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Development of a New Truss Quadratic Quadrilateral Macro-Element

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ABSTRACT

Macro-elements are one of the powerful means in reducing the number of equations to be solved in finite element analysis .In the proposed method, several finite truss elements will be transformed into a single element called the macro-element. This is done by equating the potential energy of the macro-element to the potential energy of the equivalent truss finite elements. If the order of the macro-element function corresponds to the order of the structural behavior it models, an exact solution is achieved.

In this paper, a truss quadratic quadrilateral macro-element is developed. The developed macro-element was tested and the results were compared with the results of conventional finite element solutions. Excellent results were achieved with a substantial reduction in the number of equations.

KEYWORDS: Truss finite element, Quadratic quadrilateral macro-element.

INTRODUCTION

The analysis of large structural systems using the conventional finite element method is impractical. This is because of the necessity to use a relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of degrees of freedom (d.o.f.) needed to successfully model large systems. There were many trials to overcome these problems like using sub-structuring or repeated elements, but these methods are complex and not easily programmed. The developed macro-element method used in this paper is simple and easily programmed and can be used in any type of structure.

This macro-element is based on the transformation of many structural truss elements into a single equivalent macro-element. This is done by preserving the same potential energies of the structure modeled by truss elements and the same structure modeled by macro-elements (Alani, 1983).

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FORMULATION OF MACRO-ELEMENTS

The formulation of a truss quadratic quadrilateral macro-element will be developed.

In this modeling, several basic truss elements are combined to form a macro-element (Alani, 1983, 2002).

The original structure that consists of many truss finite elements will be replaced by an equivalent model containing one or more macro-elements.

The macro- elements are assembled and analysis continued in a manner analogous to that used in the finite element method.

BASIC ASSUMPTIONS FOR MACRO-ELEMENT FORMULATION

The formulation is based on the following

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assumptions (Alani, 2006):

1. The potential and kinetic energies of the original finite element and the equivalent macro-element models are equal.
2. All the elements that are composing the macro-element must be of the same type such as truss elements, beam elements, plane stress elements, plate bending elements…etc.
3. The order of the assumed displacement field of the macro-element is at least of the same order as that of the original finite elements.
4. The macro-element behavior follows the theory which controls the behavior of the structural elements that compose the macro-element.
5. The compatibility requirements for the macro-element are the same as those of the original finite elements.

NECESSARY STEPS NEEDED FOR DEVELOPMENT OF A MACRO-ELEMENT

The necessary steps for the development of a macro-element are as follows (Alani, 2006):

Step (1) - Divide the original structure that consists of many finite elements into macro-elements.

Step (2) - Select the order of the macro-element displacement function. This step depends on the order and number of the finite-elements composing the macro-element. Accuracy of the results depends greatly on this step.

Step (3) - Set-up the stiffness matrices of the finite-elements forming the macro-element.

Step (4) - Calculate the local coordinates (ξ, η) for the nodal points of the finite elements with respect to the macro-element nodes so as to formulate the transformation matrix [T] required in the next step.

Step (5) - Formulate the transformation matrix [T], which relates the nodal degrees of freedom of the macro-element to the nodal degrees of freedom of the original structure modeled by finite elements.

The stiffness matrix of each finite element is multiplied by its corresponding transformation matrix to produce the participation of this element in establishing the macro-element stiffness matrix, as it will be seen later.

The stiffness matrix of the macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent model as follows:

$$U_o = U_m \qquad (1)$$

where:

Uo: The strain energy of the original structure modeled by many finite elements that constitute one macro-element.

Um: The strain energy of the macro-element.

$$\frac{1}{2}\lfloor q_o \rfloor [Sk_o]\{q_o\} = \frac{1}{2}\lfloor q_m \rfloor [K_m]\{q_m\} \qquad (2)$$

where:

qo: Displacement vector of the structure modeled by many finite elements that constitute one macro-element.

q m: Displacement vector of one macro-element.

[SKo]: The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element.

[Km]: The stiffness matrix of the macro-element.

Let the displacement vector of the original structure (which constitutes one macro-element), {qo}, be related to that of the macro-element, {qm}, as:

$$\{q_o\} = [T]\{q_m\} \qquad (3)$$

where [T] is the transformation matrix for the macro-element.

Substituting Eq. (3) into Eq. (2) gives:

$$\lfloor q_m \rfloor [T]^T [SKo] [T] \{q_m\} = \lfloor q_m \rfloor [K_m] \{q_m\} [T]^T$$

$$[SK_o] [T] = [K_m] \quad (4)$$

In this solution, matrix [SKo] is not needed, only [Ko], the stiffness matrix of a finite element bounded by the macro-element, is needed. To explain this, let

n be the number of finite elements comprising the macro-element;

[Te] be the finite element transformation matrix.

