Group Intuitionistic Fuzzy Topological Spaces

P. K. Sharma
Post Graduate Department of Mathematics, D.A.V. College, Jalandhar city, Punjab, India
E-mail of the corresponding author: pksharma@davjalandharc.com

Abstract
The notion of intuitionistic fuzzy set was introduced by K.T. Atanassov as a generalization of the notion of fuzzy set. Intuitionistic fuzzy topological spaces were introduced by D. Coker and studied by many eminent authors like F. Gallego Lupianez, K. Hur, J. H. Kim and J. H. Ryou. R. Biswas applied the notion of intuitionistic fuzzy set to algebra and introduced intuitionistic fuzzy subgroup of a group. In this paper, we will study intuitionistic fuzzy topology by involving the algebraic structure on it and introduce the notion of group intuitionistic fuzzy topological spaces. We will examine many properties of these spaces and obtain several results.

Keywords: Intuitionistic fuzzy topological space (IFTS), intuitionistic fuzzy subgroup (IFSG), group intuitionistic fuzzy topological space (GITFS), intuitionistic fuzzy point (IFP).

1. Introduction
The concept of fuzzy set was introduced by L.A. Zadeh [16]. Since then the concept has invaded nearly all branches of Mathematics. C.L. Chang [4] has introduced and developed the theory of fuzzy topological spaces. A. Rosenfeld [15] introduced the theory of fuzzy subgroups. Since then various notions in classical topology and group theory have been extended to fuzzy topological spaces and fuzzy group theory respectively. K.T. Atanassov [1, 2] generalised fuzzy sets to intuitionistic fuzzy sets. On the other hand, D. Coker [5] has introduced the notions of intuitionistic fuzzy topological spaces. R. Biswas [3] introduced the concept of intuitionistic fuzzy subgroup and some other concepts. The concepts of quasi-coincidence for intuitionistic fuzzy point was introduced and developed by F. Gallego Lupianez [7].

In this paper, we will study the intuitionistic fuzzy topological spaces by involving group structure on it and introduce the notion of group intuitionistic fuzzy topological spaces. It will be shown that group intuitionistic fuzzy topological spaces are different from intuitionistic fuzzy topological group, introduced by K. Hur, Y. B. Jun and J. H. Ryou in [10]. The cases when the two structures are same, will also be highlighted.

2. Preliminaries
In this section, we list some basic concepts and well known results on intuitionistic fuzzy topology and intuitionistic fuzzy groups for the sake of completeness of the topic under study.

Definition (2.1)[1] Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form A = \{ (x, \mu_A(x), \nu_A(x)) / x \in X \}, where the functions \mu_A : X \to [0,1] and \nu_A : X \to [0,1] denote the degree of membership (namely \mu_A(x)) and the degree of non-membership (namely \nu_A(x)) of each element x \in X to the set A respectively and 0 \leq \mu_A(x) + \nu_A(x) \leq 1 for each x \in X. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

Definition (2.2) [1] Let A and B be IFS's of the form A = \{ (x, \mu_A(x), \nu_A(x)) / x \in X \} and B = \{ (x, \mu_B(x), \nu_B(x)) / x \in X \}, then
(i) A \subseteq B if and only if \mu_A(x) \leq \mu_B(x) and \nu_A(x) \geq \nu_B(x) for all x \in X;
(ii) A = B if and only if A \subseteq B and B \subseteq A;
(iii) \cap A = \{ (x, \mu_A(x), \mu_B(x)) / x \in X \};
(iv) \cup A = \{ (x, \mu_A(x), \nu_B(x)) \lor \nu_B(x)) / x \in X \};
(vi) \cup A = \{ (x, \mu_A(x), \nu_B(x)) \lor \nu_B(x)) / x \in X \}.

Definition (2.3)[5] Let X be a non-empty, then a subfamily \delta \subseteq IFS(X) is said to be an intuitionistic fuzzy topology (IFT) on X, if it satisfies the following:
(i) \emptyset, 1 \in \delta;
(ii) \bigcup A \subseteq \delta, \forall A \subseteq \delta;
(iii) If A, B \in \delta be any two members, then A \cap B \in \delta.

If \delta is an intuitionistic fuzzy topology on X, then the pair (X, \delta) is called intuitionistic fuzzy topological space (IFTS). The members of \delta are called \delta-open sets. An IFS A of X is said to be \delta-closed in (X,\delta) if and only if A^c is \delta-open set in (X,\delta).

Remark (2.4)(i) The members 0, and 1, are constant intuitionistic fuzzy sets on X defined by
0 \ (x) = (0, 1); \ \forall x \in X \ \text{and} \ \frac{1}{0} \ (x) = (1, 0); \ \forall x \in X. \ \text{(ii) For the sake of simplicity, we denote the IFS} \ A = \{(x, \mu_A(x)), \nu_A(x)) \mid x \in X\} \ \text{by} \ A = (\mu_A, \nu_A).

**Definition (2.5)** [14] Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let A and B be IFSs of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined as

$$
\mu_{f(A)}(y) = \left\{ \begin{array}{ll}
\sup \{ \mu_A(x) : x \in f^{-1}(y) \} & \text{and} \ \nu_{f(A)}(y) = \left\{ \begin{array}{ll}
\inf \{ \nu_A(x) : x \in f^{-1}(y) \} & 0; \\
1 & \text{otherwise}
\end{array} \right. \\
\end{array} \right.
$$

the pre-image of B under $f$ is denoted by $f^{-1}(B)$ and is defined as

$$
f^{-1}(B) = \{ x \mid f(x) \in B \}; \ \forall x \in X
$$

**Remark (2.6)**

Note that $\mu_A(x) \leq \mu_{f(A)}(f(x))$ and $\nu_A(x) \geq \nu_{f(A)}(f(x))$; \ \forall x \in X, and equality hold when $f$ is bijective.

**Definition (2.7)** [5] Let $(X_1, \delta_1)$ and $(X_2, \delta_2)$ be IFTSs. Then the function $f : (X_1, \delta_1) \to (X_2, \delta_2)$ is called

(i) *intuitionistic fuzzy continuous*: if and only if for every $B \in \delta_2 \implies f^{-1}(B) \in \delta_1$.

(ii) *intuitionistic fuzzy open*: if and only if for every $A \in \delta_1 \implies f(A) \in \delta_2$.

(iii) *intuitionistic fuzzy closed*: if and only if image of every $\delta_1$-closed set is $\delta_2$-closed set.

(iv) *intuitionistic fuzzy homeomorphism*: if and only if $f$ is bijective, intuitionistic fuzzy continuous and intuitionistic fuzzy open (or intuitionistic fuzzy closed).

**Definition (2.8)** [10] Let A be a IFS in X and $\delta$ be an IFT on X. Then the induced intuitionistic fuzzy topology on A is the family of subsets of A which are the intersections with A of $\delta$-open sets in X. The induced intuitionistic fuzzy topology is denoted by $\delta_A$ i.e. $\delta_A = \{ A \cap U : \forall U \in \delta \}$, and the pair $(A, \delta_A)$ is called an intuitionistic fuzzy topological subspace of $(X, \delta)$.

