Group Intuitionistic Fuzzy Topological Spaces

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Abstract
The notion of intuitionistic fuzzy set was introduced by K.T. Atanassov as a generalization of the notion of fuzzy set. Intuitionistic fuzzy topological spaces were introduced by D. Coker and studied by many eminent authors like F. Gallego Lupianez, K. Hur, J. H. Kim and J. H. Ryou. R. Biswas applied the notion of intuitionistic fuzzy set to algebra and introduced intuitionistic fuzzy subgroup of a group. In this paper, we will study intuitionistic fuzzy topology by involving the algebraic structure on it and introduce the notion of group intuitionistic fuzzy topological spaces. We will examine many properties of these spaces and obtain several results.
Keywords: Intuitionistic fuzzy topological space (IFTS), intuitionistic fuzzy subgroup (IFSG), group intuitionistic fuzzy topological space (GIFTS), intuitionistic fuzzy point (IFP).

1. Introduction

The concept of fuzzy set was introduced by L.A. Zadeh [16]. Since then the concept has invaded nearly all branches of Mathematics. C.L. Chang [4] has introduced and developed the theory of fuzzy topological spaces. A. Rosenfeld [15] introduced the theory of fuzzy subgroups. Since then various notions in classical topology and group theory have been extended to fuzzy topological spaces and fuzzy group theory respectively. K.T. Atanassov [1, 2] generalised fuzzy sets to intuitionistic fuzzy sets. On the other hand, D. Coker [5] has introduced the notions of intuitionistic fuzzy topological spaces. R. Biswas [3] introduced the concept of intuitionistic fuzzy subgroup and some other concepts. The concepts of quasi-coincidence for intuitionistic fuzzy point was introduced and developed by F. Gallego Lupianez [7].

In this paper, we will study the intuitionistic fuzzy topological spaces by involving group structure on it and introduce the notion of group intuitionistic fuzzy topological spaces. It will be shown that group intuitionistic fuzzy topological spaces are different from intuitionistic fuzzy topological group, introduced by K. Hur, Y. B. Jun and J. H. Ryou in [10]. The cases when the two structures are same, will also be highlighted.

2. Preliminaries

In this section, we list some basic concepts and well known results on intuitionistic fuzzy topology and intuitionistic fuzzy groups for the sake of completeness of the topic under study.

Definition (2.1)[1] Let X be a non-empty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

Definition (2.2) [1] Let A and B be IFS's of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$;
(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(iii) $A^c = \{ \langle x, \nu_A(x), \mu_A(x)\rangle / x \in X \}$;
(iv) $A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle / x \in X \}$;
(v) $A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle / x \in X \}$.

Definition (2.3)[5] Let X be a non-empty, then a subfamily $\delta \subseteq$ IFS(X) is said to be an intuitionistic fuzzy topology (IFT) on X, if it satisfies the following:

(i) $\underset{\sim}{0}, \underset{\sim}{1} \in \delta$;
(ii) If $\{ A_i : i \in \Lambda\} \subseteq \delta$, then $\bigcup_{i \in \Lambda} A_i \in \delta$;
(iii) If $A, B \in \delta$ be any two members, then $A \cap B \in \delta$.

If δ is an intuitionistic fuzzy topology on X, then the pair (X, δ) is called intuitionistic fuzzy topological space (IFTS). The members of δ are called δ-open sets. An IFS A of X is said to be δ-closed in (X,δ) if and only if A^c is δ-open set in (X,δ).

Remark (2.4)(i)The members $0_\sim$ and $1_\sim$ are constant intuitionistic fuzzy sets on X defined by

$\underline{0}(x) = (0, 1);\ \forall x \in X$ and $\underline{1}(x) = (1,0);\ \forall x \in X$. (ii) For the sake of simplicity, we denote the IFS $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ by $A= \langle \mu_A, \nu_A\rangle$.

