 = \frac{g}{\rho^{x_{12}} d^{x_{22}} \sigma^{x_{32}}}$$

The solution is: $$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 3 & 1 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Therefore, the dimensionless groups are:

$$N_1 = \frac{h}{d}, \quad N_2 = \frac{g}{\rho^{-1}d^{-2}\sigma} = \frac{\rho g d^2}{\sigma} = \frac{\gamma d^2}{\sigma} \quad and\ N_3 = \beta$$

The following table shows the dimensionless groups obtained for all the repeated variables' groups.

Repeated variables' groups	Dimensionless groups		
$\rho d\sigma$	$\frac{h}{d}$	$\frac{\gamma d^2}{\sigma}$	β
ρdg	$\frac{h}{d}$	$\frac{\sigma}{\gamma d^2}$	β
σdg	$\frac{h}{d}$	$\frac{\gamma d^2}{\sigma}$	β
$\rho g\sigma$	$\frac{d\sqrt{\gamma}}{\sqrt{\sigma}}$	$\frac{h\sqrt{\gamma}}{\sqrt{\sigma}}$	β

For the dimensionless groups $N_1 = \frac{d\sqrt{\gamma}}{\sqrt{\sigma}}$ and $N_2 = \frac{h\sqrt{\gamma}}{\sqrt{\sigma}}$. Since any dimensionless group can be inverted or raised to any power without affecting its dimensionless status, then:

$$N_3 = \left(\frac{1}{N_1}\right)^2 = \frac{\sigma}{\gamma d^2}$$

and since the dimensionless groups can be recombined to obtain other forms, then:

$$N_4 = \frac{N_2}{N_1} = \frac{h}{d}.$$

Therefore, the dimensionless groups obtained from the 4 repeated variables' groups are the same which are:

$$N_1 = \frac{h}{d}, \quad N_2 = \frac{\sigma}{\gamma d^2} \quad and \quad N_3 = \beta$$

f (N_1, N_2, N_3) = 0. Since N_1 contains the dependent variable (h), then $h = df_1\left(\frac{\sigma}{\gamma d^2}, \beta\right)$.

Example 3 (White, 1999, Example 5.5, page 291)

Assume that the tip deflection δ of a cantilever beam is a function of the tip load P, beam length L, area moment of inertia I and material modulus of elasticity E. With dimensional analysis, determine the general form of the equation.

Solution

The variables and their dimensions are written as follows:

[δ]	[P]	[L]	[I]	[E]
L	F	L	L^4	FL^{-2}

The number of basic dimensions repeated more than once is 2. Therefore, the number of repeated variables r = 2. The number of unrepeated variables is 3. Therefore, the number of dimensionless groups m=3.

The number of variables (after excluding the dependent variable δ) is 4, then the number of repeated variables' groups is $C_2^4 = 6$, and the repeated variables' groups are: LI, EL, EI, PL, PE, PI.

Case 1: LI as repeated variables

The dimensional matrix of the repeated variables (**A**)

	L	I
F	0	0
L	1	4

The dimensional matrix of the unrepeated variables (**B**)

	δ	P	E
F	0	1	1
L	1	0	-2

The dimensional matrix of the repeated variables is singular since $|\mathbf{A}| = 0$. This means that the repeated variables form a dimensionless group. The dimensionless group is I/L^4. Therefore, the repeated variables' group LI is excluded.

Case 2: EL as repeated variables

The dimensional matrix of the repeated variables (**A**)

The dimensional matrix of the unrepeated variables (**B**)

	E	L
F	1	0
L	-2	1

	δ	P	I
F	0	1	0
L	1	0	4

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

The dimensionless groups are:

$$N_1 = \frac{\delta}{E^{x_{11}} L^{x_{21}}},\quad N_2 = \frac{P}{E^{x_{12}} L^{x_{22}}} \quad and$$

$$N_3 = \frac{I}{E^{x_{13}} L^{x_{23}}}$$

The solution is:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

Therefore, the dimensionless groups are:

$$N_1 = \frac{\delta}{L},\quad N_2 = \frac{P}{EL^2} \quad and \quad N_3 = \frac{I}{L^4}$$

f (N_1, N_2, N_3) = 0. Since N_1 contains the dependent variable (δ), then $\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$

For small elastic deflections, δ is proportional to load *P* and inversely proportional to moment of inertia *I*. Since *P* and *I* occur separately, this means that N_1 must be proportional to N_2 and inversely proportional to N_3. Thus, for these conditions,

$$\frac{\delta}{L} = (constant)\frac{P}{EL^2}\frac{L^4}{I} \quad or \quad \delta = (constant)\frac{PL^3}{EI}$$

This could not be predicted by a pure dimensional analysis. Strength-of-materials theory predicts that the value of the constant is ⅓.