**Definition (2.9)** [11] For any $p, q \in [0,1]$ and $x \in X$, a fuzzy set $x_{(p,q)}$ in X is called an intuitionistic fuzzy point (IFP) in X if

$$
x_{(p,q)}(y) = \begin{cases} 
(p,q) & \text{if } y = x \\
(0,1) & \text{otherwise}
\end{cases}, \ \forall y \in X.
$$

The intuitionistic fuzzy point $x_{(p,q)}$ is said to be contained in an intuitionistic fuzzy set A, denoted by $x_{(p,q)} \in A$, if and only if $\mu_A(x) \geq p$ and $\nu_A(x) \leq q$.

In particular, if $x_{(p,q)} \subseteq y_{(r,s)} \iff x = y$ and $p \leq r$, $q \geq s$.

The intuitionistic fuzzy characteristic mapping of a subset A of a set X is denoted by $\chi_A$ and is defined as

$$
\chi_A(x) = \begin{cases} 
(1,0) & \text{if } x \in A \\
(0,1) & \text{otherwise}
\end{cases}, \ \forall x \in X.
$$

Obviously, an intuitionistic characteristic function $\chi_A$ is also an intuitionistic fuzzy set on X and for any non-empty subsets A and B of a set X, we have $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.

**Proposition (2.10)** Let G be a group. Then the subfamily T of IFSs $A = (\mu_A, \nu_A)$ on G such that $\mu_A(x) = \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A(x^{-1})$; \ \forall x \in G. Then T is an IFT on G.

**Proof.** Straightforward.

**Remark (2.11)** In the definition (2.3) if condition (i) is replaced by (i') \ \forall p, q \in [0,1] such that $p + q \leq 1$ and $x \in X$, an intuitionistic fuzzy point $x_{(p,q)}$ in X, defined by

$$
x_{(p,q)}(y) = \begin{cases} 
(p,q) & \text{if } y = x \\
(0,1) & \text{otherwise}
\end{cases}, \ \forall y \in X,
$$

(called constant intuitionistic fuzzy set in X), are in $\delta$, then $\delta$ is called fully stratified intuitionistic fuzzy topology and the pair $(X, \delta)$ is called fully stratified intuitionistic fuzzy topological space (see [12]).

**Definition (2.12)** [3, 13] An IFS $A = (\mu_A, \nu_A)$ of a group G is said to be intuitionistic fuzzy subgroup (IFSG) of G if

(i) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$  \ (ii) $\mu_A(x^{-1}) = \mu_A(x)$;

(iii) $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\}$  \ (iv) $\nu_A(x^{-1}) = \nu_A(x)$; \ \forall x, y \in G.

Or equivalently $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Then the following results are easy to verify

**Result (2.13)** [9] (i) If H is a subset of a group G, then $\chi_H$ is a IFSG of G if and only if H is a subgroup of G.

(ii) All constant intuitionistic fuzzy sets of a group G are IFSGs of G.
(iii) If A is an IFSG of a group G such that
\[ x_{(p,q)} \in A \text{ then } \left(x^{-1}\right)_{(p,q)} \in A. \]

**Definition (2.14)**[12] Let G be a group and \( \delta \) be an IFT on G. Let A, B \( \in \delta \). We define
\[ AB = \left\{ \mu_{AB}(a), \mu_{B}(b) \right\} \text{ and } \quad A^{-1} = \left\{ \nu_{A^{-1}}, \nu_{A^{-1}} \right\} \]
by the respective formula:
\[ \mu_{AB}(x) = \text{Sup} \left\{ \mu_{A}(a), \mu_{B}(b) \right\} \quad \text{and} \quad \nu_{AB}(x) = \text{Inf} \left\{ \nu_{A}(a), \nu_{B}(b) \right\}. \]
\[ \mu_{A^{-1}}(x) = \mu_{A}(x^{-1}) \quad \text{and} \quad \nu_{A^{-1}}(x) = \nu_{A}(x^{-1}); \quad \text{for } x \in G. \]

**Definition (2.15)**[12] Let G be a group and let \((G, \delta)\) be a fully stratified IFTS. Then \((G, \delta)\) is an intuitionistic fuzzy topological group if it satisfies the following conditions:
(i) The mapping \( f: (G, \delta) \times (G, \delta) \rightarrow (G, \delta) \) defined by \( f(x, y) = xy \) is intuitionistic fuzzy continuous;
(ii) The mapping \( g: (G, \delta) \rightarrow (G, \delta) \) defined by \( g(x) = x^{-1} \) is intuitionistic fuzzy continuous.

**Example (2.16)**
(i) Let G be a group and \( \delta \) be the collection of all constant intuitionistic fuzzy sets in G. Then \((G, \delta)\) is an intuitionistic fuzzy topological group.
(ii) Let G be a group and \( \delta = \text{IFS}(G) \) be a discrete intuitionistic fuzzy topology on G. Then \((G, \delta)\) is an intuitionistic fuzzy topological group.

### 3. Group Intuitionistic Fuzzy Topology

In this section, we will introduce group intuitionistic fuzzy topological spaces and give basic properties of this structure. We also discuss similarities with and difference from other intuitionistic fuzzy topological spaces.

**Definition (3.1)** Let G be a group. Then the collection
\[ T = \left\{ A \in \text{IFS}(G) \mid x_{(p,q)} \in A \Rightarrow \left(x^{-1}\right)_{(p,q)} \in A \right\} \]
is an intuitionistic fuzzy topology on G and G with this intuitionistic fuzzy topology is called **group intuitionistic fuzzy topological space** (GIFTS). It is denoted by \((G, T)\).

**Proof.** Since 0 and 1 are constant intuitionistic fuzzy sets and are therefore IFSGs on G. Therefore,
\[ x_{(p,q)} \in 0 \Rightarrow \left(x^{-1}\right)_{(p,q)} \in 0 \quad \text{and} \quad x_{(p,q)} \in 1 \Rightarrow \left(x^{-1}\right)_{(p,q)} \in 1 \]
holds, i.e., \( 0 \in T \) and \( 1 \in T \).

Next, let \( A, B \in T \) be any two members.
Let \( x_{(p,q)} \in A \cap B \Rightarrow \mu_{A \cap B}(x) \geq p \quad \text{and} \quad \nu_{A \cap B}(x) \leq q \]
\[ \Rightarrow \min\left\{\mu_{A}(x), \mu_{B}(x)\right\} \geq p \quad \text{and} \quad \max\left\{\nu_{A}(x), \nu_{B}(x)\right\} \leq q \]
\[ \Rightarrow \mu_{A}(x) \geq p \quad \text{and} \quad \mu_{B}(x) \geq p \quad \text{and} \quad \nu_{A}(x) \leq q \quad \text{and} \quad \nu_{B}(x) \leq q \]
\[ \Rightarrow \mu_{A}(x) \geq p \quad \text{and} \quad \nu_{A}(x) \leq q \quad \text{and} \quad \mu_{B}(x) \geq p \quad \text{and} \quad \nu_{B}(x) \leq q \]
\[ \Rightarrow x_{(p,q)} \in A \quad \text{and} \quad x_{(p,q)} \in B \]
\[ \Rightarrow \left(x^{-1}\right)_{(p,q)} \in A \quad \text{and} \quad \left(x^{-1}\right)_{(p,q)} \in B \]
\[ \Rightarrow \left(x^{-1}\right)_{(p,q)} \in A \cap B \]
Thus, \( A \cap B \in T \).

