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Exact Solutions for Fixed-Fixed Anisotropic Beams under Uniform Load by Using Maple

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ABSTRACT

The approximate solutions of stresses and displacements were obtained for fixed-fixed anisotropic beams subjected to uniform load. A stress function involving un-known coefficients was constructed, and the general expressions of stress and displacement were obtained by means of airy stress function method. Two types of the description for the fixed end boundary condition were considered. The introduced unknown coefficients in stress function were determined by using the boundary conditions. The approximate solutions for stresses and displacements were finally obtained. Numerical tests show that the solutions agree with the FEM results. These solutions are achieved by using Maple software.

KEYWORDS: Fixed-fixed beam, Anisotropy, Stress function, Approximate solutions, Maple.

INTRODUCTION

The plane stress problem of beams is a classical subject in the elasticity theory and is also frequently encountered in practical cases. Isotropic beams have been investigated by Timoshenko and Goodier (1970) for many cases, such as tension, shearing, pure bending, bending of a cantilever subjected to a transverse load at the end, bending of a simply supported beam under uniform load and other cases of continuously loaded beams. Lekhnitskii (1968) studied the deformations of anisotropic beams including tension, shearing, pure bending, bending of a cantilever loaded at the end, bending of simply supported beams and cantilever beams under uniform load or linearly distributed load. For uniformly loaded, both ends fixed beams, Gere and Timoshenko (1984) presented the expressions of deflection and stress by employing Euler-Bernoulli beam theory. Approximate solutions of fixed-fixed anisotropic beam subjected to uniform load are reported in this paper.

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Basic Equations

In x-y plane, the basic functions for anisotropic material can be expressed as:

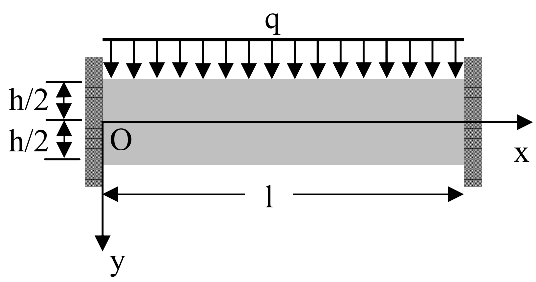

Figure 1: Fixed-fixed beam subjected to a uniform load

$$\frac{\partial u}{\partial x} = s_{11}\sigma_x + s_{12}\sigma_y + s_{16}\tau_{xy}$$
$$\frac{\partial v}{\partial y} = s_{21}\sigma_x + s_{22}\sigma_y + s_{26}\tau_{xy} \quad (1)$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = s_{61}\sigma_x + s_{62}\sigma_y + s_{66}\tau_{xy}$$

u, v, σ_x, σ_y, τ_{xy} are the components of displacement and stress, respectively, s_{ij} are elastic compliance constants, $s_{ij}=s_{ji}$. Stress function Φ with:

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2\Phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} \tag{2}$$

in which the stress function Φ must satisfy the following compatibility equation:

$$s_{22}\frac{\partial^4\Phi}{\partial x^4} - 2s_{26}\frac{\partial^4\Phi}{\partial x^3\partial y} + 2(s_{12}+s_{66})\frac{\partial^4\Phi}{\partial x^2\partial y^2} - 2s_{16}\frac{\partial^4\Phi}{\partial x\partial y^3} + s_{11}\frac{\partial^4\Phi}{\partial y^4} = 0 \tag{3}$$

Approximate Solutions for Fixed-Fixed Anisotropic Beam by Using Maple Software

Consider a fixed-fixed beam with rectangular cross-section subjected to a uniform load q as shown in Figure 1. Suppose that the width of the beam is unit, and the length and height are, respectively, *l* and h.

