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Comparative Analysis of Methods of Estimating 2- Parameter Weibull Distribution

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Abstract
This work focused on comparing three methods of estimating 2-parameter Weibull distribution by using the Mean Squared Error (MSE) as test criterion. Three methods of estimation were used, namely, maximum likelihood estimator, method of moments and least squares methods. The method of moments was selected as the best method based on the selection test criterion.
Keywords: Weibull, Moments, Estimators, and Model

1.0 INTRODUCTION: The Weilbull distribution by Professor Wallodi Weilbull in 1951 is a popular distribution for analyzing real life data. The distribution otherwise known as "life data analysis" is widely used in reliability analysis.

1.1 THE WEIBULL DISTRIBUTION MODEL

The general form of a Weilbull density function (pdf) (Mann and Singpurwalla 1974) is given by

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x_t-v}{\alpha}\right)^{\beta-1}\ell^{-\left(\frac{x_t-v}{\alpha}\right)^{\beta}} \quad \beta>0, x>v\geq 0 \quad .. \quad . \quad (1)$$

The cdf of the Weibull distribution is mathematically given as:

$$f(x)=1-\ell^{-\left(\frac{x_t-v}{\alpha}\right)^{\beta}} \quad . \quad . \quad . \quad (2)$$

Where; x_t =data vector (weekly squared stock price returns of Cornerstone Insurance Company, PLC, Port Harcourt)
β = the shape parameter of the distribution
α = the scale parameter of the distribution (spread)
v = is the location parameter
The Weibull shape parameter, β , indicates whether Weibull function is increasing, decreasing or constant. For $0<\beta<1$ indicates that the Weibull distribution has a decreasing function, $\beta>1$ indicates an increasing function and $\beta=1$ shows that Weibull function is constant and reduces to an exponential distribution. The Weibull scale parameter called α is a measure of the scale or spread in the distribution of sampled data (Johnson et al 1994).

2.0 THE SCOPE OF THE STUDY

This study will consider the various analytical methods of estimating a 2–parameter Weibull Distribution.

3.0 AIMS AND OBJECTIVES

The objective of this study is to discriminate among the three analytical methods of estimating a 2-parameter Weibull distribution and select the best.

4.0 METHODS OF ESTIMATION

In this section, we shall discuss some of the analytical methods used in estimating Weibull parameters. The analytical methods used in this study shall include that of the Maximum Likelihood Estimator (MLE) the Method of Moments (MOM), and the Least Square Method (LSM).

4.1 MAXIMUM LIKELIHOOD ESTIMATOR (MLE)

The method of maximum likelihood estimation is a commonly used procedure for estimating parameters (Harter and Moore 1965). Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a population with probability density function $f(x, \underline{\lambda})$ where $\underline{\lambda}$ is an unknown vector of parameters, $\underline{\lambda} = (\beta, \alpha)$.

The likelihood function of n random samples $x_1, x_2, \ldots, x_n$ is defined to be the joint density function (pdf) of the n random variables say $f(x_1, x_2, \ldots, x_n; \underline{\lambda})$ which is considered to be an unknown parameter $\underline{\lambda}$. In particular, if $x_1, x_2, \ldots x_n$ is a random sample from the density function $f(x_n, \underline{\lambda})$, then the likelihood function is $L = f(\alpha, \beta) = \prod_{i=1}^{n} f(x_t, \underline{\lambda})$ . . . (3)

Equation (3) is the likelihood function of α and β. The maximum likelihood of $\underline{\lambda}$, maximizes L or equivalently, the logarithm of L given by the equation $\frac{\partial \log \lambda}{\partial \lambda} = 0$, see for example, Johnson et al (1994) and Mood et al (1974) where solutions that are not functions of the sample values, $x_1, x_2, \ldots, x_n$ are not admissible, nor are solutions which are in the parameter space. Now, we apply the MLE to estimate the parameters of Weibull distribution, namely α and β respectively where $v = 0$.