Let \( x_{(p,q)} \in \bigcup_{i=\Lambda} A_{i} \) implies \( \mu_{\bigcup_{i=\Lambda} A_{i}}(x) \geq p \) and \( \nu_{\bigcup_{i=\Lambda} A_{i}}(x) \leq q \)

Further, let \( \{ A_{i} : i \in \Lambda \} \subseteq T \).
Sup \( \{\mu_{A_{i}}(x) : i \in \Lambda\} \geq p \) and \( \text{Inf}\{\nu_{A_{i}}(x) : i \in \Lambda\} \leq q \).

As both the sets \( \{\mu_{A_{i}}(x) : i \in \Lambda\} \) and \( \{\nu_{A_{i}}(x) : i \in \Lambda\} \) are subset of a bounded set \([0,1] \).

Therefore there exist some j and k in \( \Lambda \) such that \( \mu_{A_{i}}(x) \geq p \) and \( \nu_{A_{i}}(x) \leq q \).
Now, \( \mu_A(x) \geq p \Rightarrow \nu_A(x) \leq 1 - p \) and \( \nu_A(x) \leq q \Rightarrow \mu_A(x) \geq 1 - q \) and so 
\[ x_{(p,1-p)} \in A_j \text{ and so } (x^{-1})_{(p,1-p)} \in A_j \subseteq \bigcup_{i \in \Lambda} A_i. \text{ Similarly, } (x^{-1})_{(1-q,0)} \in A_k \subseteq \bigcup_{i \in \Lambda} A_i. \]
i.e., \((x^{-1})_{(p,1-p)} \) and \((x^{-1})_{(1-q,0)} \) \( \in \bigcup_{i \in \Lambda} A_i \).

So, \( \mu \bigcup_{i \in \Lambda} (x^{-1})_{(p,1-p)} \geq p \) and \( \nu \bigcup_{i \in \Lambda} (x^{-1})_{(1-q,0)} \leq 1 - p \) and also \( \mu \bigcup_{i \in \Lambda} (x^{-1})_{(1-q,0)} \geq 1 - q \) and \( \nu \bigcup_{i \in \Lambda} (x^{-1})_{(p,1-p)} \leq q \)

Therefore, we have \( \mu \bigcup_{i \in \Lambda} (x^{-1})_{(p,1-p)} \geq p \) and \( \nu \bigcup_{i \in \Lambda} (x^{-1})_{(1-q,0)} \leq q \). i.e. \( (x^{-1})_{(p,1-p)} \) \( \in \bigcup_{i \in \Lambda} A_i \).

Thus \( x_{(p,q)} \in \bigcup_{i \in \Lambda} A_i \Rightarrow (x^{-1})_{(p,q)} \in \bigcup_{i \in \Lambda} A_i. \) So, \( \bigcup_{i \in \Lambda} A_i \in T. \)

Hence \( T \) is an intuitionistic fuzzy topology on \( G. \)

**Remark (3.2)(i)** An IFS \( A \) of a group intuitionistic fuzzy topological space \((G, T)\) is called \( T \)-closed if \( A \in T. \)

One can easily verify that 0 and 1. are \( T \)-closed and if \( A, B \) are \( T \)-closed, then \( A \cap B \) is also \( T \)-closed and if \( \bigcap_{i \in \Lambda} A_i \)

{\{ A_i : i \in \Lambda \}} is an arbitrary family of \( T \)-closed sets, then \( \bigcap_{i \in \Lambda} A_i \) is also \( T \)-closed set.

(i) When \( GIFT \) is clearly understood from the content and there is no confusion about it, we may denote the GIFTS \((G,T)\) simply by the symbol \( G \), which is used for the underlying set of elements of \( G. \)

It is easy to observe that the \( GIFT \) as defined in Definition (3.1) is finer than the IFT on the group \( G \) as defined in proposition (2.10).

**Theorem (3.3)** An IFS \( A= \langle \mu_A, \nu_A \rangle \) of the GIFTS \( G \) is \( T \)-closed if and only if \( A \) is \( T \)-open.

**Proof.** Let \( A \) be \( T \)-open set, then \( x_{(p,q)} \in A \Rightarrow (x^{-1})_{(p,q)} \in A \), i.e. \( \mu_A(x) \geq p \) and \( \nu_A(x) \leq q \Rightarrow \mu_A(x^{-1}) \geq p \) and \( \nu_A(x^{-1}) \leq q \).

Let \( x_{(p,q)} \in A^c \Rightarrow \nu_A(x) \leq q \Rightarrow \mu_A(x) \geq p \) and \( \mu_A(x^{-1}) \leq q \, \text{ and } \nu_A(x^{-1}) \geq p \).

\( \Rightarrow x_{(p,q)} \in A, \text{ i.e., } (x^{-1})_{(p,q)} \in A \). Hence \( A \) is \( T \)-open set, i.e., \( A \) is \( T \)-closed set.

Converse is also true.

Now, we give an example of discrete IFTS which is not a GIFTS.

**Example (3.4)** Let \( G = \langle Z, + \rangle \) be the group of integers under addition. Let \( \delta = \text{IFS}(G) \) be the discrete IFT on \( G. \)

We show that \((G, \delta)\) is not a GIFTS.

Let \( N \), the set of all natural numbers. Define an IFS \( A= \langle \mu_A, \nu_A \rangle \) on \( G \) as follows:

\[
\mu_A(x) = \begin{cases} 
p & \text{if } x \in N \\
0 & \text{if } x \in \mathbb{Z} \setminus N 
\end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 
q & \text{if } x \in N \\
1 & \text{if } x \in \mathbb{Z} \setminus N 
\end{cases},
\]

where \( p, q \in \{0,1\} \) st. \( p + q < 1 \).

Consider \( x \in N, \) then we see that \( x_{(p,q)} \in A \) but \( (x^{-1})_{(p,q)} \notin A \). Thus, \( A \notin T \), where \( T \) is the GIFTS on \( G. \) Hence \( \delta \) is not a GIFTS.

Now, we show that under certain condition on \( G, \) a GIFTS is also a discrete IFTS.

**Theorem (3.5)** Let \((G,T)\) be a GIFTS such that \( x^{-1} = x, \forall x \in G. \) Then \( T \) is a discrete IFT on \( G. \)

**Proof.** Let \( A \in \text{IFS}(G) \) and \( x \in G \) be any element. Then for any \( p, q \in \{0,1\} \) such that \( p + q \leq 1, \) we have \( x_{(p,q)} \in A \Rightarrow \mu_A(x) \geq p \) and \( \nu_A(x) \leq q, \) i.e., \( \mu_A(x^{-1}) \geq p \) and \( \nu_A(x^{-1}) \leq q \Rightarrow \left(x^{-1} \right)_{(p,q)} \in A, \) i.e., \( A \in T. \) Hence \( T \) is a discrete IFT on \( G. \)

**Remark (3.6)(i)** If there exist at least one element \( x \) in a group \( G \) such that \( x^{-1} \neq x, \) then the GIFTS on \( G \) is not a discrete IFTS.