Every time, [Te] carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite element under consideration. The transformation for each finite element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of [Km], and:

$$\sum_{e=1}^{n} [T_e]^T [K_o] [T_e] = [K_m] \quad (5)$$

The transformation matrix [T] is simply the evaluation of the shape functions of the macro-element at the nodes of the finite elements. This evaluation is based on local coordinates for the nodal points of the finite elements with respect to the macro-element nodes.

To form a general transformation matrix Ti corresponding to an arbitrary nodal point j of the original structure within a certain macro- element, consider the notation Nk_i^j which means the shape function K of node i of this macro-element is evaluated at point j using its local coordinates within the macro – element, then the transformation matrix will depend on the macro-element type as will be seen latter.

Step (6)- Construction of the macro-element nodal load vector.

The external loading is applied at the nodes of the finite element model.

However, these nodes may not necessarily coincide with the macro-element nodes. It is required to calculate the equivalent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed to equivalent macro-element structural load vector by equating the external work done on the original structure modeled by finite elements and that of the macro-element model as follows:

$$Wo = Wm \quad (6)$$

where:

Wo: the external work done on the original structure that constitutes one macro-element.

Wm: the external work done on the macro-element.

$$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\} \quad (7)$$

where:

{Fo}: the assembled nodal load vector of the finite elements constituting one macro-element.

{Fm}: the equivalent nodal load vector of the macro-element.

Substituting Eq. (3) into Eq. (7) gives:

$$\lfloor q_m \rfloor [T]^T \{F_o] = \lfloor q_m \rfloor \{F_m\}$$

$$[T]^T \{F_o\} = \{F_m\} \quad (8)$$

where [T] is the same transformation matrix used in deriving [Km].

Step (7) - Assemble all the macro-element stiffness matrices into a structural stiffness matrix and also construct the macro-element structural load vector.

Step (8) - Apply the boundary conditions which will be at the macro-element nodes. Other boundary conditions corresponding to the eliminated nodes of the finite- elements of the original structure will be ignored.

Step (9) - Solve for the equivalent model nodal displacements in a straight forward manner.

Step (10) - Using results obtained in step (9), the displacements at any point inside the macro-elements may be calculated making use of the macro-element shape function.

Step (11) - After the structure is analyzed for nodal displacements, the stresses at selected points in each macro-element may be obtained in the usual manner.

Formulation of Truss Quadratic Quadrilateral Macro-Element

Truss problems in two dimensions are plane stress problems in general. In order to demonstrate the formulation, let a simple problem be considered. The procedure will then be generalized to any problem. In this problem, the structure consists of repeated cells called repeated elements. Each repeated cell is composed of many truss elements. It is not necessary to have the structure consist of repeated elements as will be shown later. Consider the structure shown in Fig. 1 which consist of many truss elements and denote this system as case a.

This structure will be modeled by two repeated macro-elements as shown in Fig. 2. Denote this system as case b.

A single macro-element is isolated as shown in Fig. 3. The order of the element function to be selected depends largely on the configuration of the original structure. It is then evident that a quadrilateral element with quadratic interpolation functions models the structure exactly.

For quadratic curved-sided quadrilateral elements as shown in Fig. 4, the shape functions are (Cook, 1983):

