Damage Mechanics in a Uniaxially-Loaded Elastic Tapered Bar

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ABSTRACT

The principles of damage mechanics are used to predict the displacements and stresses in a uniaxially-loaded one-dimensional elastic tapered bar. The variation of the damage variable along the length of the bar is studied. A random distribution of the damage variable along the length of the bar is also considered. It is shown how the displacements and stresses are obtained in closed-form solutions whenever possible. Otherwise, finite element analysis is employed to solve the resulting problem. The computer algebra system MAPLE is used to write a symbolic finite element program specifically for this problem with the random distribution of the damage variable for which there is no closed form solution.

KEYWORDS: Damage mechanics, Uniaxially-loaded, One-dimensional tapered bar.

INTRODUCTION

Kachanov (1958) pioneered the subject of damage mechanics by introducing the concept of effective stress. This concept is based on considering a fictitious undamaged configuration of a body and comparing it with the actual damaged configuration. The damage variable was defined in terms of both the damaged and effective cross-sectional areas of the body. Kachanov (1958) originally formulated his theory using simple uniaxial tension. Following Kachanov's work, researchers in different fields applied damage mechanics to their areas in fields like brittle materials (Krajcinovic and Foneska, 1981; Krajcinovic, 1988) and ductile materials (Lemaitre, 1984, 1985, 1986; Kachanov, 1986; Murakami, 1988). In the 1990s, applications of damage mechanics to plasticity and composite materials have appeared (Voyiadjis and Kattan, 1990, 1993, 1999; Kattan and Voyiadjis, 1990, 1993a, 1993b, 1996, 2001; Voyiadjis and Park, 1997a, 1997b; Voyiadjis and Thiagarajan, 1996; Voyiadjis et al., 1995).

A uniaxially-loaded one-dimensional elastic tapered bar is considered in this work. It is assumed that the bar is damaged and we seek the displacements and stresses in the damaged bar. The principles of continuum damage mechanics are used to predict the displacements and stresses in this case. However, the nature of the damage is not explored as the formulation is formal and can be applied to any source of damage. It is seen that for some damage variations, a closed-form solution can be obtained. However, if one assumes a random distribution of the damage variable along the length of the bar, finite element analysis is employed to solve the resulting problem. A symbolic finite element program is written specifically for this problem using the computer algebra system MAPLE.

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BRIEF REVIEW OF DAMAGE MECHANICS

The principles of damage mechanics are first reviewed for the case of uniaxial tension. In this case, isotropic damage is assumed throughout the analysis. Consider a one-dimensional elastic tapered bar subjected to a uniaxial tensile force T as shown in Figure 1a. The cross-sectional area of the bar is $A(x)$ at a distance x from the origin. The uniaxial stress $\sigma(x)$ in the bar is found easily from the formula $T = \sigma(x)A(x)$. In order to use the principles of damage mechanics, we consider a fictitious undamaged configuration of the bar as shown in Figure 1b. In this configuration, all types of damage including both voids and cracks, are removed from the bar. The effective cross-sectional area of the bar in this configuration is denoted by $\overline{A}(x)$ and the effective uniaxial stress is $\overline{\sigma}(x)$. The bars in both the damaged configuration and the effective undamaged configuration are subjected to the same tensile force T. Therefore, considering the effective undamaged configuration, we have the formula $T = \overline{\sigma}(x)\overline{A}(x)$. Equating the two expression of T obtained from both configurations, one obtains the following expression for the effective uniaxial stress $\overline{\sigma}(x)$:

$$\overline{\sigma}(x) = \frac{A(x)}{\overline{A}(x)}\sigma(x). \tag{1}$$

Next, one uses the definition of the damage variable $\phi(x)$ as originally proposed by Kachanov (1958):

$$\phi(x) = \frac{A(x) - \overline{A}(x)}{A(x)}. \tag{2}$$

Thus, the damage variable is defined as the ratio of the total area of voids and cracks to the total area. Its value ranges from zero (for the case of an undamaged specimen) to one (for the case of complete rupture). Substituting for $A(x)/\overline{A}(x)$ from equation (2) into equation (1), one obtains the following expression for the effective uniaxial stress:

$$\overline{\sigma}(x) = \frac{\sigma(x)}{1-\phi(x)}. \tag{3}$$

Equation (3) above was originally derived by Kachanov in 1958 for cylindrical bars. It is clear from equation (3) that the case of complete rupture ($\phi(x) = 1$) is unattainable, because the damage variable $\phi(x)$ is not allowed to take the value 1 in the denominator.