Definition (2.5) [14] Let X and Y be two non-empty sets and $f: X \to Y$ be a mapping. Let A and B be IFSs of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined as

$$\mu_{f(A)}(y) = \begin{cases} Sup\{\ \mu_A(x) : x \in f^{-1}(y)\} \\ 0 \ ; \quad otherwise \end{cases} \text{ and } \nu_{f(A)}(y) = \begin{cases} Inf\{\ \nu_A(x) : x \in f^{-1}(y)\} \\ 1 \ ; \quad otherwise \end{cases}.$$

Also the pre-image of B under f is denoted by $f^{-1}(B)$ and is defined as

$$f^{?}(B)(x) = B(f(x))\ ;\ \forall x \in X$$

Remark (2.6) Note that $\mu_A(x) \le \mu_{f(A)}(f(x))$ and $\nu_A(x) \ge \nu_{f(A)}(f(x))\ ;\ \forall x \in X$, and equality hold when f is bijective.

Definition (2.7)[5] Let (X_1,δ_1) and (Y_2, δ_2) be IFTSs. Then the function $f: (X_1, \delta_1) \to (Y_2, \delta_2)$ is called

(i) *intuitionistic fuzzy continuous:* if and only if for every $B \in \delta_2 \Rightarrow f^{-1}(B) \in \delta_1$.

(ii) *intuitionistic fuzzy open:* if and only if for every $A \in \delta_1 \Rightarrow f(A) \in \delta_2$.

(iii) *intuitionistic fuzzy closed:* if and only if image of every δ_1-closed set is δ_2-closed set.

(iv) *intuitionistic fuzzy homeomorphism:* if and only if f is bijective, intuitionistic fuzzy continuous and intuitionistic fuzzy open (or intuitionistic fuzzy closed).

Definition (2.8)[10] Let A be a IFS in X and δ be an IFT on X. Then the induced intuitionistic fuzzy topology on A is the family of subsets of A which are the intersections with A of δ-open sets in X. The induced intuitionistic fuzzy topology is denoted by δ_A i.e. $\delta_A = \{A \cap U : \forall U \in \delta\}$, and the pair (A, δ_A) is called an intuitionistic fuzzy topological subspace of (X, δ).

Definition (2.9)[11] For any p, q $\in [0,1]$ and $x \in X$, a fuzzy set $x_{(p,q)}$ in X is called an intuitionistic fuzzy point (IFP) in X if

$$x_{(p,q)}(y) = \begin{cases} (p,q) & ;\ if\ y = x \\ (0,1) & otherwise \end{cases},\ \forall y \in X.$$

The intuitionistic fuzzy point $x_{(p,q)}$ is said to be contained in an intuitionistic fuzzy set A, denoted by $x_{(p,q)} \in A$, if and only if $\mu_A(x) \ge p$ and $\nu_A(x) \le q$.

In particular, if $x_{(p,q)} \subseteq y_{(r,s)} \Leftrightarrow x = y$ and $p \le r$, $q \ge s$.

The intuitionistic fuzzy characteristic mapping of a subset A of a set X is denoted by χ_A is defined as

$$\chi_A(x) = \begin{cases} (1,0) & if\ x \in A \\ (0,1) & otherwise \end{cases},\ \forall x \in X.$$

Obviously, a intuitionistic characteristic function χ_A is also an intuitionistic fuzzy set on X and for any non-empty subsets A and B of a set X, we have $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.

Proposition (2.10) Let G be a group. Then the subfamily T of IFSs $A = \langle \mu_A, \nu_A\rangle$ on G such that $\mu_A(x) = \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A(x^{-1})\ ;\ \forall x \in G$. Then T is an IFT on G.

Proof. Straightforward.