The stress function is recommended in a polynomial form as:

$$\Phi = Ay^5 + Bxy^4 + Cy^4 + Dx^2y^3 + Exy^3 + Fy^3 + Gxy^2 + Hy^2 + Lx^2y + Jxy + Kx^2 \tag{4}$$

```
> phi:=A*y^5+B*x*y^4+C*y^4+D*x^2*y^3+E*x*y^3+F*y^3+G*x*y^2+H*y^2+L*x^2*y+J*x*y+K*x^2;
```

$$\phi := A\,y^5 + B\,x\,y^4 + C\,y^4 + D\,x^2\,y^3 + E\,x\,y^3 + F\,y^3 + G\,x\,y^2 + H\,y^2 + L\,x^2\,y + J\,x\,y + K\,x^2$$

A, B, C, D, E, F, G, H, L, J and K are 11 unknown constants. Substituting Eq.(4) into Eq.(3) yields:

```
> phisub:=s22*(diff(phi,x,x,x,x))-2*s26*(diff(phi,x,x,x,y))+2*(s12+s66)*(diff(phi,x,x,y,y))-2*s16*(diff(phi,x,y,y,y))+s11*(diff(phi,y,y,y,y))=0;
```

$$phisub := 24\,(s12+s66)\,D\,y - 2\,s16\,(24\,B\,y + 12\,D\,x + 6\,E) + s11\,(120\,A\,y + 24\,B\,x + 24\,C) = 0$$

The parameters must be satisfied that *phisup* is equal to 0 for the arbitrary values x, y. Three equations can be derived as:

```
> result1:=diff(phisub,y);
```

$$result1 := 24\,(s12+s66)\,D - 48\,s16\,B + 120\,s11\,A = 0$$

```
> result2:=diff(phisub,x);
```

$$result2 := -24\,s16\,D + 24\,s11\,B = 0$$

```
> result3:=subs(x=0,y=0,phisub);
```

$$result3 := -12\,s16\,E + 24\,s11\,C = 0$$

(5)

The substitution of Eq.(4) into Eq.(2) gives the expressions of the stress as:

```
> sigmax:=diff(phi,y,y);
```

$$sigmax := 20\,A\,y^3 + 12\,B\,x\,y^2 + 12\,C\,y^2 + 6\,D\,x^2\,y + 6\,E\,x\,y + 6\,F\,y + 2\,G\,x + 2\,H$$

```
> sigmay:=diff(phi,x,x);
```

$$sigmay := 2\,D\,y^3 + 2\,L\,y + 2\,K \quad (6)$$

```
> toxy:=-diff(phi,x,y);
```

$$toxy := -4\,B\,y^3 - 6\,D\,x\,y^2 - 3\,E\,y^2 - 2\,G\,y - 2\,L\,x - J$$

By substitution of Eq.(6) into Eq.(1) and integration, the expressions of the displacement are then obtained as:

```
> dudx:=s11*sigmax+s12*sigmay+s16*toxy;
```

$$dudx := s11\,(20\,A\,y^3 + 12\,B\,x\,y^2 + 12\,C\,y^2 + 6\,D\,x^2\,y + 6\,E\,x\,y + 6\,F\,y + 2\,G\,x + 2\,H) + s12\,(2\,D\,y^3 + 2\,L\,y + 2\,K) + s16\,(-4\,B\,y^3 - 6\,D\,x\,y^2 - 3\,E\,y^2 - 2\,G\,y - 2\,L\,x - J)$$

```
> dvdy:=s12*sigmax+s22*sigmay+s26*toxy;
```

$$dvdy := s12\,(20\,A\,y^3 + 12\,B\,x\,y^2 + 12\,C\,y^2 + 6\,D\,x^2\,y + 6\,E\,x\,y + 6\,F\,y + 2\,G\,x + 2\,H) + s22\,(2\,D\,y^3 + 2\,L\,y + 2\,K) + s26\,(-4\,B\,y^3 - 6\,D\,x\,y^2 - 3\,E\,y^2 - 2\,G\,y - 2\,L\,x - J)$$

```
> dudyplusdvdx:=s16*sigmax+s26*sigmay+s66*toxy;
```

$$dudyplusdvdx := s16\,(20\,A\,y^3 + 12\,B\,x\,y^2 + 12\,C\,y^2 + 6\,D\,x^2\,y + 6\,E\,x\,y + 6\,F\,y + 2\,G\,x + 2\,H) + s26\,(2\,D\,y^3 + 2\,L\,y + 2\,K) + s66\,(-4\,B\,y^3 - 6\,D\,x\,y^2 - 3\,E\,y^2 - 2\,G\,y - 2\,L\,x - J)$$