(ii) If \( |G| \geq 2, \) then the GIFTS(G,T) on \( G \) is not an indiscrete IFTS, i.e., there exist \( A \in \text{IFS}(G), \) which is different from \( \emptyset \) and \( G, \) such that \( A \notin T. \)

**Theorem (3.7)** Every intuitionistic fuzzy subgroup of a group \( G \) is \( T \)-open set in GIFTS.

**Proof.** Let \( A = \langle \mu_A, \nu_A \rangle \) be any IFSG of the group \( G. \) To show that \( A \in T. \)
Let \( x_{(p,q)} \in A \Rightarrow \mu_A(x) \geq p \) and \( \nu_A(x) \leq q \), i.e., \( \mu_A(x^{-1}) \geq p \) and \( \nu_A(x^{-1}) \leq q \) \( \Rightarrow (x^{-1})_{(p,q)} \in A \). So \( A \in T \).

**Remark (3.8)** Converse of theorem (3.7) is not true i.e. T-open set need not be IFSG of \( G \).

**Example (3.9)** Consider the group \( G = \{ e, a, b, ab \} \), where \( a^2 = b^2 = e \) and \( ab = ba \) be the Klein four group.

Define an IFS \( A = \{ \mu_A, \nu_A \} \) of \( G \) by

\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x = e \\
0.3 & \text{if } x = a \\
0.4 & \text{if } x = b \\
0.2 & \text{if } x = ab 
\end{cases} 
\quad \text{and} \quad \nu_A(x) = \begin{cases} 
0 & \text{if } x = e \\
0.4 & \text{if } x = a \\
0.3 & \text{if } x = b \\
0.5 & \text{if } x = ab 
\end{cases}
\]

Clearly, \( A \) is not an IFSG of \( G \). But \( A \) is T-open set, where \( T \) is a GIFTS on \( G \).

Now, the question arises, when a T-open set of \( G \) is an IFSG of \( G \). In this direction, we first prove the following lemma.

**Lemma (3.10)** Let \( A \) be a intuitionistic fuzzy subgroup of \( G \), then for all \( x_{(p,q)}, y_{(r,s)} \in A \) we have \( (xy)_{(u,v)} \in A \), where \( u = \min\{p, r\} \) and \( v = \max\{q, s\} \).

**Proof.**

Let \( A = \{ \mu_A, \nu_A \} \) be an IFSG of \( G \) and \( x_{(p,q)}, y_{(r,s)} \in A \), then we have \( \mu_A(x) \geq p \); \( \nu_A(x) \leq q \) and \( \mu_A(y) \geq r \); \( \nu_A(y) \leq s \). Therefore

\[
\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{p, r\} = u \quad \text{and} \quad \nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \max\{q, s\} = v
\]

implies \( \mu_A(xy) \geq u \) and \( \nu_A(xy) \leq v \), i.e., \( (xy)_{(u,v)} \in A \).

**Theorem (3.11)** Let \( A \) be a T-open set in GIFTS \((G,T)\), then \( A \) is an IFSG of \( G \) if and only if for all \( x_{(p,q)}, y_{(r,s)} \in A \) we have \( (xy)_{(u,v)} \in A \), where \( u = \min\{p, r\} \) and \( v = \max\{q, s\} \).

**Proof.** When \( A \in T \) and also an IFSG of \( G \), then the result follows by lemma (3.10).

Conversely, let \( A = \{ \mu_A, \nu_A \} \) be a T-open set such that \( \forall x_{(p,q)}, y_{(r,s)} \in A \) we have \( (xy)_{(u,v)} \in A \), where \( u = \min\{p, r\} \) and \( v = \max\{q, s\} \).

To show that \( A \) is an IFSG of \( G \).

Let \( x, y \in G \) be any element and suppose that \( \mu_A(x) = p, \nu_A(x) = q \) and \( \mu_B(y) = r, \nu_B(y) = s \).

**Case(i)** when \( p \neq r \) or \( q \neq s \) or both

**Subcase(i)** when \( p \neq r \) and \( q = s \), wlog let \( p < r \) and \( q = s \), then \( \min\{p, r\} = u \).

Now, \( \mu_A(x) = p \) and \( \mu_B(y) = r \) \( \Rightarrow x_{(p,q)} \in A \) and \( y_{(r,s)} \in A \). As \( A \in T \), \( \Rightarrow (xy^{-1})_{(p,q)} \in A \).

Thus \( x_{(p,q)}, (y^{-1})_{(r,s)} \in A \Rightarrow (xy)^{-1}_{(p,q)} \in A \). i.e. \( \mu_A((xy)^{-1}) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_B(y)\} \) and \( \nu_A((xy)^{-1}) \leq q = \max\{q, s\} = \max\{\nu_A(x), \nu_B(y)\} \).

**Subcase(ii)** when \( p = r \) and \( q \neq s \), wlog let \( q < s \), then \( \max\{q, s\} = s \).

Now, \( \nu_A(x) = q \) and \( \nu_B(y) = s \) \( \Rightarrow x_{(p,q)} \in A \) and \( y_{(r,s)} \in A \). As \( A \in T \), \( \Rightarrow (y^{r^{-1}})_{(p,q)} \in A \).

Thus \( x_{(p,q)}, (y^{r^{-1}})_{(r,s)} \in A \Rightarrow (y^{-1})_{(r,s)} \in A \). i.e. \( \mu_A((y^{-1})) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_B(y)\} \) and \( \nu_A((y^{-1})) \leq s = \max\{q, s\} = \max\{\nu_A(x), \nu_B(y)\} \).

**Subcase(iii)** when \( p \neq r \) and \( q = s \), wlog let \( p < r \), \( q = s \), then \( \min\{p, r\} = p \) and \( \max\{q, s\} = s \).

Now, \( \mu_A(x) = p \) and \( \mu_B(y) = r \) \( \Rightarrow y_{(r,s)} \in A \) and \( v_{A}(x) = q \) \( \Rightarrow (xy^{-1})_{(p,q)} \in A \).

**Case(ii)** when \( p = r \) and \( q = s \) \( \Rightarrow p = r \) and \( q = s \), then \( \min\{p, r\} = p \) and \( \max\{q, s\} = s \).

Thus \( x_{(p,q)}, (y^{r^{-1}})_{(r,s)} \in A \Rightarrow (y^{r^{-1}})_{(r,s)} \in A \). i.e. \( \mu_A((y^{r^{-1}})) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_B(y)\} \) and \( \nu_A((y^{r^{-1}})) \leq s = \max\{q, s\} = \max\{\nu_A(x), \nu_B(y)\} \).

Thus in all the cases, we see that \( A \) is an IFSG of \( G \).

**Remark (3.12)** The GIFTS on the group \( G \) is fully stratified intuitionistic fuzzy topological space by remark (2.11)(ii) and Theorem (3.7).
Now, \( \mu_{AB}(xy) = \text{Sup}\{\min\{\mu_A(x), \mu_B(y)\}\} \geq u \Rightarrow \mu_{AB}(xy) \geq u \) and 
\( \nu_{AB}(xy) = \text{Inf}\{\max\{\nu_A(x), \nu_B(y)\}\} \leq v \Rightarrow \nu_{AB}(xy) \leq v \).