Remark (2.11) In the definition (2.3) if condition (i) is replaced by (i)′ ∀ p, q $\in [0,1]$ such that $p + q \le 1$ and $x \in X$, an intuitionistic fuzzy point $x_{(p,q)}$ in X, defined by

$$x_{(p,q)}(y) = \begin{cases} (p,q) & ;\ if\ y = x \\ (0,1) & ;otherwise \end{cases},\ \forall y \in X,$$

(called constant intuitionistic fuzzy set in X), are in δ, then δ is called fully stratified intuitionistic fuzzy topology and the pair (X, δ) is called fully stratified intuitionistic fuzzy topological space (see [12]).

Definition (2.12)[3, 13] An IFS $A= \langle \mu_A, \nu_A\rangle$ of a group G is said to be intuitionistic fuzzy subgroup (IFSG) of G if

(i) $\mu_A(xy) \ge \min\{\mu_A(x), \mu_A(y)\}$ (ii) $\mu_A(x^{-1}) = \mu_A(x)$;

(iii) $\nu_A(xy) \le \max\{\nu_A(x), \nu_A(y)\}$ (iv) $\nu_A(x^{-1}) = \nu_A(x)$; $\forall x, y \in G$

Or equivalently $\mu_A(xy^{-1}) \ge \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \le \max\{\nu_A(x), \nu_A(y)\}$.

Then the following results are easy to verify

Result (2.13)[9] (i) If H is a subset of a group G, then χ_H is a IFSG of G if and only if H is a subgroup of G.

(ii) All constant intuitionistic fuzzy sets of a group G are IFSGs of G.

(iii) If A is an IFSG of a group G such that $x_{(p,q)} \in A$ then $\left(x^{-1}\right)_{(p,q)} \in$ A.

Definition (2.14)[12] Let G be a group and δ be an IFT on G. Let A, B ∈δ. We define $AB = \langle \mu_{AB}, \nu_{AB} \rangle$ and $A^{-1} = \langle \mu_{A^{-1}}, \nu_{A^{-1}} \rangle$ by the respective formula:

$$\mu_{AB}(x) = \underset{x=ab}{Sup}\{\mu_A(a), \mu_B(b)\} \quad \text{and} \quad \nu_{AB}(x) = \underset{x=ab}{Inf}\{\nu_A(a), \nu_B(b)\}.$$

$$\mu_{A^{-1}}(x) = \mu_A(x^{-1}) \text{ and } \nu_{A^{-1}}(x) = \nu_A(x^{-1}); \text{ for } x \in G.$$

Definition (2.15)[12] Let G be a group and let (G, δ) be a fully stratified IFTS. Then (G, δ) is an intuitionistic fuzzy topological group if it satisfies the following conditions:
(i) The mapping f: (G, δ) × (G, δ) → (G, δ) defined by $f(x, y) = xy$ is intuitionistic fuzzy continuous;
(ii) The mapping g : (G, δ) → (G, δ) defined by $g(x) = x^{-1}$ is intuitionistic fuzzy continuous.

Example (2.16)
(i) Let G be a group and δ be the collection of all constant intuitionistic fuzzy sets in G. Then (G,δ) is a intuitionistic fuzzy topological group.
(ii) Let G be a group and δ = IFS(G) be a discrete intuitionistic fuzzy topology on G. Then (G, δ) is a intuitionistic fuzzy topological group.

3. Group Intuitionistic Fuzzy Topology

In this section, we will introduce group intuitionistic fuzzy topological spaces and give basic properties of this structure. We also discuss similarities with and difference from other intuitionistic fuzzy topological spaces.

Definition (3.1) Let G be a group. Then the collection $T = \left\{A : A \in IFS(G) \ s.t. \ ?x_{(p,q)} \in A \Rightarrow \left(x^{-1}\right)_{(p,q)} \in A\right\}$ is an intuitionistic fuzzy topology on G and G with this intuitionistic fuzzy topology is called **group intuitionistic fuzzy topological space** (GIFTS). It is denoted by (G,T).