and

```
> . . . . . .
```

$$\mathrm{trialu}(x,y) := (5\,s16\,A-(s12+s66)\,B+s26\,\mathrm{D})\,y^4 + ((20\,s11\,A-4\,s16\,B+2\,s12\,\mathrm{D})\,x+4\,s16\,C-(s12+s66)\,E)\,y^3 + (3\,(2\,s11\,B-s16\,\mathrm{D})\,x^2+3\,(4\,s11\,C-s16\,E)\,x+3\,s16\,F-(s12+s66)\,G + 2\,s26\,L)\,y^2+(2\,s11\,\mathrm{D}\,x^3+3\,s11\,E\,x^2+(6\,s11\,F+2\,s12\,L-2\,s16\,G)\,x)\,y + (s11\,G-s16\,L)\,x^2+(2\,s11\,H-s16\,J+2\,s12\,K)\,x+w\,y+u0$$

```
> . . . . . .
```

$$\mathrm{trialv}(x,y) := (5\,s12\,A+.5\,s22\,\mathrm{D}-s26\,B)\,y^4 + (2\,(2\,s12\,B-s26\,\mathrm{D})\,x+4\,s12\,C-s26\,E)\,y^3 + (3\,s12\,(\mathrm{D}\,x^2+E\,x+F)-s26\,G+s22\,L)\,y^2 + (2\,(s12\,G-s26\,L)\,x+2\,s12\,H-s26\,J+2\,s22\,K)\,y-.5\,s11\,\mathrm{D}\,x^4-s11\,E\,x^3 - (3\,s11\,F-2\,s16\,G+(s12+s66)\,L)\,x^2+(2\,s16\,H+2\,s26\,K-s66\,J)\,x-w\,x + v0 \qquad (7)$$

where uo, vo and ω are arbitrary constants. So, the displacement components involve 14 undetermined constants. The boundary conditions **BC1** for the ***Timoshenko***'s theory in (Timeoshenko and Goodier, 1970) can be represented as:

$$y=h/2 \;\rightarrow\; \sigma_y=0 \;\oplus\; y=-h/2 \;\rightarrow\; \sigma_y=-q \;\oplus\; y=\pm h/2 \;\rightarrow\; \tau_{xy}=0$$

$$x=0,\; y=0,\; \rightarrow\; u=0, v=0, \frac{\partial v}{\partial x}=0 \;\oplus\; x=l,\; y=0,\; \rightarrow\; u=0, v=0, \frac{\partial v}{\partial x}=0$$

$$\text{and}\quad x=l/2,\; y=0 \;\rightarrow\; u=0$$

We obtain 11 equations equal to 0:

```
> equation1:=subs(y=h/2,sigmay);
```

$$equation1 := \frac{1}{4}\mathrm{D}\,h^3+L\,h+2\,K$$

```
> equation2:=subs(y=-h/2,sigmay+q);
```

$$equation2 := -\frac{1}{4}\mathrm{D}\,h^3-L\,h+2\,K+q$$

```
> equation3:=subs(y=h/2,toxy);
```

$$equation3 := -\frac{1}{2} B\, h^3 - \frac{3}{2} \mathrm{D}\, x\, h^2 - \frac{3}{4} E\, h^2 - G\, h - 2\, L\, x - J$$

```
> equation4:=subs(y=-h/2,toxy);
```

$$equation4 := \frac{1}{2} B\, h^3 - \frac{3}{2} \mathrm{D}\, x\, h^2 - \frac{3}{4} E\, h^2 + G\, h - 2\, L\, x - J$$

```
> equation5:=subs(x=0,y=0,trialu(x,y));
```

$$equation5 := u0$$

```
> equation6:=subs(x=0,y=0,trialv(x,y));
```

$$equation6 := v0$$

```
> equation7:=subs(x=0,y=0,diff(trialv(x,y),x));
```

$$equation7 := 2\, s16\, H + 2\, s26\, K - s66\, J - w$$

```
> equation8:=subs(x=l,y=0,trialu(x,y));
```

$$equation8 := (s11\, G - s16\, L)\, l^2 + (2\, s11\, H - s16\, J + 2\, s12\, K)\, l + u0$$

```
> equation9:=subs(x=l,y=0,trialv(x,y));
```