Consider the Weilbull pdf given in equation (3), the likelihood function will be given as:

$$L(x_1, x_2, \ldots, x_n; \beta, \alpha) = \prod_{i=1}^{n} \left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta-1} \ell^{-\left(\frac{x_i}{\alpha}\right)^{\beta}}$$

$$= \left(\frac{\beta}{\alpha}\right)^{n} \left(\frac{1}{\alpha}\right)^{\beta-1} \prod_{i=1}^{n} (x_i)^{\beta-1} \ell^{-\left(\frac{x_i}{\alpha}\right)^{\beta}}$$

$$= \left(\frac{\beta}{\alpha}\right)^{n} \left(\frac{1}{\alpha}\right)^{n\beta-n} \sum_{i=1}^{n} x_i^{(\beta-1)} \ell^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}} \quad (4)$$

Taking the algorithm of both sides and differentiating partially w.r.t β and α in turn and equating to zero, we obtain the estimating equations as follows:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} \div \sum_{i=1}^{n} x_i^{\beta} \ln xi = 0. \quad (5)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{-n}{\alpha} + \frac{1}{\alpha^2} \sum_{i=1}^{n} x_i^{\beta} = 0. \quad (6)$$

On eliminating α in equations (5) and (6) and by simplification, we obtain

$$\ln x_i - \frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^{n} \ln x_i = 0. \quad (7)$$

Hence $$\hat{\beta}_{mle} = \frac{\sum_{i=1}^{n} \ln x_i}{n} \quad (8)$$

α is now estimated using equation (6) as thus:

$$-\alpha^2 n \div \alpha \sum_{i=1}^{n} x_i{}^{\beta} = 0$$

So that

$$\hat{\alpha}_{mle} = \frac{\sum_{i=1}^{n} x_i^{\hat{\beta}}}{n} \tag{9}$$

4.2 METHOD OF MOMENTS (MOM)

The method of moments is another technique commonly used in the field of parameter estimation. Let $x_1, x_2, \ldots, x_n$ be a random sample and then an unbiased estimator for the k^{th} moment is given by;

$$m_k = \frac{1}{n}\sum_{i=1}^{n} x_t^{k}, \tag{10}$$

Where $\hat{m}_k$ stands for the estimate of k^{th} moment. In Weibull, the k^{th} moment follows from equation (10) (Al-Fawzan 2000) as

$$\mu_k = \left(\frac{1}{\alpha^{\beta}}\right)^{-\frac{k}{\beta}} \Gamma\left(1+\frac{k}{\beta}\right) \tag{11}$$

Where Γ is a gamma function evaluated at the value of $\left(1+\frac{1}{\beta}\right)$ which provides the values $\Gamma(k)$ at any value of k. From (11), we can find the 1st and 2nd moments as follows:

$$\hat{m}_1 = \hat{\mu}_1 = \left(\frac{1}{\alpha}\right)^{\frac{1}{\beta}} \Gamma\left(1+\frac{1}{\beta}\right) \tag{12}$$

and

$$m_2 = \hat{\mu} + \sigma^2 = \left[\frac{1}{\alpha}\right]^{\frac{2}{\beta}} \left\{\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\} \tag{14}$$

When we divide m_1 by the square of $m_{2,}$ we get an expression which is a function of β only as in Al-Fawzan (2000)

$$\frac{\mu_2}{\sigma^2+\mu^2} = \frac{\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{1}{\beta}\right)}{\Gamma\left(1+\frac{2}{\beta}\right)} \tag{15}$$

Where

$$\hat{\mu} = \sum_{i=1}^{n} \ln\left(\frac{s_t}{s_{t-1}}\right) = \mathrm{E}(\mathrm{X}_t) = \frac{1}{n}\sum_{t=1}^{n} \mathrm{X}_t, \quad \sigma^2 = \mathrm{E}(\mathrm{X}_t^2) - (\mathrm{E}(\mathrm{X}_t))^2, \text{ and } \mathrm{Z} = \frac{1}{\beta}.$$

Equation (15) is transformed in order to estimate β and α respectively as in Nwobi (1984):

$$\frac{\mu^2}{\sigma^2+\mu^2} = \frac{\Gamma(1+\mathrm{Z})\Gamma(1+\mathrm{Z})}{\Gamma(1+2\mathrm{Z})} \tag{16}$$

The value of the scale parameter α_{mom} can be estimated, thus

$$\hat{\alpha}_{mom} = \frac{\hat{\mu}}{\Gamma\left(1+\frac{1}{\beta}\right)} \tag{17}$$

Where $\hat{\mu}$ is the mean of the original data.