Thus, \( \mu_{AB}(xy) \geq u \) and \( \nu_{AB}(xy) \leq v \Rightarrow (xy)_{(\mu,\nu)} \in AB \).

**Theorem (3.14)** If \( A \) and \( B \) are \( T \)-open sets in \( G \), then \( AB \) is also \( T \)-open set provided \( G \) is an abelian group.

**Proof.** Let \( x_{(p,q)} \in A \) and \( y_{(r,s)} \in B \), then by lemma (3.13), we have 
\( (xy)_{(p,r)} \in AB \), where \( p = \text{min}\{p, r\} \) and \( v = \text{max}\{q, s\} \).

Also, because \( A, B \in T \), \( (x^-)^{-1}_{(p,q)} \in A \) and \( (y^-)^{-1}_{(r,s)} \in B \). Again by using lemma (3.13), we have 
\( (x^-)^{-1}_{(p,q)} \in AB \Rightarrow ((y^-)^{-1})_{(r,s)} \in AB \), i.e., \( ((y^-)^{-1})_{(r,s)} \in AB \) [As \( G \) is abelian ]

Thus, \( (xy)_{(\mu,\nu)} \in AB \Rightarrow (xy)_{(\mu,\nu)} \in AB \) in \( T \).

**Theorem (3.15)** Let \( H \) be a subgroup of a GIFTS \( (G, T) \), then the relative IFT \( T_H \) on \( H \) is a GIF on \( H \).

This subset \( H \) of \( G \) has an intuitionistic fuzzy characteristic function \( \chi_H \) defined as:

\[
\chi_H(x) = \begin{cases} 
(1,0) & \text{if } x \in H \\
(0,1) & \text{if } x \notin H
\end{cases}
\]

Clearly, \( \chi_H(x) = (1,0), \forall x \in H \) is a constant intuitionistic fuzzy set on \( H \).

Let \( T_H = \{ A \cap \chi_H : \forall A \in T \} \). Then we claim that \( T_H \) is a GIF on \( H \).

For, choose \( 0 \in T \), then as \( 0 \cap \chi_H(x) = (\min\{0,1\}, \max\{1,1\}) = (0,1) = 0 \), so we have \( 0 \in T_H \). To Show that \( H \in T_H \), choose \( \chi_H \) the characteristic function for \( G \), then \( (\chi_H \cap \chi_H)(x) = \chi_H(x) \), so we have \( H \in T_H \).

Further, let \( \{A_i \cap \chi_H : i \in \Lambda\} \subseteq T_H \). Then, \( \bigcap_{i \in \Lambda} A_i \cap \chi_H \) is an IFT on \( H \).

Also, let \( A_1 \cap \chi_H \) and \( A_2 \cap \chi_H \) be any two members of \( T_H \). Then 
\( (A_1 \cap \chi_H) \cap (A_2 \cap \chi_H) = (A_1 \cap A_2) \cap \chi_H \in T_H \). Hence \( T_H \) is an IFT on \( H \).

Let \( A \cap \chi_H \in T_H \), let \( x_{(p,q)} \in (A \cap \chi_H) \Rightarrow \mu_{A \cap \chi_H}(x) \geq p \) and \( v_{A \cap \chi_H}(x) \leq q \).

\( \Rightarrow \min\{\mu_A(x), \mu_{\chi_H}(x)\} \geq p \) and \( \max\{v_A(x), v_{\chi_H}(x)\} \leq q \).

As \( x \in H \) therefore, \( \mu_{\chi_H}(x) = 1 \) and \( v_{\chi_H}(x) = 0 \), so \( \mu_A(x) = \min\{\mu_A(x), \mu_{\chi_H}(x)\} \geq p \) and \( v_A(x) = \max\{v_A(x), v_{\chi_H}(x)\} \leq q \).

Also, \( \mu_{\chi_H}(x^-) = 1 \) and \( v_{\chi_H}(x^-) = 0 \), as such \( \min\{\mu_A(x^-), \mu_{\chi_H}(x^-)\} \geq p \) and \( \max\{v_A(x^-), v_{\chi_H}(x^-)\} \leq q \).

Hence relative intuitionistic fuzzy topology on \( H \) is the GIF on \( H \).

### 4. Functions on Group Intuitionistic Fuzzy Topological Spaces

**Theorem (4.1)** Let \( f: G \to G' \) is a group isomorphism and \( T, T' \) be GIFTS on groups \( G \) and \( G' \) respectively. Then \( f \) is an intuitionistic fuzzy homeomorphism between GIFTS \( (G, T) \) and \( (G', T') \).

**Proof.** Let \( f : G \to G' \) be an group isomorphism, where \( G \) and \( G' \) are groups. We show that \( f \) is an intuitionistic fuzzy homeomorphism.

For intuitionistic fuzzy continuity: let \( B \) be any \( T' \)-open set and let \( x_{(p,q)} \in f^{-1}(B) \). Then 
\( \mu_{f^{-1}(B)}(x) \geq p \) and \( v_{f^{-1}(B)}(x) \leq q \) \( \Rightarrow \mu_{f}(f(x)) \geq p \) and \( v_{f}(f(x)) \leq q \).

i.e., \( f(x)_{(p,q)} \in B \). As \( B \in T' \), we have \( f(x)_{(p,q)} \in B \) if \( f(x)_{(p,q)} \in B \) \( \Rightarrow \mu_{f}(f(x)) \geq p \) and \( v_{f}(f(x)) \leq q \) \( \Rightarrow f(x)_{(p,q)} \in B \).

For intuitionistic fuzzy openness: let \( A \in T \) and \( y_{(r,s)} \in f(A) \) \( \Rightarrow \mu_{f(A)}(y) \geq r \) and \( v_{f(A)}(y) \leq s \) \( \Rightarrow \mu_{A}(x) \geq r \) and \( v_{A}(x) \leq s \).

[\( \therefore f \) is isomorphism \( \mu_{f(A)}(y), v_{f(A)}(y) = (\mu_{f(A)}(f(x)), v_{f(A)}(f(x))) = (\mu_{A}(x), v_{A}(x)) \)]
Next, we show that the topology $T_{y}$ is $T$-open set so $(x^{-1})_{(r,s)} \in A$. i.e., $\mu_{x}(x^{-1}) \geq r$ and $V_{x}(x^{-1}) \leq s$

Again, as $f$ is isomorphism, we have $f(x^{-1}) = y^{-1}$ and $\mu_{x}(x^{-1}) \geq r$ and $V_{x}(x^{-1}) \leq s$

Thus, $\mu_{f(x)}(y^{-1}) = \mu_{f(x)}(f(x^{-1})) = \mu_{x}(x^{-1}) \geq r$ and $V_{f(x)}(y^{-1}) = V_{f(x)}(f(x^{-1})) = V_{x}(x^{-1}) \leq s$

Thus, $(y^{-1})_{(r,s)} \in f(A)$. So, $f$ is an intuitionistic fuzzy open mapping.

Hence $f$ is an intuitionistic fuzzy homeomorphism.

**Theorem (4.2)** Let $f: (G,T) \rightarrow (G',T')$ be a mapping. Then $f$ is intuitionistic fuzzy continuous if and only if $f(x^{-1}) = f((x^{-1}))^{-1}$, $\forall x \in G$.