$$equation9 := -.5\, s11\, \mathrm{D}\, l^4 - s11\, E\, l^3 - (3\, s11\, F - 2\, s16\, G + (s12 + s66)\, L)\, l^2 + (2\, s16\, H + 2\, s26\, K - s66\, J)\, l - w\, l + v0$$

```
> equation10:=subs(x=l,y=0,diff(trialv(x,y),x));
```

$$equation10 := -2.0\, s11\, \mathrm{D}\, l^3 - 3\, s11\, E\, l^2 - 2\, (3\, s11\, F - 2\, s16\, G + (s12 + s66)\, L)\, l + 2\, s16\, H + 2\, s26\, K - s66\, J - w$$

```
> equation11:=subs(x=l/2,y=0,trialu(x,y));
```

$$equation11 := \frac{1}{4} (s11\, G - s16\, L)\, l^2 + \frac{1}{2} (2\, s11\, H - s16\, J + 2\, s12\, K)\, l + u0$$

Together with the 3 equations in (5), we get 14 equations to determine 14 unknown constants:

> . . . . .

$$\{G = \frac{s16\, q}{s11\, h}, C = \frac{s16\, l\, q}{s11\, h^3}, E = 2.\, \frac{l\, q}{h^3}, v0 = 0., u0 = 0., D = -2.\, \frac{q}{h^3}, L = \frac{q}{h},$$
$$K = -.2500000000\, q, w = -.5000000000\, \frac{q\,(-2.\, x\, s16^2 + 3.\, l\, s16^2 - 1.\, s16\, s12\, h + 2.\, s11\, s66\, x - 3.\, s11\, s66\, l + s26\, h\, s11)}{h\, s11},$$
$$J = .5000000000\, \frac{q\,(2.\, x - 3.\, l)}{h}, H = .2500000000\, \frac{q\,(2.\, x\, s16 - 3.\, l\, s16 + s12\, h)}{h\, s11},$$
$$A = .4000000000\, \frac{(s11\, s12 + s11\, s66 - 2.\, s16^2)\, q}{s11^2\, h^3},$$
$$F = -.3333333333\, \frac{q\,(s11^2\, l^2 + s11\, s12\, h^2 + s11\, s66\, h^2 - 2.\, h^2\, s16^2)}{h^3\, s11^2},$$
$$B = -2.\, \frac{s16\, q}{s11\, h^3}\}$$

The components of stress:

> . . . . .

$$sigmaxsub := 8.000000000\, \frac{(s11\, s12 + s11\, s66 - 2.\, s16^2)\, q\, y^3}{s11^2\, h^3} - \frac{24.\, s16\, q\, x\, y^2}{s11\, h^3} + \frac{12\, s16\, q\, l\, y^2}{s11\, h^3} - \frac{12\, q\, x^2\, y}{h^3} + \frac{12\, q\, l\, x\, y}{h^3} - \frac{2.000000000\, q\,(s11^2\, l^2 + s11\, s12\, h^2 + s11\, s66\, h^2 - 2.\, h^2\, s16^2)\, y}{h^3\, s11^2} + \frac{2\, s16\, q\, x}{s11\, h} + \frac{.5000000000\, q\,(2.\, x\, s16 - 3.\, l\, s16 + s12\, h)}{h\, s11} \quad (8a)$$

> . . . . .

$$sigmaysub := -4\, \frac{q\, y^3}{h^3} + \frac{2\, q\, y}{h} - .5000000000\, q \quad (8b)$$

> . . . . .

$$toxysub := 8.\frac{s16\,q\,y^3}{s11\,h^3} + \frac{12\,q\,x\,y^2}{h^3} - \frac{6\,q\,l\,y^2}{h^3} - \frac{2\,s16\,q\,y}{s11\,h} - \frac{2\,q\,x}{h} - \frac{.5000000000\,q\,(2.\,x - 3.\,l)}{h} \tag{8c}$$