Proof. Since $0_{\sim}$ and $1_{\sim}$ are constant intuitionistic fuzzy sets and are therefore IFSGs on G. Therefore, $x_{(p,q)} \in \underset{\sim}{0} \Rightarrow \left(x^{-1}\right)_{(p,q)} \in \underset{\sim}{0}$ and $x_{(p,q)} \in \underset{\sim}{1} \Rightarrow \left(x^{-1}\right)_{(p,q)} \in \underset{\sim}{1}$ holds, i.e., $\underset{\sim}{0} \in$ T and $\underset{\sim}{1} \in$ T.

Next, let A , B ∈ T be any two members.

$$\begin{aligned}
\text{Let } x_{(p,q)} \in A \cap B \quad &\Rightarrow \quad \mu_{A\cap B}(x) \ge p \quad \text{and} \quad \nu_{A\cap B}(x) \le q \\
&\Rightarrow \min\{\mu_A(x), \mu_B(x)\} \ge p \quad \text{and} \quad \max\{\nu_A(x), \nu_B(x)\} \le q \\
&\Rightarrow \mu_A(x) \ge p \text{ and } \mu_B(x) \ge p \text{ and } \nu_A(x) \le q \text{ and } \nu_B(x) \le q \\
&\Rightarrow \mu_A(x) \ge p \text{ and } \nu_A(x) \le q \text{ and } \mu_B(x) \ge p \text{ and } \nu_B(x) \le q \\
&\Rightarrow x_{(p,q)} \in A \quad \text{and} \quad x_{(p,q)} \in B \\
&\Rightarrow \left(x^{-1}\right)_{(p,q)} \in A \text{ and } \left(x^{-1}\right)_{(p,q)} \in B \\
&\Rightarrow \left(x^{-1}\right)_{(p,q)} \in A \cap B
\end{aligned}$$

Thus, $A \cap B \in$ T.

Further, let { A_i : i ∈ Λ } ⊆ T . Let $x_{(p,q)} \in \bigcup_{i\in\Lambda} A_i$ implies $\mu_{\bigcup_{i\in\Lambda} A_i}(x) \ge p$ and $\nu_{\bigcup_{i\in\Lambda} A_i}(x) \le q$

$\text{Sup}\{\ \mu_{A_i}(x) : i \in \Lambda\ \} \ge p$ and $\text{Inf}\{\ \nu_{A_i}(x) : i \in \Lambda\ \} \le q.$

As both the sets $\{\ \mu_{A_i}(x) : i \in \Lambda\ \}$ and $\{\ \nu_{A_i}(x) : i \in \Lambda\ \}$ are subset of a bounded set [0,1].

Therefore there exist some j and k in Λ such that $\mu_{A_j}(x) \ge p$ and $\nu_{A_k}(x) \le q$.

Now, $\mu_{A_j}(x) \geq p \Rightarrow \nu_{A_j}(x) \leq 1-p$ and $\nu_{A_k}(x) \leq q \Rightarrow \mu_{A_k}(x) \geq 1-q$ and so $x_{(p,1-p)} \in A_j$ and so $(x^{-1})_{(p,1-p)} \in A_j \subseteq \bigcup_{i\in\Lambda} A_i$. Similarly, $(x^{-1})_{(1-q,q)} \in A_k \subseteq \bigcup_{i\in\Lambda} A_i$.

i.e., $(x^{-1})_{(p,1-p)}$ and $(x^{-1})_{(1-q,q)} \in \bigcup_{i\in\Lambda} A_i$.

So, $\mu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \geq p$ and $\nu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \leq 1-p$ and also $\mu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \geq 1-q$ and $\nu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \leq q$

Therefore, we have $\mu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \geq p$ and $\nu_{\bigcup_{i\in\Lambda} A_i}(x^{-1}) \leq q$. i.e. $(x^{-1})_{(p,q)} \in \bigcup_{i\in\Lambda} A_i$.