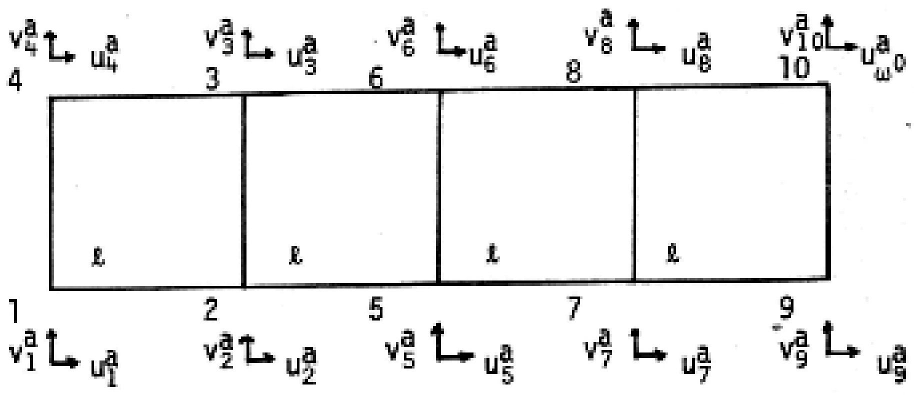

Figure (1): Case a

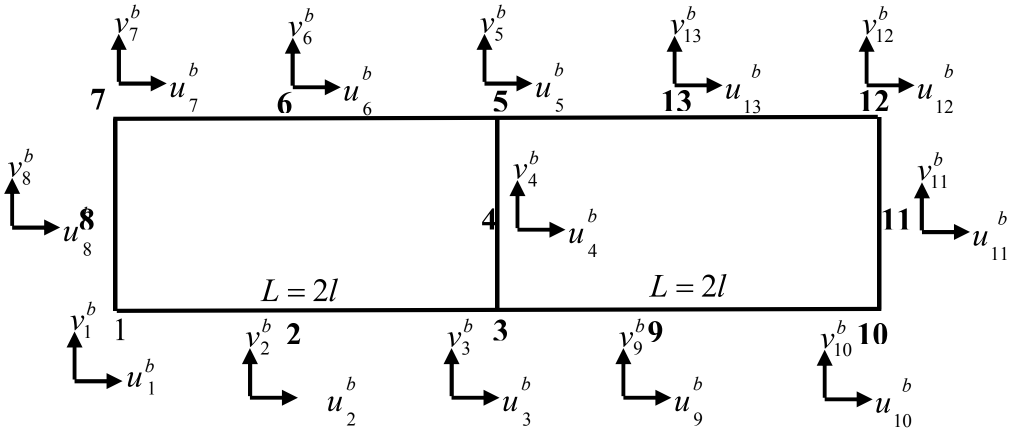

Figure (2): Case b

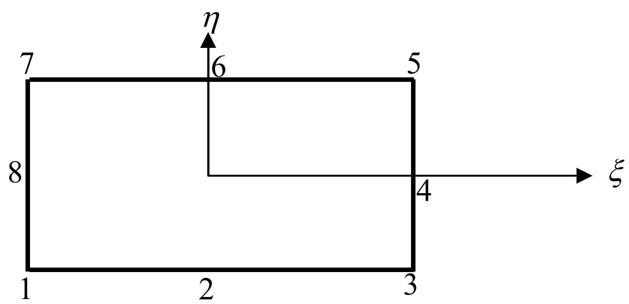

Figure (3): Eight-node quadratic quadrilateral isoparametric element

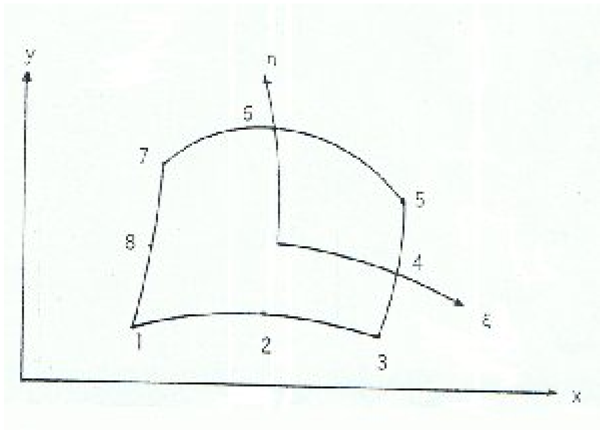

Figure (4): Quadratic curved-sided quadrilateral isoparametric element

Corner nodes:

$$\left.\begin{array}{l} N_i = 1/4\,(1+\xi_0)(1+\eta_0)(\xi_0+\eta_0-1) \\ \text{Mid-side nodes :} \\ \text{For } \xi_i = 0\,;\ N_i = 1/2\,(1-\xi^2)\,(1+\eta_0) \\ \text{For } \eta_i = 0\,;\ N_i = 1/2\,(1+\xi_0)\,(1-\eta^2) \end{array}\right\} \quad \ldots\ldots\ldots\ldots(9)$$

where : $\xi_0 = \xi * \xi_i$; $\eta_0 = \eta * \eta_i$ and ξ_i , η_i are the values of ξ , η at the nodes (joints) of the original truss structure with respect to the macro-element enclosed in those joints.

In this paper, the deformed configuration of the original structure is represented by quadratic quadrilateral macro-element and the displacement functions are :

$$\left.\begin{array}{l} u(\xi,\eta) = \sum_{i=1}^{n} N_i * u_i \\ \\ v(\xi,\eta) = \sum_{i=1}^{n} N_i * v_i \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Now, the formulation of transformation matrix, which is steps 5 will be done, and then we will return to steps 3 and 4.

The displacement relations between cases a and b are:

$\{q_0\} = [T]\ \{q_m\}$

where:

$$\left.\begin{array}{l} \{q_0\}^T = \left[u_1^a v_1^a u_2^a v_2^a u_3^a v_3^a \ldots\ldots\ldots\ldots\ldots u_6^a v_6^a\right] \\ \\ \{q_m\}^T = \left[u_1^b v_1^b u_2^b v_2^b u_3^b v_3^b \ldots\ldots\ldots\ldots\ldots u_6^b v_6^b\right] \end{array}\right\} \quad \ldots\ldots(11)$$

In general, the relations between case a and case b are :

$$u_k^a = u_{(\xi,\eta)}^b \quad \text{and} \quad v_k^a = v_{(\xi,\eta)}^b \qquad \ldots\ldots (12)$$

where k=1, 2, 3,……….........., n

ξ,η are the local coordinates of node k of the original structure within the macro-element.