4.3 THE LEAST SQUARES METHOD (LSM)

The third estimation technique among the analytical methods for estimating a 2-parameter Weibull is the **Least Squares Method**. It is commonly applied in engineering and mathematics problems that are often not thought of as an estimation problem. We assume that there is a linear relationship between two values.
Considering

$$\mathrm{Y} = \alpha + \beta x_t \tag{18}$$

$$Y = \ln\left[\ln\left(\frac{1}{1-F(T)}\right)\right]; \quad m = \beta, X = \ln x \text{ and } b = \beta \ln \alpha$$

Assume that a set of data pairs $\left(y_{1,}y_2\right),\left(x_{1,}x_2\right)\ldots\left(x_n,x_y\right)$ were obtained and plotted.

According to the least squares principle which minimizes the vertical distance between the data points and the straight line fitted to the data, the best fitting line to this data is the straight line:

$$Y = \alpha + \beta x$$

Such that $\sum\limits_{t=1}^{n}\left(y_t - \alpha + \beta X_t\right)^2 = \min(\alpha,\beta) = \sum_{t=1}^{n}\left(y_t - \alpha + \beta X_t\right)^2$

Where $\hat{\alpha}$ and $\hat{\beta}$ are the least estimates of α and β, and n is the number of data points.

To obtain $\hat{\alpha}$ and $\hat{\beta}$ we let $Q = \sum_{t=1}^{n}\left(y_t - \alpha + \beta X_t\right)^2$; and differentiating Q with respect to β and equating to zero yields the following system of equations;

$$\frac{\partial Q}{\partial \alpha} = 2\sum_{t=1}^{n}\left(y_t - \alpha + \beta X_t\right)^2 = 0 \tag{19}$$

and

$$\frac{\partial Q}{\partial \beta} = 2\sum_{t=1}^{n}\left(y_t - \alpha + \beta X_t\right)^2 X_t = 0 \tag{20}$$

Expanding and solving equations (19) and (20) simultaneously, we have

$$\hat{\beta}_{LSM} = \frac{\sum_{t=1}^{n} x_t y_t - \sum_{t=1}^{n} x_t \sum_{t=1}^{n} y_t}{\sum_{t=1}^{n} x_t^2 - \frac{\left(\sum_{t=1}^{n} X_t\right)^2}{n}} \tag{21}$$

and

$$\hat{\alpha}_{LSM} = \frac{\sum_{t=1}^{n} Y_t}{n} - \hat{\beta}\frac{\sum_{t=1}^{n} X_t}{n} = \bar{y}_t - \hat{\beta}\bar{x}_t. \tag{22}$$

5.0 METHODS OF ANALYSIS

We have discussed the three analytical methods for estimating a 2-parameter Weibull namely α and β. To compare the analytical methods, the Mean Squared Error (MSE) test criterion will be used. The MSE test criterion defined by (Al-Fawzan, 2000) is given as

$$MSE = \sum_{t=1}^{n} \frac{\left\{\hat{F}(x_t) - F(x_t)\right\}^2}{N} \tag{23}$$

Where $\hat{F}(x_i) = 1 - e^{-\left(\frac{x_i}{\alpha}\right)^{\hat{\beta}}}$ and F(x)=cumulative frequency of rank (i)

Such that the method with minimum Mean Squared Error ($MSE_{(min)}$) becomes the best method for Weibull parameter estimations and hence satisfies the objective of this study.

5.1 COMPUTATIONAL RESULTS

5.1.1 Maximum Likelihood Estimator

To obtain the estimates of $\hat{\beta}_{MLE}$ and $\hat{\alpha}_{MLE}$ we use the Equations (8) and (9) respectively as:

$$\hat{\beta}_{MLE} = \frac{\sum_{t=1}^{n} \ln(X_i)}{n} = \frac{261.2817}{99} = 2.6392$$

$$\hat{\alpha}_{MLE} = \frac{\sum_{t=1}^{n} X_i^{\hat{\beta}}}{n} = \frac{3731.11}{99} = 37.6880$$

$$F(X_t) = CF = 1 + 2 + 3 +,\dots,+99$$

$$\therefore \hat{\alpha}_{MLE} = 37.6880,\ \hat{\beta}_{MLE} = 2.6392,\ MSE = 1.3549 \times 10^4$$

5.1.2 Method of Moments

Using Equation (10), $\mathrm{E}(X_t) = \hat{\mu} = \frac{261.2817}{99} = 2.6392$

$$\sigma^2 = \sum_{t=1}^{n} \frac{(x_t - \bar{x}_t)^2}{n-1} = \frac{522.6559}{98} = 5.3332$$

$$F(X_t) = 1 + 2 + 3 +,\dots,+99$$

To estimate the 2-parameters Weibull distribution, we use the two computed values of $\hat{\mu}$ and σ^2.

