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# The Minimum and Maximum Number of Irreducible Representations of Prime Degree for a Non Abelian Group Using its Centre

Jelten, B. Naphtali[1*], Momoh, S.U[2]
1. Department of Remedial Sciences, University of Jos, P.M.B. 2984, Plateau State, Nigeria
2. Department of Mathematics, University of Jos, P.M.B. 2984, Plateau State, Nigeria
*E-mail of the corresponding author: jeltenn2012@gmail.com

**Abstract**

A formula for calculating the number $s$ of irreducible representations of degree $r$ for $r$ a prime number was derived. Let $|G| = r^w$ and $|Z(G)| = r^t$ with $s = t <$ w. We have that when $w \leq 2$, then $s = t = 0$. Furthermore min($s$) is attained when $w = 3$, $s = t = 1$ and max($s$), which is equal to max($t$) , is attained when $w \geq 4$ and $s = t =$ w - 1. In this paper we are interested in the centre of finite non abelian groups. In particular, groups of prime power order. These groups have the property that the centre is non trivial. So, we are concerned with finite groups where the size of the centre is at least 2. An upper bound for the size of the centre of finite non abelian groups has been achieved by Cody (2010) as $|Z(G)| \leq ¼|G|$.

**Key words:** Non abelian, centre, conjugacy, irreducible representation

## 1 Introduction

Denote by $Z(G)$ the centre of a group $G$. We intend to see how $Z(G)$ determines the degrees of the irreducible representations of a group $G$. We require the following preliminaries.

*1.1 Definition*

The centre $Z(G)$ of a group is the set of elements $z$ in $Z(G)$ that commute with every element $q$ in $G$. That is:

$$Z(G) = \{z \in G : zq = qz\}, \text{ for all } q \in G.$$

We note that $Z(G)$ is a commutative normal subgroup of $G$. The quotient of $G$ by $Z(G)$ is isomorphic to the inner automorphism of $G$.

The group $G$ has trivial centre if $Z(G) = \{e\}$ where $e$ is the identity element of $G$. We call the elements $x \in Z(G)$ central elements and the elements $y \in G- Z(G)$ non central elements.

*1.2 Definitions*

(i) If $a, q \in G$, we say that $a$ is conjugate to $q$ if there exists an element $g \in G$ such that $gq = ag$. The conjugacy class of $a$ denoted by $C(a)$ is the set of all elements of $G$ that are conjugates to $a$. That is:

$$C(a) = \{g^{-1}ag : g \in G\}.$$

(ii) The centralizer $C_G(q)$ of an element $q$ in $G$ is the set of all elements $g \in G$ that commute with $q$. Equivalently we write:

$$C_G(q) = \{g \in G: gq = qg, \text{ for any } q \in G\}$$

The index of $C_G(q)$ in $G$ is the size of the conjugacy class $C(q)$ of $q$ in $G$. That is:

$$|C(q)| = |G: C_G(q)|.$$

In particular $|C(q)|$ divides $|G|$. If $q \in Z(G)$ then $C_G(\mathrm{q}) = G$.

Next we make an important remark

*1.3 Remark*

The centre $Z(G)$ of $G$ is maximum when $G$ is abelian. That is $Z(G) = G$ and $G/Z(G) = \{e\}$. The centre $Z(G)$ is minimum when $Z(G) = \{e\}$. In this case $G/Z(G) = G$.

However our work is based on non abelian groups. That is, where :

$$\{e\} < Z(G) < G.$$

We observe that:

(i) $z \in Z(G)$ if and only if its conjugacy class has one element which is $z$ itself.
(ii) Conjugate elements lie in the same conjugacy class and have the same order.
(iii) The conjugacy classes of a group are disjoint and the union of all the conjugacy classes forms the group.

The next theorem is the classical Lagrange's Theorem.

*1.4 Theorem*

If $G$ is finite and $H$ is a subgroup of G then $|H|$ divides $|G|$.

More over the number of distinct left cosets of $H$ in $G$ is denoted by $|G:H|$and

$$|G:H|= |G|/|H|.$$

From 1.2, we have the following:

*1.5 Proposition*

If $a, q \in G$ then:

(i) either $C(a) = C(q)$ or $C(a) \cap C(q) = \varnothing$;

(ii) if a is conjugate to $q$ in $G$, then $a^c$ is conjugate to $q^c$ in $G$ for every integer $c$. Moreover, $a$ and $q$ have the same order.

In 1.6 we determine the size of the conjugacy class of an element of $G$.

*1.6 Theorem*

Let $|G| < \infty$, $q \in G$ then the conjugacy *class* $C(q)$ of $q$ in $G$ is given by:

$$|C(q)| = |G:C_G(q)| = |G|/|C_G(q)|.$$

Consequently if $a$, $q$ are in the same conjugacy class then $|C_G(q)| = |C_G(a)|$. So if

$$C(q) =\{q_i , 1 \le i \le k\}, \text{ then}$$

$$\sum |C(q)| = |G: C_G(q_i)||C_G(q_i)| = |G|$$

The next Theorem presents the class equation for finite groups whose proof follows readily from 1.6.