At any point within the macro-element:

$$\left.\begin{array}{ll} u_{(\xi,\eta)}^b = \sum_{1}^{8} N_i u_i^b\,; & \text{therefore,}\quad u_k^a = \sum_{1}^{8} N_i u_i^b \\ v_{(\xi,\eta)}^b = \sum_{1}^{8} N_i v_i^b\,; & \text{therefore,}\quad v_k^a = \sum_{1}^{8} N_i v_i^b \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Then, the general form of transformation matrix is as shown in Eqn. 14 below.

$$\begin{Bmatrix} u_1^a \\ v_1^a \\ u_2^a \\ v_2^a \\ . \\ . \\ u_{n-1}^a \\ v_{n-1}^a \\ u_n^a \\ v_n^a \end{Bmatrix} = \begin{bmatrix} N_1|_1^a & 0 & N_2|_1^a & 0 & N_3|_1^a & 0 & N_4|_1^a & 0 \\ 0 & N_1|_1^a & 0 & N_2|_1^a & 0 & N_3|_1^a & 0 & N_4|_1^a \\ N_1|_2^a & 0 & N_2|_2^a & 0 & N_3|_2^a & 0 & N_4|_2^a & 0 \\ 0 & N_1|_2^a & 0 & N_2|_2^a & 0 & N_3|_2^a & 0 & N_4|_2^a \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ N_1|_{n-1}^a & 0 & N_2|_{n-1}^a & 0 & N_3|_{n-1}^a & 0 & N_4|_{n-1}^a & 0 \\ 0 & N_1|_{n-1}^a & 0 & N_2|_{n-1}^a & 0 & N_3|_{n-1}^a & 0 & N_4|_{n-1}^a \\ N_1|_n^a & 0 & N_2|_n^a & 0 & N_3|_n^a & 0 & N_4|_n^a & 0 \\ 0 & N_1|_n^a & 0 & N_2|_n^a & 0 & N_3|_n^a & 0 & N_4|_n^a \end{bmatrix} \begin{Bmatrix} u_1^b \\ v_1^b \\ u_2^b \\ v_2^b \\ u_3^b \\ v_3^b \\ u_4^b \\ v_4^b \\ . \\ . \\ u_8^b \\ v_8^b \end{Bmatrix} \quad \ldots.(14)$$

where

$N_i \Big|_j^a$ is the shape function i evaluated at node j of case a.

As an example, let us find $_3$ and their relations with u^b. This d.o.f. $_3$ is acting at node 3 of system a. According to the macro-element, this node has ξ = 0.0, η=+1, then:

$$u_3^a = u_{(0,1)}^b = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 + N_7u_7 + N_8u_8$$

Therefore,

u_3^a = 1/4 (1- ξ)(1- η)(- ξ- η-1)u_1 +1/2 (1- $ξ^2$)(1- η) u_2 +1/4(1+ ξ)(1- η)(ξ- η-1)u_3 +1/2(1- $ξ^2$)(1- η)u_4 + 1/4 (1+ ξ)(1+ η)(ξ+ η-1)u_5 + 1/2(1- $ξ^2$)(1+ η) u_6+1/4(1- ξ)(1+ η)(- ξ+ η-1)u_7 +1/2 (1- ξ)(1- $η^2$)u_8

=1/4(1-0)(1-1)(0-1-1) u_1+1/2(1-0)(1-1) u_2 +1/4 (1+0)(1-1)(1-1-1) u_3 +1/2 (1+0)(1-1) u_4 +1/4(1+0)(1+1)(0+1-1) u_5 +1/2 (1-0)(1+1) u_6 +1/4(1-0)(1+1)(0+1-1) u_7 +1/2(1-0)(1-1) u_8.

Therefore, $u_3^a = u_6^b$

By the same way, all the elements of transformation matrix [T] will be evaluated. Now, the construction of the transformation matrix [T] is done. Step 3 requires setting –up the stiffness matrices of the finite elements forming the macro-element. The assumptions are based on the notion that the structure is a truss. The formulation of the stiffness matrix of the truss element required will be done . The displacement function of a truss element is linear. This is because a truss element has two nodes with one degree of freedom per node. See Fig.5.

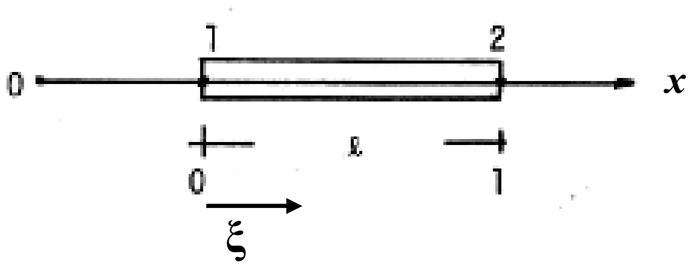

Figure (5): Truss element in local

The shape functions in dimensionless coordinates are:

$$N_1 = 1 - \xi$$

$$N_2 = \xi$$

The displacement function is:

$$u = N_1u_1 + N_2u_2$$

$$u = u(\xi) \text{ and } \xi = \xi(x) \qquad (15\text{-a})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\ \frac{\partial \xi}{\partial x}$$