**Proof.** Firstly, let $f(x^{-1}) = f((x^{-1}))^{-1}$. As $x \in G$, suppose that $B$ be an $T$-open set such that $f(x^{-1}) \subseteq B$, i.e., $(f(x^{-1}))^{-1} \subseteq B$.

Let $A \subseteq T'$ be any element. Then $(x^{-1})_{(r,s)} \in A$. Let $x \in G$. Then $x^{-1} = y^{-1}$ and $f(x^{-1}) = f((x^{-1}))^{-1}$. As $B \subseteq A$, we have $(x^{-1})_{(r,s)} \subseteq B$, i.e., $(x^{-1})_{(r,s)} \subseteq f^{-1}(B)$. Thus $x^{-1} \in f^{-1}(B)$. So, $f^{-1}(B) \subseteq T'$.

Hence $f$ is an intuitionistic fuzzy continuous.

Conversely, let $f: (G,T) \rightarrow (G',T')$ be intuitionistic fuzzy continuous mapping.

To show $f(x^{-1}) = f((x^{-1}))^{-1}$, $\forall x \in G$.

Let $x \in G \Rightarrow f(x) \in G'$. Suppose that $B \subseteq T'$ is an $T'$-open set such that $f(x) \subseteq B$.

Let $\mu_{x}(f(x)) \geq r$ and $V_{x}(f(x)) \leq s$, i.e., $\mu_{f^{-1}(x)}(x^{-1}) \geq r$ and $V_{f^{-1}(x)}(x^{-1}) \leq s$, i.e., $(x^{-1})_{(r,s)} \subseteq f^{-1}(B)$. Thus $x^{-1} \in f^{-1}(B)$. So, $f^{-1}(B) \subseteq T'$.

Hence $f$ is an intuitionistic fuzzy continuous. 

As the map $f$ is intuitionistic fuzzy continuous, $f^{-1}(B) \subseteq T'$, so $(x^{-1})_{(r,s)} \subseteq f^{-1}(B)$.

Let $H = \{ f(x) \mid f(x) \subseteq B \}$. Clearly, $H \subseteq G'$. Define the IFS $C$ on $G'$ as follow:

$$
\mu_{C}(y) = \begin{cases} 
 r & \text{if } y \in H \\
 s & \text{if } y \notin H \\
 0 & \text{if } y \notin H, \quad \text{where } r, s \in (0,1] \text{ such that } r + s \leq 1.
\end{cases}
$$

Obviously, $C \subseteq T'$. As proved earlier, $(f(x))_{(r,s)} \in C \Rightarrow (f(x))_{(r,s)} \in C$

This means that $f(x) = f((x^{-1}))^{-1}$. Hence proved.

**Theorem (4.3)** Group intuitionistic fuzzy topological space is an intuitionistic fuzzy topological property.

**Proof.** Let $(G,T)$ be a GIFTS and $f : (G,T) \rightarrow (X,T')$ be a intuitionistic fuzzy homeomorphism. We will show that $(X,T')$ is also a GIFTS. To prove this, we first show that $X$ is a group under the operation:

$$
y_{1}y_{2} = f(x_{1})f(x_{2}) = f(x_{1}x_{2}), \quad \forall y_{1}, y_{2} \in X,
$$

Closure property: let $y_{1}, y_{2} \in X$, then $y_{1}y_{2} = f((y_{1})^{-1}y_{2}) = f(x_{2}x_{1}) \subseteq X$.

Associativity: let $y_{1}, y_{2}, y_{3} \in X$, then

$$
y_{1}y_{2}y_{3} = y_{1}(f(x_{2}x_{3})) = f(x_{2})(f(x_{3})) = f(x_{2}x_{3})x_{1} = (f(x_{1})f(x_{2}))(f(x_{3})) = (y_{1}y_{2})y_{3}.
$$

Existence of identity element: let $e$ be the identity element of $G$.

We show that $f(e)$ is the identity element of $X$.

Let $y \in X$ be any element. Then $f(y) = f(f(x)) = f(f^{-1}(y)) = f(f^{-1}(x)) = y$

Similarly, we can show that $f(e)^{-1} = y$.

Existence of inverse: let $y \in X$. Then $\exists x \in G$ such that $y = f(x)$. We show that $y^{-1} = f(x^{-1})$ is the inverse of $y$. Now, $y^{-1} = f(x^{-1}) = f^{-1}(f(x)) = f^{-1}(f(x^{-1})) = f(x^{-1}) = f(e)$

Similarly, we can show that $y^{-1} = f(e)$. Hence $X$ is a group.

Next, we show that the topology $T'$ is the GIFT on $X$.

Let $T''$ be the GIFT on $X$. Then we show that $T' = T''$.

Let $y \in X$ be any element then $\exists x \in G$ such that $f(x) = y$. It follows that $y = f(x) = f^{-1}(f(x)) = f^{-1}(y)$.
Let $B \in T'$. As $f$ is intuitionistic fuzzy continuous $\therefore f^{-1}(B) \in T$. As $T$ is a GIFT on $G$

Therefore $x(p,q) \in f^{-1}(B) \Rightarrow (x^{-1})_{p,q} \in f^{-1}(B)$

i.e. $\mu_{f^{-1}(B)}(x) \geq p$ and $\nu_{f^{-1}(B)}(x) \leq q \Rightarrow \mu_{f^{-1}(B)}(x^{-1}) \geq p$ and $\nu_{f^{-1}(B)}(x^{-1}) \leq q$

i.e. $\mu_y(f(x)) \geq p$ and $\nu_y(f(x)) \leq q \Rightarrow \mu_y(f(x^{-1})) \geq p$ and $\nu_y(f(x^{-1})) \leq q$

i.e. $\mu_y(y) \geq p$ and $\nu_y(y) \leq q \Rightarrow \mu_y(y^{-1}) \geq p$ and $\nu_y(y^{-1}) \leq q$

i.e. $y_{(p,q)} \in B \Rightarrow (y^{-1})_{(p,q)} \in B$ . So $B \in T''$ [ $\therefore T''$ is a GIFT on $X$]

Thus $B \in T' \Rightarrow B \subseteq T''$. Conversely, let $B \in T''$. Then for any element $y \in X$, $\exists$ unique $x \in G$ such that $f(x) = y$

Now, $y_{(p,q)} \in B \Rightarrow (y^{-1})_{(p,q)} \in B$ i.e. $\mu_y(y) \geq p$ and $\nu_y(y) \leq q \Rightarrow \mu_y(y^{-1}) \geq p$ and $\nu_y(y^{-1}) \leq q$

i.e. $\mu_y(f(x)) \geq p$ and $\nu_y(f(x)) \leq q \Rightarrow \mu_y(f(x^{-1})) \geq p$ and $\nu_y(f(x^{-1})) \leq q$

i.e. $\mu_y(y) \geq p$ and $\nu_y(y) \leq q \Rightarrow \mu_y(y^{-1}) \geq p$ and $\nu_y(y^{-1}) \leq q$

i.e. $x(p,q) \in f^{-1}(B) \Rightarrow (x^{-1})_{p,q} \in f^{-1}(B)$ i.e. $f^{-1}(B) \in T$.