The other dimensionless groups for the remaining repeated variables' groups can be obtained following the same procedure. The following table shows all the dimensionless groups for all the repeated variables' groups.

Repeated variables' groups	**Dimensionless groups**		
EL	$\frac{\delta}{L}$	$\frac{P}{EL^2}$	$\frac{I}{L^4}$
EI	$\frac{\delta}{I^{1/4}}$	$\frac{P}{EI^{1/2}}$	$\frac{L}{I^{1/4}}$
PL	$\frac{\delta}{L}$	$\frac{EL^2}{P}$	$\frac{I}{L^4}$
PE	$\frac{\delta E^{1/2}}{P^{1/2}}$	$\frac{LE^{1/2}}{P^{1/2}}$	$\frac{EI^2}{P^2}$
PI	$\frac{\delta}{I^{1/4}}$	$\frac{EI^{1/2}}{P}$	$\frac{L}{I^{1/4}}$

Example 4 (Kreith et al., 1999, Example 3.3.2, page 3-32)

The local heat transfer coefficient *h* may be postulated to be a function of the inside diameter *D* of the pipe, the space-mean of the time-mean velocity *V*, the density *ρ*, the dynamic viscosity *μ*, the specific heat capacity c_p and the thermal conductivity *k* of the fluid. Determine the h equation.

Solution

The variables and their dimensions are written as follows:

[h]	[D]	[V]	[ρ]	[μ]	[k]	[c_p]
$MT^{-3}\theta^{-1}$	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$	$MLT^{-3}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$

The number of basic dimensions repeated more than once is 4. Therefore, the number of repeated variables r = 4. The number of unrepeated variables is 3. Therefore, the number of dimensionless groups m = 3.

The number of variables (after excluding the dependent variable h) is 6, then the number of repeated variables' groups is $C_4^6 = 15$, and the repeated variables' groups are: ρVDk, ρVDμ, ρVDc_p, ρVμk, ρVμc_p, ρVkc_p, ρDμk, ρDμc_p, ρDkc_p, ρμkc_p, VDμk, VDμc_p, VDkc_p, Vμkc_p, Dμkc_p.

In order to shorten the research pages, only the first repeated variables' group, ρVDk, is considered.

The dimensional matrix of the repeated variables (**A**)

	ρ	V	D	k
M	1	0	0	1
L	-3	1	1	1
T	0	-1	0	-3
θ	0	0	0	-1

The dimensional matrix of the unrepeated variables (**B**)

	h	μ	c_p
M	1	1	0
L	0	-1	2
T	-3	-1	-2
θ	-1	0	-1

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The dimensionless groups are:

$$N_1 = \frac{\text{h}}{\rho^{x_{11}} V^{x_{21}} D^{x_{31}} k^{x_{41}}},\ N_2 = \frac{\mu}{\rho^{x_{12}} V^{x_{22}} D^{x_{32}} k^{x_{42}}} \quad and \quad N_3 = \frac{c_p}{\rho^{x_{13}} V^{x_{23}} D^{x_{33}} k^{x_{43}}}$$

The solution is:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ -3 & -1 & -2 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, the dimensionless groups are:

$$N_1 = \frac{\text{h}}{D^{-1}k},\ N_2 = \frac{\mu}{\rho VD} \quad and \quad N_3 = \frac{c_p\,\rho VD}{k}$$

f (N_1, N_2, N_3) = 0. Since N_1 contains the dependent variable h, then $h = \frac{k}{D} f\left(\frac{\rho VD}{\mu}, \frac{c_p\,\rho VD}{k}\right)$.

CONCLUSION

The new method is simple and can handle any number of variables. It needs only finding the inverse of the repeated variables' matrix. Since the maximum order of the repeated variables' matrix is (4x4), it is easy to find its inverse. If this method is programmed, then time is saved and the error that may occur at any step of hand calculations is avoided. The use of combinations is very useful in the dimensional analysis.

RECOMMENDATION

Constructing software for the new method of dimensional analysis is required. This will be original since, for the knowledge of the researcher, there is no software for dimensional analysis.

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