*1.7 Theorem*

$$|G| = \sum|G: C_G(q_i)| \qquad (i),$$

where the sum runs over the elements from each conjugacy class of $G$.

From 1.6, equation ($i$) becomes

$$|G|= |Z(G)| + \sum|G: C_G(q_i)| \qquad (ii)$$

The sum in (*ii*) runs over $q_i$ from each conjugacy class such that $q_i$ is not an element of $Z(G)$. From 1.2 and equation (*ii*) above we have:

$$|G|=|Z(G)| + \sum|C(q_i)| \qquad (iii)$$

*1.8 Remark*

In the abelian environment, the sum in equation (*iii*) above is zero. Consequently, the class equation is relevant only when we are in the non abelian environment.

The motivation for this work is from Audu, M.S and Momoh, S.U (1993) who achieved a lower bound for the minimum number of faithful generators of a transitive and faithful $p$ - group. However, in this work the main task is to determine the minimum and maximum number of irreducible representations of prime degree for a non abelian group using its centre.

A main theorem for this consideration is due to Cody, C (2010) in which a bound for the size of the centre of a non abelian group was given. Precisely, Cody worked on the commutativity of non abelian groups with interest in measuring the probability that pairs of elements of $G$ selected at random commute. He discovered that this probability is 1 for abelian groups. In the non abelian case, he discovered that the probability is 5/8. This occurs when the order of the centre of the group is at most ¼ of the order of $G$.

For any non abelian group, the maximum size of the centre is given from Cody (2010) by:

*1.9 Theorem*

If G is a finite non abelian group, then the maximum possible order of the centre of $G$ is $\frac{1}{4}|G|$.That is, $|Z(G)| \leq 1/4|G|$.

**Proof**

Let $z\in Z(G)$. Since $G$ is non abelian, $Z(G) \neq G$. Thus there exists an element $q\in G$ such that $q$ is not in the centre. This implies that $C_G(q) \neq G$ and $C_G(q) \neq Z(G)$. Since $z\in Z(G)$ every element in G commutes with $z$, so $qz = zq$. It follows that

$z \in C_G(q)$. Since $q \in C_G(q)$, we have that $Z(G)$ is a proper subset of $C_G(q)$. Since a group that is a subset of a subgroup under the same operation is itself a subgroup of the subgroup, we find that $Z(G)$ is a proper subgroup of $C_G(q)$. By 1.4, it follows that:

$$|Z(G)| \leq 1/2|C_G(q)|.$$

Now, since we assumed $C_G(q) \neq G$, then $C_G(q)$ is a proper subset of $G$. Therefore by 1.4 and the fact that the centralizer of any group element is a subgroup of $G$, we find that $|C_G(q)| \leq 1/2|G|$.That is:

$$\begin{aligned}|Z(G)| &\leq 1/2|C_G(q)| \\ &\leq 1/2(1/2|G|) \\ &\leq 1/4|G|.\end{aligned}$$

We have a corollary to 1.7 which requires the following definition.

*1.10 Definition*

Let $r$ be a prime number. Then $G$ is called an $r$ - group if the order of $G$ is a positive power of $r$.

From Houshang and Hamid (2009) we have:

*1.11 Corollary*

If $|G|$ is a power of a prime $r$ then $G$ has a non trivial centre.

**Proof**

Let $G$ be the union between its centre and the conjugacy classes say $J_i$ of size greater than 1.

Thus from equation (*iii*) of 1.7

$$|G| = |Z(G)| + \sum|C(J_i)|$$

Each conjugacy class $J_i$ has size of a power $w$ say of prime $r$ such that $w \geq 1$. In this case $w = 0$ for classes whose elements are central elements. Since each conjugacy class $J_i$ has size a power of $r$ then $|J_i|$ is divisible by $r$. Furthermore as r divides |G| it follows that r also divides $|Z(G)|$. Accordingly $Z(G)$ is non- trivial.

What follows is the definition of an important concept..

*1.12 Definition*

A representation of a group $G$ over a field $F$ is a homomorphism $\pi$ from $G$ to $GL(n, F)$, the group of $n$ by $n$ invertible matrices with entries in $F$ for some integer $n$ . Here $n$ is the degree of $\pi$, and we write:

$\pi: G \rightarrow GL\ (n, F)$.

Consequently we say that $\pi$ is a representation of $G$ if $(aq)\pi = (a)\ \pi(q)\pi$, for all $a, q \in G$.

A representation without proper sub representations is said to be an irreducible representation.

*1.13 Definition*

The character $X_c(q)$ of a representation $\pi$ is the trace of $q\pi$ namely:

$$X_c(q) = tr\ (q\pi) \text{ for all } q \in G\ .$$

The character $X_c(q)$ is said to be irreducible (respectively reducible) if it is the character of an irreducible (respectively reducible) representation. We call characters of degree 1 linear or trivial characters.

We outline the following from Herstein I.N (1964) for reference.

*1.14 Theorem*

(i) Every group of order $r^2$ where r is a prime is abelian;

(ii) Let $n_i$ be the degrees of the irreducible representation of $G$, then $|G|=\sum n_i^2$ .