The components of displacement:

$$u := \Bigg(2.000000000\,\frac{s16\,(s11\,s12 + s11\,s66 - 2.\,s16^2)\,q}{s11^2\,h^3} + \frac{2.\,(s12 + s66)\,s16\,q}{s11\,h^3} - \frac{2\,s26\,q}{h^3}\Bigg)y^4 + \Bigg(\Bigg(8.000000000\,\frac{(s11\,s12 + s11\,s66 - 2.\,s16^2)\,q}{s11\,h^3} + \frac{8.\,s16^2\,q}{s11\,h^3} - \frac{4\,s12\,q}{h^3}\Bigg)x + \frac{4\,s16^2\,q\,l}{s11\,h^3} - \frac{2\,(s12 + s66)\,q\,l}{h^3}\Bigg)y^3 + \Bigg(-6.\,\frac{s16\,q\,x^2}{h^3} + \frac{6\,s16\,q\,l\,x}{h^3} - \frac{.9999999999\,s16\,q\,(s11^2\,l^2 + s11\,s12\,h^2 + s11\,s66\,h^2 - 2.\,h^2\,s16^2)}{h^3\,s11^2} - \frac{(s12 + s66)\,s16\,q}{s11\,h} + \frac{2\,s26\,q}{h}\Bigg)y^2 + \Bigg(-4\,\frac{s11\,q\,x^3}{h^3} + \frac{6\,s11\,q\,l\,x^2}{h^3} + \Bigg(-2.000000000\,\frac{q\,(s11^2\,l^2 + s11\,s12\,h^2 + s11\,s66\,h^2 - 2.\,h^2\,s16^2)}{s11\,h^3} + \frac{2\,s12\,q}{h} - \frac{2\,s16^2\,q}{s11\,h}\Bigg)x\Bigg)y + \Bigg(.5000000000\,\frac{q\,(2.\,x\,s16 - 3.\,l\,s16 + s12\,h)}{h} - \frac{.5000000000\,s16\,q\,(2.\,x - 3.\,l)}{h} - .5000000000\,q\,s12\Bigg)x - .5000000000\,q\,(-2.\,x\,s16^2 + 3.\,l\,s16^2 - 1.\,s16\,s12\,h + 2.\,s11\,s66\,x - 3.\,s11\,s66\,l + s26\,h\,s11)\,y/(h\,s11) \tag{9a}$$

$$v := \left(2.000000000\,\frac{s12\,(s11\,s12 + s11\,s66 - 2.\,s16^2)\,q}{s11^2\,h^3} - \frac{1.0\,s22\,q}{h^3} + \frac{2.\,s26\,s16\,q}{s11\,h^3}\right) y^4 + \left(2\left(-4.\,\frac{s12\,s16\,q}{s11\,h^3} + \frac{2\,s26\,q}{h^3}\right)x + \frac{4\,s12\,s16\,q\,l}{s11\,h^3} - \frac{2\,s26\,q\,l}{h^3}\right)y^3 + \left(3\,s12\left(-2\,\frac{q\,x^2}{h^3} + \frac{2\,q\,l\,x}{h^3} - \frac{.3333333333\,q\,(s11^2\,l^2 + s11\,s12\,h^2 + s11\,s66\,h^2 - 2.\,h^2\,s16^2)}{h^3\,s11^2}\right) - \frac{s26\,s16\,q}{s11\,h} + \frac{s22\,q}{h}\right)y^2 + \left(2\left(\frac{s12\,s16\,q}{s11\,h} - \frac{s26\,q}{h}\right)x + \frac{.5000000000\,s12\,q\,(2.\,x\,s16 - 3.\,l\,s16 + s12\,h)}{h\,s11} - \frac{.5000000000\,s26\,q\,(2.\,x - 3.\,l)}{h} - .5000000000\,s22\,q\right)y + \frac{1.0\,s11\,q\,x^4}{h^3} - \frac{2\,s11\,q\,l\,x^3}{h^3} - \left(-.9999999999\,\frac{q\,(s11^2\,l^2 + s11\,s12\,h^2 + s11\,s66\,h^2 - 2.\,h^2\,s16^2)}{s11\,h^3} - \frac{2\,s16^2\,q}{s11\,h} + \frac{(s12 + s66)\,q}{h}\right)x^2 + \left(.5000000000\,\frac{s16\,q\,(2.\,x\,s16 - 3.\,l\,s16 + s12\,h)}{h\,s11} - .5000000000\,s26\,q - \frac{.5000000000\,s66\,q\,(2.\,x - 3.\,l)}{h}\right)x + .5000000000\,q\,(-2.\,x\,s16^2 + 3.\,l\,s16^2 - 1.\,s16\,s12\,h + 2.\,s11\,s66\,x - 3.\,s11\,s66\,l + s26\,h\,s11)\,x/(h\,s11) \tag{9b}$$