For the uniaxial tension case shown in Figure 1, the constitutive relation is Hooke's law of linear elasticity given by:

$$\sigma(x) = E(x)\,\varepsilon(x); \tag{4}$$

where $\varepsilon(x)$ is the strain and $E(x)$ is the modulus of elasticity (Young's modulus). The same form of the linear elastic constitutive relation applies to the effective (undamaged) state; i.e.,

$$\overline{\sigma}(x) = \overline{E}\,\overline{\varepsilon}(x); \tag{5}$$

where $\overline{\varepsilon}(x)$ and $\overline{E}$ are the effective counterparts of $\varepsilon(x)$ and $E(x)$, respectively. Next, we will derive the necessary transformation equations between the damaged and the hypothetical undamaged states of the material. In the derivation, the following assumptions are incorporated: (1) The elastic deformations are small (infinitesimal) compared with the plastic deformations (finite), and (2) there exists an elastic strain energy scalar function $U(x)$. This function is assumed based on the linear relation between the Cauchy stress $\sigma(x)$ and the engineering strain $\varepsilon(x)$ given by equation (4). The elastic strain energy function $U(x)$ is defined by:

$$U(x) = \frac{1}{2}\sigma(x)\,\varepsilon(x). \tag{6}$$

It is clear from equations (4) and (6) that $\sigma(x) = dU(x)/d\varepsilon(x)$ and $\varepsilon(x) = dU(x)/d\sigma(x)$. Sidoroff (1981) proposed the hypothesis of elastic energy equivalence. This latter hypothesis assumes that "the elastic energy for a damaged material is equivalent in form to that of the undamaged (effective) material, except that the stress is replaced by the effective stress in the energy formulation". Thus, according to this hypothesis, the elastic strain energy $U(x)=\frac{1}{2}\sigma(x)\varepsilon(x)$ is equated to the effective elastic strain energy $\overline{U}(x)=\frac{1}{2}\overline{\sigma}(x)\overline{\varepsilon}(x)$ as follows:

$$\frac{1}{2}\sigma(x)\varepsilon(x)=\frac{1}{2}\overline{\sigma}(x)\overline{\varepsilon}(x). \qquad (7)$$