As $f$ is open mapping. Therefore $f(f^{-1}(B)) \in T'$. i.e. $B \in T''$.

Thus $B \in T' \Rightarrow B \in T''$ so $T'' \subseteq T$. Hence $T' = T''$.

Corollary (4.4) Let $f: (G,T) \rightarrow (G',T')$ be a intuitionistic fuzzy continuous mapping and $K = \{x \in G : f(x) = e'\}$. Then the topology induced on the set $K$ is a GIFT, where $e'$ is the identity element of $G'$.

Proof. Let $T_k = \{A \cap \chi_k : \forall A \in T\}$. Then we claim that $T_k$ is a GIFT on $K$.

Let $x \in K$ be any element, then $f(x) = e'$. Now $f(x^{-1}) = f(f(x))^{-1} = \{e'\}^{-1} = e' \Rightarrow x^{-1} \in K$.

Let $x(p,q) \in (A \cap \chi_k) \Rightarrow \mu_{A \cap \chi_k}(x) \geq p$ and $\nu_{A \cap \chi_k}(x) \leq q$.

$\Rightarrow \min \{\mu_A(x), \mu_{\chi_k}(x)\} \geq p$ and $\max \{\nu_A(x), \nu_{\chi_k}(x)\} \leq q$

$\Rightarrow \mu_A(x) \geq p$ and $\nu_A(x) \leq q$ [As $\mu_{\chi_k}(x) = 1$ and $\nu_{\chi_k}(x) = 0$].

So $x(p,q) \in A$. As $A \in T \Rightarrow (x^{-1})_{p,q} \in A$, i.e., $\mu_A(x^{-1}) \geq p$ and $\nu_A(x^{-1}) \leq q$, also $\mu_{A \cap \chi_k}(x^{-1}) = 1$ and $\nu_{A \cap \chi_k}(x^{-1}) = 0$. Therefore $\min \{\mu_A(x^{-1}), \mu_{A \cap \chi_k}(x^{-1})\} \geq p$ and $\max \{\nu_A(x^{-1}), \nu_{A \cap \chi_k}(x^{-1})\} \leq q$

i.e., $\mu_{A \cap \chi_k}(x^{-1}) \geq p$ and $\nu_{A \cap \chi_k}(x^{-1}) \leq q$ i.e. $(x^{-1})_{p,q} \in (A \cap \chi_k)$. Hence $T_k$ is a GIFT on $K$.

Theorem (4.5) Let $f: (G,T) \rightarrow (G,T)$ be a mapping from a GIFTs $(G,T)$ onto itself defined by $f(x) = x^{-1} ; \forall x \in G$. Then $f$ is bijective and intuitionistic fuzzy continuous.

Proof. Clearly, $f$ is one-one and onto. Let $B \in T$ be any open set. Then we show that $f^{-1}(B) \in T$.

Let $x(p,q) \in f^{-1}(B) \Rightarrow \mu_{f^{-1}(B)}(x) \geq p$ and $\nu_{f^{-1}(B)}(x) \leq q$

$\Rightarrow \mu_y(f(x)) \geq p$ and $\nu_y(f(x)) \leq q \Rightarrow \mu_y(f(x^{-1})) \geq p$ and $\nu_y(f(x^{-1})) \leq q$ i.e., $\mu_y(f(x^{-1})) \geq p$ and $\nu_y(f(x^{-1})) \leq q \Rightarrow \mu_y(f(x^{-1})) \geq p$ and $\nu_y(f(x^{-1})) \leq q$,

i.e., $(x^{-1})_{p,q} \in f^{-1}(B)$. Thus, $x(p,q) \in f^{-1}(B) \Rightarrow (x^{-1})_{p,q} \in f^{-1}(B)$.

So, $f^{-1}(B) \in T$. Hence $f$ is intuitionistic fuzzy continuous.

5. Product of Group Intuitionistic Fuzzy Topological Spaces

In this section, we study the product of group intuitionistic fuzzy topological spaces. We also discuss similarities with difference from product of intuitionistic fuzzy topological spaces and also from the intuitionistic fuzzy topological groups.

Theorem (5.1) Let $(G_1, T_1)$ and $(G_2, T_2)$ be two GIFTs. Then product intuitionistic fuzzy topology on $G_1 \times G_2$ is contained in the GIFT on $G_1 \times G_2$. The equality between the two topologies need not hold.

Proof. We know that $G_1 \times G_2$ is a group under the point wise operation defined by

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2, y_1y_2) ; \forall (x_1, y_1), (x_2, y_2) \in G_1 \times G_2.$$
The element \((e_1, e_2)\) is the identity of \(G_1 \times G_2\), where \(e_1, e_2\) are the identities of \(G_1, G_2\) respectively. Let \((x, y)\) be any element of \(G_1 \times G_2\). Then \((x^{-1}, y^{-1})\) is the inverse of \((x, y)\) in \(G_1 \times G_2\). Let \(T'\) be the product IFT on \(G_1 \times G_2\) and \(T\) be the GIFT on \(G_1 \times G_2\). We show that \(T' \subseteq T\). Let \(A \in T'\) and let \(z = (x, y)\) be any element of \(G_1 \times G_2\). Then \(z_{(p,q)} \in A = B \times C\), where \(B \in T_1\), \(C \in T_2\) if \(
abla_{B \circ C}(x, y) \geq p\) and \(\nabla_{B \circ C}(x, y) \leq q\)
\[
\Rightarrow \min \{ \mu_B(x), \mu_C(y) \} \geq p \quad \text{and} \quad \max \{ \nabla_B(x), \nabla_C(y) \} \leq q
\]
\[
\Rightarrow \mu_B(x) \geq p, \mu_C(y) \geq p \quad \text{and} \quad \nabla_B(x) \leq q, \nabla_C(y) \leq q
\]
\[
\Rightarrow \mu_B(x) \geq p, \nabla_B(x) \leq q \quad \text{and} \quad \mu_C(y) \geq p, \nabla_C(y) \leq q
\]
\[
\Rightarrow x_{(p,q)} \in B \quad \text{and} \quad y_{(p,q)} \in C \Rightarrow (x^{-1}_{(p,q)}) \in B \quad \text{and} \quad (y^{-1}_{(p,q)}) \in C \quad [\because B \in T_1, C \in T_2]\n\]
\[
\Rightarrow \mu_B(x^{-1}) \geq p, \nabla_B(x^{-1}) \leq q \quad \text{and} \quad \mu_C(y^{-1}) \geq p, \nabla_C(y^{-1}) \leq q
\]
\[
\Rightarrow \Rightarrow \min \{ \mu_B(x^{-1}), \mu_C(y^{-1}) \} \geq p \quad \text{and} \quad \max \{ \nabla_B(x^{-1}), \nabla_C(y^{-1}) \} \leq q
\]
\[
\Rightarrow \mu_{B \circ C}(x^{-1}, y^{-1}) \geq p \quad \text{and} \quad \nabla_{B \circ C}(x^{-1}, y^{-1}) \leq q, \text{i.e., } \mu_{B \circ C}(z^{-1}) \geq p \quad \text{and} \quad \nabla_{B \circ C}(z^{-1}) \leq q
\]
\[
\Rightarrow \mu_A(z^{-1}) \geq p \quad \text{and} \quad \nabla_A(z^{-1}) \leq q, \text{i.e., } (z^{-1})_{(p,q)} \in A
\]

Thus \(z_{(p,q)} \in A \Rightarrow (z^{-1})_{(p,q)} \in A\). So, \(A \in T'\). Therefore, \(T' \subseteq T\).