The boundary conditions **BC2** for the ***Goodier*** 's theory in (Timoshenko and Goodier, 1970) can be represented as:

$$y = h/2 \;\rightarrow\; \sigma_y = 0 \quad \oplus \quad y = -h/2 \;\rightarrow\; \sigma_y = -q \quad \oplus \quad y = \pm h/2 \;\rightarrow\; \tau_{xy} = 0$$

$$x = 0,\; y = 0,\; \rightarrow\; u = 0, v = 0, \frac{\partial u}{\partial y} = 0 \quad \oplus \quad x = l,\; y = 0,\; \rightarrow\; u = 0, v = 0, \frac{\partial u}{\partial y} = 0$$

$$\text{and} \quad x = l/2,\; y = 0,\; \rightarrow\; u = 0$$

Similarly, we will have the results of stresses and displacements.

Example and comparison of the results of BC1, BC2 and FEM

Suppose that the geometric parameters of the beam are: span 10 m, height 1m and width unit. The uniform load intensity is q $=10^7$ N/m. The material properties are: $s_{11} = 11.162\text{x}10^{-12}$, $s_{12} = -4.557\times 10^{-12}$, $s_{16} = 1.847\times 10^{-12}$, $s_{22} = 11.970\times 10^{-12}$, $s_{26} = 2.171\times 10^{-12}$, $s_{66} = 33.778\times 10^{-12}$ (unit: $m^2.N^{-1}$). Figure 2 shows the curve of displacement component v at y=0 (the deflection of the neutral axis) and Figure 3 shows the curve of displacement component u at y = −h/2, for BC1, BC2 and FEM finite element method. The FEM results are achieved by ABAQUS. The boundary conditions for FEM are treated as: *(i)* x = 0, l, - h/2 ≤ y ≤ h/2, u = v = 0; *(ii)* y = h/2, 0 ≤ x ≤ l, $\tau_{xy} = \sigma_y = 0$, *(iii)* y = -h/2, 0 ≤ x ≤ l, $\sigma_y = 10^7$ Pa, $\tau_{xy} = 0$. The Quad4 element of 0.01 m× 0.01 m is employed and the total elements for the whole beam are 1000.

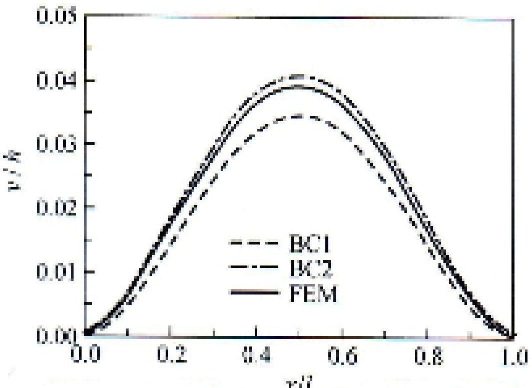

Figure 2: Dimensionless displacement component v at y =0

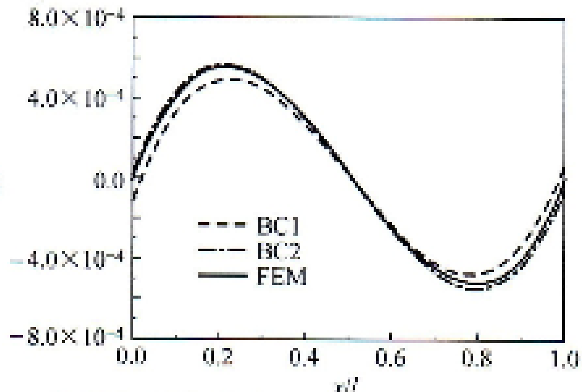

Figure 3: Dimensionless displacement component u at y = −h/2

CONCLUTION

The approximate solutions for fixed-fixed anisotropic beam subjected to uniform load are presented in this paper. The solutions supply a classical example for the elasticity theory. Numerical tests show that the solutions agree with the FEM results. The approximate solutions of the two types of description for fixed-end boundary provide a theoretical range for FEM results.

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