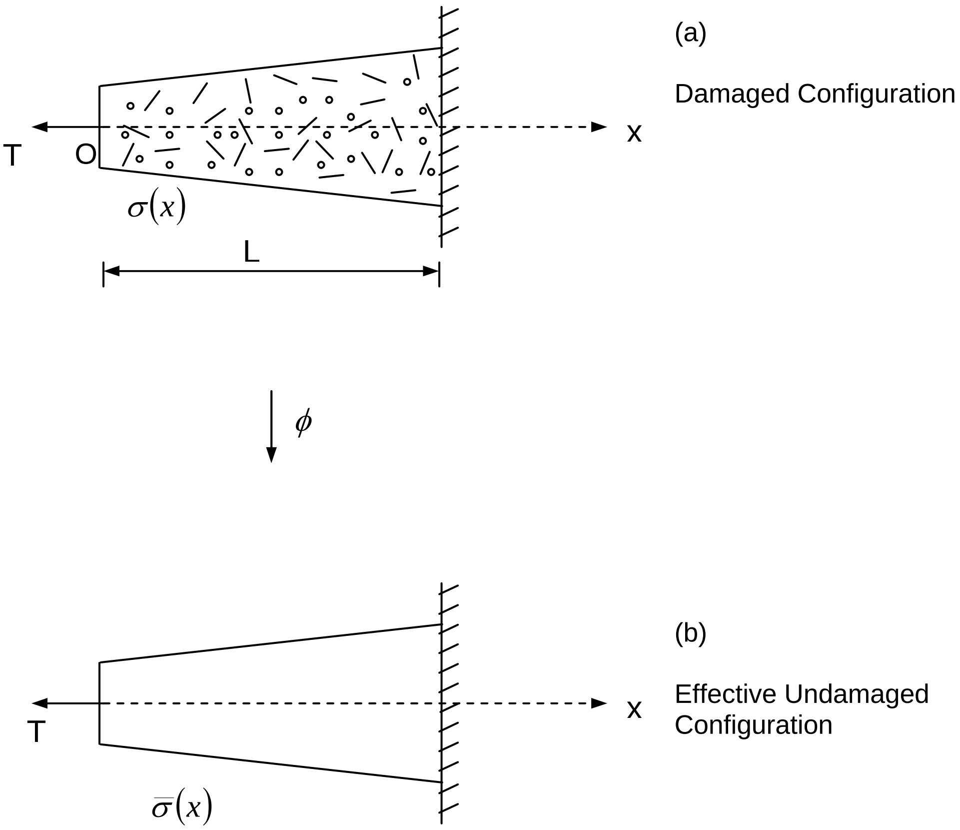

Figure (1): A One-Dimensional Tapered Bar Subjected to Uniaxial Tension

Substituting equation (3) into equation (7) and simplifying, we obtain the following relation between the strain $\varepsilon(x)$ and the effective strain $\overline{\varepsilon}(x)$:

$$\overline{\varepsilon}(x)=[1-\phi(x)]\varepsilon(x). \qquad (8)$$

Continuing further, we substitute equations (3) and (8) into equation (5), simplify the result and compare it with equation (4) to obtain:

$$E(x)=\overline{E}[1-\phi(x)]^2. \qquad (9)$$

Equation (9) represents the transformation law for the modulus of elasticity. It is clear now that Young's modulus for the damaged material depends on the value of the damage variable $\phi(x)$. Solving equation (9) for $\phi\ (x)$, one obtains:

$$\phi(x)=1-\sqrt{\frac{E(x)}{\overline{E}}}. \qquad (10)$$

Once the values of $E(x)$ are measured

experimentally for the damaged material, one can use equation (10) to obtain values of the damage variable $\phi(x)$. It should be noted that the value of $\overline{E}$ is constant for the effective (undamaged) material.

ANALYTICAL SOLUTION

We are interested in determining the displacement u_0 at the left end of the bar at the point of application of the load. The displacement u_0 can be determined using the following formula from mechanics of materials:

$$u_0 = \int_0^L \varepsilon(x)\, dx; \qquad (11)$$

where L is the length of the bar. Substituting for $\varepsilon(x)$ from equation (8), for the strain $\overline{\varepsilon}(x)$ from equation (5) and for the stress $\overline{\sigma}(x) = T / \overline{A}(x)$, we obtain:

$$u_0 = \frac{T}{\overline{E}} \int_0^L \frac{dx}{\overline{A}(x)[1-\phi(x)]}. \qquad (12)$$

Considering the bar to have a circular cross-section with a diameter d_A at the left end and a diameter d_B at the right end, we can write the following formula for the effective cross-sectional area $\overline{A}(x)$ at a distance x from the origin:

$$\overline{A}(x) = \frac{\pi}{4}[d_A + \frac{x}{L}(d_B - d_A)]^2. \qquad (13)$$

Next, we will explore several alternatives for the distribution of the damage variable $\phi(x)$ along the length of the bar. We will consider the following three cases:

1. $\phi(x) = \phi_0$ - The damage variable is constant along the length of the bar.
2. $\phi(x) = \phi_0 + \frac{x}{L}(\phi_1 - \phi_0)$ - The damage variable varies linearly along the length of the bar from a value of ϕ_0 at the left end to a value of ϕ_1 at the right end.
3. $\phi(x)$ varies as a random function along the length of the bar.

We can obtain analytical solutions for cases (1) and (2) only. Case (3) needs to be solved using an approximate solution like the finite element method (FEM). Considering case (1), using $\phi(x) = \phi_0$, substituting for $\overline{A}(x)$ from equation (13) into equation (12) and integrating, we obtain the following expression for the displacement at the left end of the bar:

$$u_0 = \frac{4TL}{\pi \overline{E} d_A d_B (1-\phi_0)}. \qquad (14)$$

Next, we consider case (2), where a linear distribution of damage is assumed along the length of the bar. Using $\phi(x) = \phi_0 + \frac{x}{L}(\phi_1 - \phi_0)$, substituting for $\overline{A}(x)$ from equation (13) into equation (12) and integrating, we obtain the following expression for the displacement at the left end of the bar:

$$u_0 = \frac{4TL}{\pi \overline{E}[(1-\phi_1)d_A - (1-\phi_0)d_B]}[\frac{d_A - d_B}{d_A d_B} + \frac{\phi_0 - \phi_1}{(1-\phi_1)d_A - (1-\phi_0)d_B} \ln [\frac{(1-\phi_1)d_A}{(1-\phi_0)d_B}]. \qquad (15)$$