But,

$$\frac{x - x_1}{\ell} = \xi \qquad (15\text{-b})$$

$$\partial x = \ell d\xi, \quad \frac{\partial \xi}{\partial x} = \frac{1}{\ell} \qquad (15\text{-c})$$

$$\varepsilon = \frac{\partial u}{\partial x} = \frac{1}{\ell}\ \frac{\partial u}{\partial \xi} \qquad (15\text{-d})$$

$$\sigma = E\varepsilon$$

Then, the total potential energy is:

$$\Pi = 1/2\int_{vol} \varepsilon^T E\,\varepsilon dv - \sum_1^m F_i u \quad \text{i = 1, 2.m} \qquad (15\text{-e})$$

where m is the number of concentrated loads at the nodes of the truss structure. But,

$$\varepsilon = \frac{1}{\ell}\ \frac{\partial u}{\partial \xi} = \frac{1}{\ell}\left\{ (\frac{\partial N_1}{\partial \xi}u_1 + \frac{\partial N_2}{\partial \xi}u_2) \right\} \qquad \text{(15-f)}$$

$$= \frac{1}{\ell}\ \begin{bmatrix} -1 & 1 \end{bmatrix} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{u} \qquad \text{(15-g)}$$

$$\therefore \Pi = \tfrac{1}{2}\{u\}^T \left[\int_{vol} \frac{1}{\ell}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{E}{\ell}\begin{bmatrix} -1 & 1 \end{bmatrix} \right] dv\,\{u\} - \sum_{1}^{m} \begin{bmatrix} N_1 N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} F_i \qquad \text{(15 – h)}$$

If the cross-section of the element is assured constant all over the length, then:

$$\Pi = 1/2\frac{AE}{\ell}\{u\}^T \left[\int_0^1 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} (\ell d\xi) \right] \{u\} - \{u\}^T \sum_1^m [N]^T \ F_i \qquad \text{(15-i)}$$

Minimize the total potential energy with respect to $\{u\}^T$

$$\frac{\partial \Pi}{\partial \{u\}^T} = 0 \qquad \text{(15-j)}$$

$$\frac{AE}{l}\int_0^l \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi\{u\ \} - \sum_1^m [N\ ]^T F_i = 0 \qquad \text{(15-k)}$$

$$\underbrace{\frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{K_e} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\sum_1^m [N\ ]^T F_i}_{F_e} \qquad \text{(15-l)}$$

where

$[K_e]$ is the element stiffness matrix in local coordinates.

$\{F_e\}$ is the load vector applied at the truss nodes. To transform the element stiffness matrix to global coordinates, a coordinate transformation matrix is needed. In two-dimensional problems, this has the form:

$$[\lambda] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad \text{(15-m)}$$

where θ is the angle between the local axis and the global axis.

$$[K_0] = [\lambda]^T [K_e][\lambda] \qquad \text{(15-n)}$$

$$[K_0] = \frac{AE}{\ell}\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \qquad \text{(15-o)}$$

$$[K_0] = \frac{AE}{\ell}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \qquad \text{(15-p)}$$

Now, the formulation of the macro-element is:

$$\left.\begin{array}{l} [\ \ ] = \ [\ \ ][\ \ ] \\ h \qquad\qquad\qquad : \\ \{\ \ \} = \ [\ \ ] \end{array}\right\} \qquad \text{(16)}$$

where F_{ix}, F_{iy} are the components of the load F_i in the x and y global directions. Construction of the transformation matrix T is based on the evaluation of the shape functions at specific points inside the macro-element. These points are the nodes of the original structure; i.e., truss nodes.

It is then clear that a local coordinate for each of those nodes is required. There are two methods to do this. The first is the closed form solution, and the second is an iterative scheme. Although the closed form solution of the local coordinates is readily formulated for quadratic curved-sided quadrilateral elements, it is more complex to formulate for higher order elements.

Closed Form Solution for (ξ and η) for Quadratic Isoparametric Quadrilateral Macro-elements

Step 4 required calculating the local coordinates (ξ and η) for the nodal points of the truss finite elements with respect to the macro-element nodes. The relations between the local coordinates (ξ and η) and the global coordinates (X , Y) of any point in the region of the macro-element are :

$$X = \sum_{1}^{8} N_i X_i \qquad (17)$$

$$Y = \sum_{1}^{8} N_i Y_i \qquad (18)$$

where X_i and Y_i are the values of the global coordinates of node i, and *(X , Y)* are the values of the required point in global coordinates. In the above two Eqns., the unknowns are ξ and η only. The shape functions of the quadratic isoparametric quadrilateral element are shown in Eqn. 9. Expanding and rearranging them gives (Smith and Griffith, 2004).