$\hat{\beta}_{MOM}$ is estimated using Equation (14) as

$$\frac{\mu^2}{\sigma^2 + \mu^2} = \frac{\Gamma(1+Z)\Gamma(1+Z)}{\Gamma(1+2Z)}$$

$$\frac{(2.6392)^2}{5.3332 + (2.6392)^2} = \frac{\Gamma(1+Z)\Gamma(1+Z)}{\Gamma(1+2Z)}$$

$$\frac{6.9654}{12.2986} = 0.5664$$

$$\frac{\Gamma(1+Z)\Gamma(1+Z)}{\Gamma(1+2Z)} = 0.5664$$

From the Gamma Function Table, Z lies between 0.86 and 0.87 and using their linear application, we obtain Z as;

$$0.86-0.5734$$
$$Z-0.5664$$
$$\frac{0.86-Z}{Z-0.87}=\frac{0.5734-0.5664}{0.5664-0.5679}$$
$$Z=0.8745$$

Since $Z=\frac{1}{\beta}=1.1435$

$$\therefore \hat{\beta}_{MOM}=1.1435$$

To obtain the estimate of $\hat{\alpha}_{MOM}$, we use Equation (17)

$$\hat{\alpha}_{MOM}=\frac{\mu}{\Gamma\left(1+\frac{1}{\beta}\right)}$$
$$=\frac{2.6392}{\Gamma\left(1+\frac{1}{1.1435}\right)}=\frac{2.6392}{\Gamma(1.8745)}$$

From the value of the gamma function $\Gamma_{(a)}$ table

$$\Gamma(1.8745)=0.9518$$
$$\hat{\alpha}_{MOM}=\frac{2.6392}{0.9518}=2.7730$$
$$\therefore \hat{\alpha}_{MOM}=2.7730$$

Hence, our estimates for the method of moments are as follows:

$\hat{\alpha}_{MOM}=2.7730,\ \hat{\beta}_{MOM}=1.1435$ and $MSE=1.3488\times10^4$

5.1.3 THE LEAST SQUARE METHOD (LSM)

Let $y=\ln\left\{\ln\left(\frac{1}{1-F(X)}\right)\right\}$ and $x=\ln\left(r_t^2\right)$

Where F(X) =Cumulative Frequency of i, $\bar{y}=\frac{\sum_{t=1}^{n} y}{n}, \bar{x}=\frac{\sum_{t=1}^{n} x}{n}$

$$\sum_{t=1}^{n} y=431.60, \bar{y}=4.3602,$$

$$\bar{x}=2.6392, \sum_{t=1}^{n} x=261.2817, \sum_{t=1}^{n} xy=90.77, \left(\sum_{t=1}^{n} x\right)^2=68268.13, \sum_{t=1}^{n} x^2=1212.23$$

To estimate $\hat{\beta}_{LSM}=0.2488$

We substitute into Equation (22) to get $\hat{\alpha}_{LSM}$

$$\hat{\alpha}_{LSM}=03.7036$$

Hence, our estimates for the Least Square Method are given as follows

$$\hat{\alpha}_{LSM} = 3.7036, \ \hat{\beta}_{LSM} = 0.2488, \ MSE = 1.3549 \times 10^4$$

5.1.4 COMPARISON OF ANALYTICAL METHODS

Table : Summary Of Results And Comparison Of The Analytical Method Of Weibull Parametric Estimation

Method of estimation	$\hat{\alpha}$	$\hat{\beta}$	MSE
MLE	37.6880	2.6392	1.3549×10^4
MOM	2.7730	1.1435	1.3488×10^4
LSM	3.7036	0.2488	1.3549×10^4

From Table 1, it is obviously seen that MOM is the best method since it has the minimum Mean Squared Error of 1.3488×10^4.

6.0 CONCLUSION

In this paper, we have presented the analytical methods for estimating a 2-parameter Weibull Distribution. We have seen from the results shown in Table 1 that the method of moments (MOM) achieves the best result and hence satisfies the objective of the study.

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