The above expression reduces to equation (14) when $\phi_0 = \phi_1$. For a prismatic bar with $d_A = d_B$, the above expression reduces to:

$$u_0 = \frac{4Tl}{\pi \overline{E} d_A^2 (\phi_0 - \phi_1)} \ln (\frac{1-\phi_1}{1-\phi_0}). \qquad (16)$$

Both equations (14) and (15) are exact analytical solutions for the tapered bar. However, if the distribution of the damage variable $\phi(x)$ along the bar is not analytically defined but as a random function, then we cannot obtain an analytical expression for the solution. In this case, we need to resort to an approximate solution method like the finite element method.

FINITE ELEMENT SOLUTION

In this section, finite element analysis is used to determine the displacements and stresses in the bar when the damage variable is randomly distributed along the length of the bar. If the bar is discretized into n elements, then we will assume that the value of the damage variable is ϕ_1 in element 1, ϕ_2 in element 2, ϕ_3 in element 3,… and so on until we reach the value ϕ_n in element n. The values $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are random numbers in the range $0 < \phi_i < 1$. To illustrate how the solution is obtained in this case, we will first use two elements followed by a program that can solve the problem for any number of elements.

Figure 2 shows the bar discretized into two finite elements with the value of the damage variable being ϕ_1 in element 1 and ϕ_2 in element 2. The average effective cross-sectional area for each element can be obtained as follows:

$$\overline{A}_1 = \frac{\pi}{32}(5d_A^2 + 2d_A d_B + d_B^2); \qquad (17a)$$

$$\overline{A}_2 = \frac{\pi}{32}(d_A^2 + 2d_A d_B + 5d_B^2). \qquad (17b)$$

The stiffness coefficient for each element is given by:

$$k_1 = \frac{EA_1}{L/2}; \qquad (18a)$$

$$k_2 = \frac{EA_2}{L/2}. \qquad (18b)$$

Substituting for E, A_1 and A_2 from equations (9) and (2), respectively and using equations (17), one obtains:

$$k_1 = \frac{\pi\overline{E}}{16L}(1-\phi_1)(5d_A^2 + 2d_A d_B + d_B^2); \qquad (19a)$$

$$k_2 = \frac{\pi\overline{E}}{16L}(1-\phi_2)(d_A^2 + 2d_A d_B + 5d_B^2). \qquad (19b)$$

Assembling the global stiffness matrix, applying the boundary conditions and solving the resulting equations, one obtains the nodal displacements as follows:

$$u_1 = \frac{-16TL}{\pi\overline{E}}\left[\frac{1}{(1-\phi_1)(5d_A^2 + 2d_A d_B + d_B^2)} + \frac{1}{(1-\phi_2)(d_A^2 + 2d_A d_B + 5d_B^2)}\right]; \qquad (20a)$$

$$u_2 = \frac{-16TL}{\pi\overline{E}(1-\phi_2)(d_A^2 + 2d_A d_B + 5d_B^2)}. \qquad (20b)$$

where the minus sign indicates that the node displacement is to the left. The stresses in the elements are then obtained as follows:

$$\sigma_1 = \frac{32T}{\pi(1-\phi_1)(5d_A^2 + 2d_A d_B + d_B^2)}; \qquad (21a)$$

$$\sigma_2 = \frac{32T}{\pi(1-\phi_2)(d_A^2 + 2d_A d_B + 5d_B^2)}. \qquad (21b)$$

The displacement at the left end from the finite element solution shown in equation (20a) is compared with the displacement at the left end from the analytical solution shown in equations (14) and (15).