$$\left.\begin{aligned}
N_1 &= 1/4\,(1-\xi)\,(1-\eta)\,(-\xi-\eta-1)\\
N_2 &= 1/2\,(1-\xi^2)\,(1-\eta)\\
N_3 &= 1/4\,(1+\xi)\,(1-\eta)\,(\xi-\eta-1)\\
N_4 &= 1/2\,(1+\xi)\,(1-\eta^2)\\
N_5 &= 1/4\,(1+\xi)\,(1+\eta)\,(\xi+\eta-1)\\
N_6 &= 1/2\,(1-\xi^2)\,(1+\eta)\\
N_7 &= 1/4\,(1-\xi)\,(1+\eta)\,(-\xi+\eta-1)\\
N_8 &= 1/2\,(1-\xi)\,(1-\eta^2)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Substituting Eqns. (19) into Eqn. (17) yields:

$$\begin{aligned}
X &= N_1 X_1 + N_2 X_2 + N_3 X_3 + N_4 X_4 + N_5 X_5 + N_6 X_6 + N_7 X_7 + N_8 X_8\\
&= [1/4\,(1-\xi)(1-\eta)(-\xi-\eta-1)]X_1 + [1/2\,(1-\xi^2)\,(1-\eta)]X_2 + [1/4\,(1+\xi)(1-\eta)(\xi-\eta-1)]X_3\\
&\quad + [1/2\,(1+\xi)\,(1-\eta^2)]X_4 + [1/4\,(1+\xi)(1+\eta)(\xi+\eta-1)]X_5 + [1/2\,(1-\xi^2)\,(1+\eta)]X_6 +\\
&\quad [1/4\,(1-\xi)(1+\eta)(-\xi+\eta-1)]X_7 + [1/2\,(1-\xi)\,(1-\eta^2)]X_8 .
\end{aligned}$$

Expand, simplify and transfer all terms with η only and X_i to left-hand side (LHS) and rearrange to get:

$$\eta^2\underbrace{\left(1/4X_1 - 1/4X_3 - 1/4X_5 - 1/4X_7 + 1/2X_8\right)}_{ALX} + \eta\underbrace{\left(1/2X_2 + 1/2X_4 - 1/2X_6\right)}_{BLX} +$$

$$\underbrace{\left(X + 1/4X_1 - 1/2X_2 + 1/4X_3 - 1/2X_4 + 1/4X_5 - 1/2X_6 + 1/4X_7 - 1/2X_8\right)}_{CLX} =$$

$$\xi\left[\underbrace{\left(1/2X_4 - 1/2X_8\right)}_{CRX} + \eta\underbrace{\left(1/4X_1 - 1/4X_3 - 1/2X_4 + 1/4X_5 - 1/4X_7\right)}_{BRX} + \eta^2\underbrace{\left(-1/4X_1 + 1/4X_3 + 1/4X_5 - 1/4X_7 + 1/2X_8\right)}_{ARX}\right]$$

$$+\,\xi^2\left[\underbrace{\left(1/4X_1 - 1/2X_2 + 1/4X_3 + 1/4X_5 - 1/2X_6 + 1/4X_7\right)}_{CCRX} + \eta\underbrace{\left(-1/4X_1 + 1/2X_2 - 1/4X_3 + 1/4X_5 - 1/2X_6 + 1/4X_7\right)}_{BBRX}\right]$$

Therefore,

$$\underbrace{(ALX)\eta^2 + (BLX)\eta + (CLX)}_{CDX} = \xi\left[\underbrace{CRX + BRX\eta + ARX\eta^2}_{BDX}\right] + \xi^2\underbrace{\left(CCRX + BBRX\right)}_{ADX}$$

Therefore,

$(ADX)\xi^2 + (BDX)\xi - CDX = 0$.

Therefore,

$$\xi = \frac{-BDX \pm \sqrt{(BDX)^2 + 4(ADX)(-CDX)}}{2ADX} \qquad (20)$$

The right-hand side of the above Eqn. is a function of η only.

In the same way, substituting Eqns. (19) into Eqn. (18) and simplifying yield:

$$\xi = \frac{-BDY \pm \sqrt{(BDY)^2 + 4(ADY)(-CDY)}}{2ADY} \qquad (21)$$

The right-hand side of the above Eqn. is a function of η only

Eqns. (20) and (21) are equal. By equating Eqn. (20) to Eqn. (21), one can solve for η . Use the solved value of η in Eqn. (17) or Eqn. (18) to get the value of ξ.