Next, we show that the equality between \(T\) and \(T'\) need not hold.

Let \(G_1 = (Z, +)\) and \(G_2 = (2Z, +)\). Let \(T_1, T_2\) be the GIFT on \(G_1\) and \(G_2\) respectively. Again suppose that \(T'\) be the product IFT on \(G_1 \times G_2\) and \(T\) be the GIFT on \(G_1 \times G_2\).

Consider the subset \(H = \{ (2, 4), (-2, -4) \}\) of \(G_1 \times G_2\). Define the IFS \(A\) on \(G_1 \times G_2\) such that
\[
\mu_A(x, y) = \begin{cases} p & \text{if } (x, y) \in H \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nabla_A(x, y) = \begin{cases} q & \text{if } (x, y) \in H \\ 1 & \text{otherwise} \end{cases}
\]
But the smallest \(T'\)-open set containing \(A\) is \(B \times C\), where \(B \in T_1\) and \(C \in T_2\) defined as:
\[
\mu_B(x) = \begin{cases} p & \text{if } x = 2, -2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nabla_B(x) = \begin{cases} q & \text{if } x = 2, -2 \\ 1 & \text{otherwise} \end{cases}
\]
\[
\mu_C(x) = \begin{cases} p & \text{if } x = 4, -4 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nabla_C(x) = \begin{cases} q & \text{if } x = 4, -4 \\ 1 & \text{otherwise} \end{cases}
\]
\[
(B \times C) (x, y) = \{ \mu_{B \circ C}(x, y), \nabla_{B \circ C}(x, y) \} = \{ \min \{ \mu_B(x), \mu_C(y) \}, \max \{ \nabla_B(x), \nabla_C(y) \} \}
\]
\[
= \begin{cases} (p, q) & \text{if } (x, y) \in (2, 4) \times (-2, -4) \\ (0, 1) & \text{otherwise} \end{cases}
\]

Thus, \(A \in T'.\) Hence, \(T' \neq T\).

**Remark (5.2)** If in the above theorem (5.1), we have \(x^{-1} = x\) and \(y^{-1} = y\) for all \(x \in G_1\) and for all \(y \in G_2\), then the product IFT on \(G_1 \times G_2\) is same as the GIFT on \(G_1 \times G_2\).

Next, we show that GIFTs \((G, T)\) is not a intuitionistic fuzzy topological group as defined in (2.13)

**Example (5.3)** Consider \(G = \{1, w, w^2\}\), where \(w\) is non-real cube root of unity. Then \((G, *)\) is a group. Consider the IFSs \(A\) and \(B\) on \(G\) as follows:
\[
A = \{ 1, 0.9, 0.1 \}, \langle w, 0.6, 0.3 \rangle, \langle w^2, 0.6, 0.3 \rangle \} \quad \text{and} \quad B = \{ 1, 0.8, 0.1 \}, \langle w, 0.3, 0.5 \rangle, \langle w^2, 0.3, 0.5 \rangle \}.
\]
Clearly \(B \subseteq A\).

Consider \(T = \{ \emptyset, G, A, B \}\). It is easy to verify that \(T\) is a GIFT on \(G\).

Define the mapping \(f : G \to G\) by \(f(a) = a^{-1}\); \(\forall a \in G\) and \(g : G \times G \to G\) by \(g(a, b) = ab\); \(\forall a, b \in G\).

Clearly \(f \) is a group homomorphism.

**Remark (5.4)** If in the above theorem (5.1), we have \(x^{-1} = x\) and \(y^{-1} = y\) for all \(x \in G_1\) and for all \(y \in G_2\), then the product IFT on \(G_1 \times G_2\) is same as the GIFT on \(G_1 \times G_2\).

Next, we show that GIFTs \((G, T)\) is not a intuitionistic fuzzy topological group as defined in (2.13)

**Example (5.3)** Consider \(G = \{1, w, w^2\}\), where \(w\) is non-real cube root of unity. Then \((G, *)\) is a group. Consider the IFSs \(A\) and \(B\) on \(G\) as follows:
\[
A = \{ 1, 0.9, 0.1 \}, \langle w, 0.6, 0.3 \rangle, \langle w^2, 0.6, 0.3 \rangle \} \quad \text{and} \quad B = \{ 1, 0.8, 0.1 \}, \langle w, 0.3, 0.5 \rangle, \langle w^2, 0.3, 0.5 \rangle \}.
\]
Clearly \(B \subseteq A\).

Consider \(T = \{ \emptyset, G, A, B \}\). It is easy to verify that \(T\) is a GIFT on \(G\).

Define the mapping \(f : G \to G\) by \(f(a) = a^{-1}\); \(\forall a \in G\) and \(g : G \times G \to G\) by \(g(a, b) = ab\); \(\forall a, b \in G\).

Clearly \(f \) is a group homomorphism.
not open set in product fuzzy topological space \((G,T)\times(G,T)\), for then \(g^{-1}(A) = C\times D\), for some \(C, D \in T\), which is not possible in this case.

Now, the question arises, when a GIFTS be a fuzzy topological group. In this direction we have the following result.

**Theorem (5.4)** Let \(G\) be a group such that \(x^{-1} = x\), for all \(x \in G\). Then the GIFTS \((G,T)\) is same as the intuitionistic fuzzy topological group.

**Proof.** It follows from Theorems (3.5) and Example (2.16)(ii)

6. **Conclusion**

In this paper, the notion of group intuitionistic fuzzy topological spaces is introduced. It has been observed that the group intuitionistic fuzzy topology is different from discrete and indiscrete intuitionistic fuzzy topology, the cases when they behave same have been examined. It is seen that group intuitionistic fuzzy topology is a hereditary property subject to the subgroup of the group \(G\). Also it is noticed that group intuitionistic fuzzy topology is a topological and productive property. Moreover, it has been seen that the notion of group intuitionistic fuzzy topology is different from that of intuitionistic fuzzy topological group as introduced by Hur, Jun and Ryou. The cases when the two are same have also been established.

7. **Scope of further study**

Many properties like intuitionistic fuzzy connectedness, intuitionistic fuzzy compactness, intuitionistic fuzzy separation axioms, convergence of sequence in group intuitionistic fuzzy topological space \((G,T)\) are yet to be examined. This work is under progress.

**Acknowledgement**

The author is highly thankful to the referee for his valuable suggestions for improving the quality of the paper.

**References**

The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage: http://www.iiste.org

CALL FOR JOURNAL PAPERS

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

Prospective authors of journals can find the submission instruction on the following page: http://www.iiste.org/journals/ All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

MORE RESOURCES

Book publication information: http://www.iiste.org/book/

Academic conference: http://www.iiste.org/conference/upcoming-conferences-call-for-paper/

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digtial Library, NewJour, Google Scholar