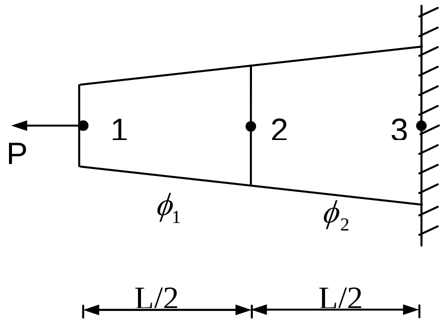

Figure (2): Finite Element Discretization Using Two Elements

For a higher number of elements, we use symbolic computation to solve the problem. A symbolic finite element program is written specifically for this problem using the computer algebra system MAPLE. In this program, the nodal displacements and element stresses are determined for any number of elements n. The following is a listing of the program:

```
fem:=proc(n elements)
with(linalg):
nn nodes:=n elements+1:
y:=vector(nn n odes):
k:=vector(n elements):
A:=vector(nn nodes):
AVR:=vector(n elements):
phi:=vector(n elements):
for i from 1 to nn nodes do
y[i]:=dA + (i-1)*(dB-dA)/n elements:
A[i]:=Pi*y[i]^2/4:
od:
for i from 1 to n elements do
AVR[i]:=(A[i]+A[i+1])/2:
k[i]:=E*AVR[i]*n elements*(1-phi[i])/L:
od:
ST:=matrix(n n nodes, n nodes):
RHS:=vector(n n nodes):
for i from 1 to n n nodes do
for j from 1 to n n nodes do
ST[i,j]:=0:
od:
RHS[i]:=0:
od:
for i from 1 to n elements do
ST[i,i]:=ST[i,i]+k[i]:
ST[i+1,i+1]:=ST[i+1,i+1]+k[i]:
ST[i,i+1]:=ST[i,i+1]-k[i]:
ST[i+1,i]:=ST[i+1,i]-k[i]:
od:
RHS[1]:=RHS[1]+T:
A:=matrix(n elements,n elements):
B:=vector(n elements):
for i from 1 to n elements do
for j from 1 to n elements do
A[i,j]:=ST[i,j]:
od:
B[i]:=RHS[i]:
od:
AINV:=matrix(n elements,n elements):
AINV:=inverse(A):
u:=vector(n elements):
sig:=vector(n elements):
u1:=vector(n n nodes):
u:=evalm(AINV &* B):
for i from 1 to ne lements do
u1[i]:=u[i]:
od:
u1[nn nodes]:=0:
for i from 1 to n elements do
sig[i]:=E*(u1[i+1]-u1[i])*n elements/L:
od:
print(u):
print(sig):
end:
```

In order to obtain the results shown above for the two-element discretization, the program is invoked by executing the command *fem(2)*. For three elements, execute the command *fem(3)*. This works for any number n of elements (e.g. for 10 elements, execute the command *fem(10)*).

The results become very complicated for any number $n>2$; therefore, we need to use some numerical values except for the damage variables. Using the following values (d_A = 50 mm, d_B = 100 mm, L = 2 m, T = 250 kN, E = 70 GPa), expressions (22a), (22b), (22c) and (22d) show the displacement at the left end for n = 2, 3, 4 and 5 elements, respectively. In the expressions shown below, we have neglected all nonlinear terms like $\phi_1\phi_2$ and $\phi_2\phi_3\phi_4$, since they are very small and can be neglected. Furthermore, if these terms are retained, the equations will become so huge that they will not fit here.

$$-\frac{64}{455\pi}\frac{-38+13\phi_1+25\phi_2}{1-\phi_1-\phi_2} \qquad (22a)$$

$$-\frac{96}{87535\pi}\frac{-5051+2550\phi_1+3526\phi_2+4026\phi_3}{1-\phi_1-\phi_2-\phi_3} \qquad (22b)$$

$$-\frac{128}{33630947\pi}\frac{-1474908+889003\phi_1+1081103\phi_2+1192295\phi_3+1262323\phi_4}{1-\phi_1-\phi_2-\phi_3-\phi_4} \qquad (22c)$$

$$-\frac{160}{4305581483\pi}\frac{-152015073+101598428\phi_1+115833716\phi_2+124799008\phi_3+130805312\phi_4+135023828\phi_5}{1-\phi_1-\phi_2-\phi_3-\phi_4-\phi_5} \qquad (22d)$$

We will not attempt to obtain a numerical solution in this work, because we have not explored the determination of the random values of the damage variable. These values can be determined using models of damage evolution which is beyond the scope of this work.

CONCLUSION

The problem of a one-dimensional uniaxially loaded elastic tapered bar is solved including the effects of damage. The problem is solved using analytical solutions where possible. A finite element solution is also included to solve the problem when the values of the damage variable are randomly determined along the length of the bar. A symbolic finite element program is written using the computer algebra system MAPLE to solve the problem for any number of finite elements.

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