Now, the values of local coordinates ξ and η for any point of coordinates X,Y are known. Therefore, $[T]$ is constructed. Also, $[K_e]$ is known. Extract from the transformation matrix $[T]$ that part corresponding to the degrees of freedom of $[K_e]$. Let this part be called $[T_e]$, then:

$$[K_m] = \sum_{e=1}^{n} [T_e]^T [K_e][T_e] \qquad (22)$$

The external loadings are applied at the truss structure nodes although they may not coincide with the macro-element nodes. Thus, a subroutine to calculate the equivalent consistent nodal load vector for the macro-element is required. This is done for each concentrated load by the evaluation of the equation shown below.

$$\{F_m\}i = [N]^T F_i \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \qquad (23)$$

$16\times1 \quad 16\times2 \quad 2\times1$

where

$$N = \begin{bmatrix} N1 & 0 & N2 & 0 & N3 & 0 ... N8 & 0 \\ 0 & N1 & 0 & N2 & 0 & N3 & 0 & N8 \end{bmatrix}$$

F_i is the force at node i. F_m is the macro-element nodal load vector from F_i only. Summing up the effect of all loadings yields:

$$\{F_m\} = \sum_{i=1}^{n} [N]^T F_i \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \qquad (24)$$

Here, the shape functions are evaluated at the local coordinate values of the point of application of F_i. Next, the total stiffness and load vector of the structure composed of macro-elements is formulated. This is done by the conventional assembling subroutine used in finite element methods. The boundary conditions are applied.

The linear system is solved using Gauss elimination or any solving subroutine. The solution is for the macro-element nodal values.

Applications

Problem No.1: A cantilever truss consists of nine truss members with a total of 12 degrees of freedom, all members are of same AE. Three degrees of freedom are constrained as shown in Fig.6.

This problem was modeled by a quadratic quadrilateral isoparametric macro-element as shown in Fig.7. The results are shown in Table 1.

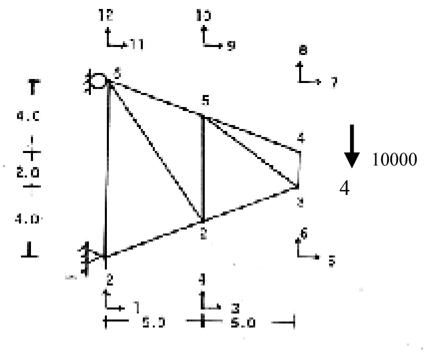

Figure (6): Cantilever truss in two dimensions: problem no. 1

Problem No. 2: A cantilever truss consists of 13 truss members with 16 degrees of freedom, all members are of same AE. Four degrees of freedom are constrained as shown in Fig. 8.

This problem was modeled by a quadratic quadrilateral isoparametric macro-element as shown in Fig.7. The results are shown in Table 2.

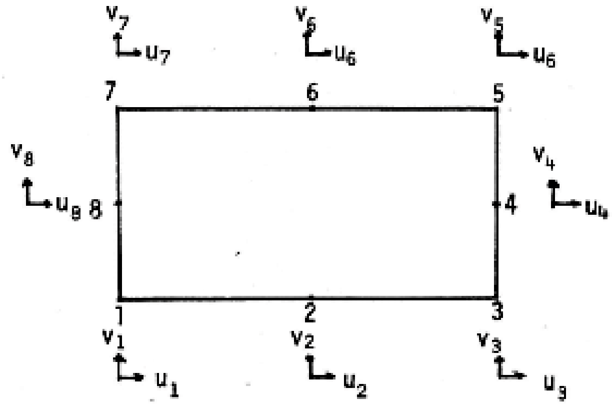

Figure (7): Macro-element method modeling of truss problem no. 1 & 2 in two dimensions

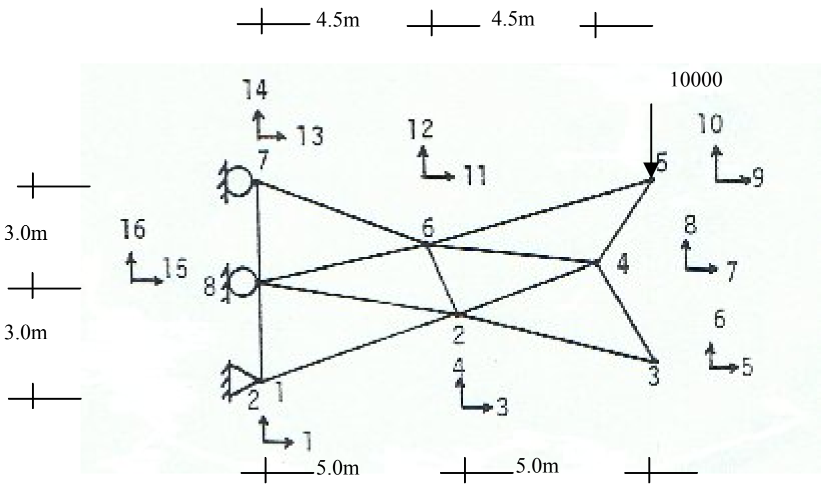

Figure (8): Cantilever truss in two dimensions, problem no. 2

Problem No. 3: Same as problem no. 1: a) Using one linear quadrilateral macro-element with 8 degrees of freedom to model the whole truss, the results are as tabulated in Table 3. b) Using two linear quadrilateral macro-elements to model the truss, the results are as tabulated in Table 2.