*1.15 Proposition*

(i) The number of the irreducible representations of any group $G$ is equal to the number of conjugacy classes of $G$;

(ii) Every irreducible representation of an abelian group $G$ over $C$, the set of complex numbers is one dimensional.

*Proof*

(i) The class functions are determined by their values on the conjugacy classes of $G$. These are complex vectors spaces. They have dimension equal to the number of conjugacy classes.

But irreducible characters form a basis for the same vector space. Thus the number of conjugacy classes and the number of irreducible characters are the same.

(ii) Since $G$ is an abelian group it has $|G|$ conjugacy classes. From ($i$) above, it shows that $G$ has |G| number of irreducible characters and from 1.14 (ii), we have that:

$$|G| = n_1^2 + n_2^2 + \ldots + n_{|G|}^2.$$

It clearly shows that this can be satisfied only when $n_i = 1$ for all $i$.

## 2 Results

We determine the minimum and maximum number of non trivial irreducible representations of prime degree for non abelian groups in the following results.

*2.1 Theorem*

Let $G$ be a finite non abelian group of order $r^w$ such that $|Z(G)| = r^t$ with $t < w$, $r$ a prime number, $t$ and $w$ positive integers. Then $G$:

(i) does not have an irreducible representation of degree $r$ greater than 1 whenever $t = 0$;

(ii) has its minimum number of irreducible representations of degree $r$ greater than 1 whenever $w = 3$.

*Proof*

(i) If $t = 0$ the centre is trivial. From 1.11 it follows that $G$ cannot have a trivial centre when $w > 1$. Accordingly $G$ has $|G|$ conjugacy classes and $|G|$ irreducible representations each of which according to 1.15 cannot be greater than 1 whenever $t = 0$.

(ii) Since $G$ is non abelian, there exists an element $g \in G$ such that $X_c(\mathrm{g}) = r > 1$ ( 1.15). Let $s$ be the number of irreducible representations of degree $r$ with $s = t < w$. Then from 1.9,

$$|Z(G)| \leq 1/4\, |G|.$$

That is $r^t \leq 1/4\, r^w$. This implies that:

$$4 \leq r^{w-t} \text{ or } 2^2 \leq r^{w-t}.$$

To find the minimum irreducible representation we let $r$ take its minimum value which is 2 (minimum prime). Then we have that $t = w - 2$. It follows that:

$$s = t = w - 2.$$

However $w \neq 0, 1, 2$ since $t > 0$ and $s > 0$. Accordingly, the minimum value of $w$ is 3 which gives the corresponding minimum values of $s$ and $t$ as $t = 1$, $s = 1$ and $r > 1$.

*2.2 Theorem*

Let $G$ be a finite non abelian group of order $r^w$ whose centre is of size $r^t$. Where $r$ is a prime number. Then the maximum number s of irreducible representation of degree r such that

$r > 1$ is:

($i$) $s = w - 1$ where $s = t < w$ and

($ii$) $w \geq 4$ when $t > 1$

*Proof:*

(*i*) Now by the definition of $G$, $t \neq w$, else we will be in the abelian environment. But from hypothesis $t < w$, this implies that the maximum value that $t$ can attain is $w$ - 1, since $t$ and $w$ are positive integers. Accordingly

$$s = w - 1.$$

(*ii*) In 2.1(*ii*) we have that the least number $s$ of irreducible representation of degree $r > 1$ is 1 when the value of $w$ is 3 and $t = 1$. But for $t$ to be greater than 1, given that $w$ must be greater than $t$, then $w \geq 4$.

## 3. Discussions

1. In theorem 2.1(*i*) $w \leq 2$, $s = t = 0$. This implies that for all $g$ in $G$, $X_c(g) = 1$. This holds in the abelian environment, where $Z(G) = G$. Here $t = w$.

**2**. From 2.1 (*ii*)

$$w = 3, s = t = 1$$

so that $|G| = r^3$ and $|Z(G)| = r^1$. There is only one irreducible representation of degree $r$, where $r$ is a prime. That is $s = 1$

If for example $r = 2$, then

$$|G| = 2^3 = 8, |Z(G)| = 2^1$$

and so $G$ has only one irreducible representation of degree 2.

**3**. For $w \geq 4$, in particular let $w = 4$. Then $|G| = r^4$ giving

$$s = w - 1 = 3.$$

Let $r = 2$ then from 2.2 (*ii*),

$|G| = 2^4 = 16$. This gives:

$$s = w - 1 = 4 - 1 = 3.$$

Accordingly $G$ has 3 irreducible representations of degree 2.

For $r = 3$, and

$$|G| = 3^3, s = 3-1 = 2.$$

So for $|G| = 27$, $G$ has 2 irreducible representations of degree 3.

## 4 Conclusion

This paper achieved the minimum and maximum number of irreducible representations of prime degree for any non abelian group by using the centre. By $w \leq 3$, we mean that the order of the group for a minimum is at most $r^3$ for a prime $r$. For the maximum the order of the group is at least $r^4$. That is $w \geq 4$. With this the minimum and maximum number of the irreducible representations can be determined with little or no knowledge of the information about the group or it's elements.

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