Table 1. Displacement results of truss problem no.1 in two dimensions

Displaces	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	-1.958×10^{-4}	-1.958×10^{-4}	0.0
4	-1.072×10^{-3}	-1.072×10^{-3}	0.0
5	-4.865×10^{-5}	-4.865×10^{-5}	0.0
6	-2.741×10^{-3}	-2.741×10^{-3}	0.0
7	-4.160×10^{-4}	-4.160×10^{-4}	0.0
8	-2.941×10^{-3}	-2.941×10^{-3}	0.0
9	2.517×10^{-4}	2.517×10^{-4}	0.0
10	-1.272×10^{-3}	-1.272×10^{-3}	0.0
11	0.0	0.0	0.0
12	-6.000×10^{-4}	-6.000×10^{-4}	0.0

Table 2. Displacement results of truss problem no. 2 and truss problem no. 3-b in two dimensions

Displaces	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	$-1.001X10^{-3}$	$-1.001X10^{-3}$	0.0
4	$-4.044X10^{-3}$	$-4.044X10^{-3}$	0.0
5	$-2.827X10^{-3}$	$-2.827X10^{-3}$	0.0
6	$-1.313X\ X10^{-2}$	$-1.313X\ X10^{-2}$	0.0
7	$-2.475X\ X10^{-4}$	$-2.475X\ X10^{-4}$	0.0
8	$-1.128\ X10^{-2}$	$-1.128\ X10^{-2}$	0.0
9	$3.535\ X10^{-3}$	$3.535\ X10^{-3}$	0.0
10	$1.55X\ X10^{-2}$	$1.55X\ X10^{-2}$	0.0
11	$1.168X\ X10^{-3}$	$1.168X\ X10^{-3}$	0.0
12	$-2.961X\ X10^{-3}$	$-2.961X\ X10^{-3}$	0.0
13	0.0	0.0	0.0

Table 3. Displacement results of truss problem no. 3-a in two dimensions

Displaces	FEM	MEM	Error %
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	-1.958×10^{-4}	-5.082×10^{-5}	74.0
4	-1.072×10^{-3}	-1.211×10^{-3}	12.96
5	-4.865×10^{-5}	-1.016×10^{-4}	52.10
6	-2.741×10^{-3}	-2.421×10^{-3}	11.67
7	-4.160×10^{-4}	-3.589×10^{-4}	13.72
8	-2.941×10^{-3}	-2.588×10^{-3}	12.00
9	2.517×10^{-4}	-1.794×10^{-4}	-
10	-1.272×10^{-3}	-1.582×10^{-3}	20.00
11	0.0	0.0	0.0
12	-6.000×10^{-4}	-5.770×10^{-4}	3.8

DISCUSSION OF RESULTS

The solved problems showed that using macro-elements in the analysis reduced the number of equations to be solved. When the order of the displacement function of the macro-element is at least the same order as the displacement function of the original truss structure, excellent results are achieved with good amount of reduction in d.o.f.

In example 1, the number of nodes in the truss structure is six nodes, and the quadratic quadrilateral isoparametric macro-element has eight nodes with quadratic shape functions. This means that the shape functions of the macro-element are of higher order than the truss displacements and can describe the displacement of the truss structure exactly, and this is the case in Table 1, where there are no errors between the two solutions.

In example 2, the number of nodes in the truss structure is eight nodes, and the quadratic quadrilateral isoparametric macro-element has eight nodes with quadratic shape functions. This means that the shape functions of the macro-element can describe the displacement of the truss structure exactly, and this is the case in Table 2, where there are no errors between the two solutions.

In example 3, the number of nodes in the truss structure is six nodes, and the linear quadrilateral macro-element has four nodes. This means that the shape functions of the macro-element can not describe the displacement of the truss structure exactly, and this is the case in Table 3, where there are large errors between the two solutions. But, when two linear quadrilateral macro-elements are used to model the truss, an exact solution is achieved. This means that even if the order of the macro-element is less than the order of the truss modeled, an exact solution can be achieved, but with increasing the number of macro-elements used in modeling.

The beautifulness of this method is that one can use different kinds of finite elements inside the macro-element if required and the method still yields good results.

CONCLUSIONS

New modeling of truss quadrilateral isoparametric macro-element based on truss type finite element was developed.

The solved examples demonstrated that using these macro-elements in the analysis largely reduced the total number of d.o.f. required to model a certain structure. This in turn reduced the total number of equations to be solved.

Reduction in total number of equations reduced computer time and memory space for storage.

At the same time, these macro-elements provided accurate results. In addition, finite elements of different sizes and material properties can easily be used inside the macro-elements if required in the analysis. This developed macro-element theory was applied to different kinds of structural elements like beams, thin plates and thick plates, and good results were achieved in terms of accuracy and time of execution. This theory can be applied to any kind of structure as long as the basic assumptions for macro-element formulation are satisfied.

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