Thus $x_{(p,q)} \in \bigcup_{i\in\Lambda} A_i \Rightarrow (x^{-1})_{(p,q)} \in \bigcup_{i\in\Lambda} A_i$. So, $\bigcup_{i\in\Lambda} A_i \in$ T.

Hence T is an intuitionistic fuzzy topology on G..

Remark (3.2)(i) An IFS A of a group intuitionistic fuzzy topological space (*G,* T) is called T- closed if $A^c \in$ T. One can easily verify that $0_\sim$ and $1_\sim$ are T-closed and if A, B are T-closed, then A ∪ B is also T-closed and if { A_i : i ∈ Λ } be an arbitrary family of T-closed sets, then $\bigcap_{i\in\Lambda} A_i$ is also T-closed set.

(i) When GIFT is clearly understood from the content and there is no confusion about it, we may denote the GIFTS (G,T) simply by the symbol G, which is used for the underlying set of elements of G.

It is easy to observe that the GIFT as defined in Definition (3.1) is finer than the IFT on the group G as defined in proposition (2.10).

Theorem (3.3) An IFS A= $\langle \mu_A, \nu_A \rangle$of the GIFTS G is T-closed if and only if A is T-open.

Proof. Let A be T-open set, then $x_{(p,q)} \in A \Rightarrow (x^{-1})_{(p,q)} \in A$ i.e. $\mu_A(x) \geq p$ and $\nu_A(x) \leq q$ $\Rightarrow \mu_A(x^{-1}) \geq p$ and $\nu_A(x^{-1}) \leq q$.

Let $x_{(p,q)} \in A^c \Rightarrow \nu_A(x) \geq p$ and $\mu_A(x) \leq q \Rightarrow \nu_A(x) \not< p$ and $\mu_A(x) \not> q$

$\Rightarrow x_{(q,p)} \notin A$, i.e., $(x^{-1})_{(q,p)} \notin A$, i.e., $\mu_A(x^{-1}) \not\geq q$ and $\nu_A(x^{-1}) \not\leq p$

$\Rightarrow \mu_A(x^{-1}) \leq q$ and $\nu_A(x^{-1}) \geq p$, i.e., $(x^{-1})_{(p,q)} \in A^c$.

Thus $x_{(p,q)} \in A^c \Rightarrow (x^{-1})_{(p,q)} \in A^c$. Hence A^c is T-open set, i.e., A is T-closed set.

Converse is also true.

Now, we give an example of discrete IFTS which is not a GIFTS.

Example (3.4) Let G = (Z , +) be the group of integers under addition. Let δ = IFS(G) be the discrete IFT on G. We show that (G, δ) is not a GIFTS.

Let N, the set of all natural numbers. Define an IFS A= $\langle \mu_A, \nu_A \rangle$ on G as follows:

$$\mu_A(x) = \begin{cases} p & if\ x \in N \\ 0 & if\ x \in Z \setminus N \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} q & if\ x \in N \\ 1 & if\ x \in Z \setminus N \end{cases}$$

, where p, q∈ (0,1] st. p + q <1.

Consider $x \in$N, then we see that $x_{(p,q)} \in$A but $(-x)_{(p,q)} \notin$A. Thus, A∉T, where T is the GIFT on G. Hence δ is not a GIFT on G.

Now, we show that under certain condition on G, a GIFTS is also a discrete IFTS.

Theorem (3.5) Let (G,T) be a GIFTS such that $x^{-1} = x$, ∀ $x \in$G. Then T is a discrete IFT on G. **Proof.** Let A∈IFS(G) and $x \in$G be any element. Then for any p, q∈[0,1] such that p + q ≤ 1, we have $x_{(p,q)} \in A \Rightarrow \mu_A(x) \geq$ p and $\nu_A(x) \leq$ q , i.e., $\mu_A(x^{-1}) \geq$ p and $\nu_A(x^{-1}) \leq$ q $\Rightarrow (x^{-1})_{(p,q)} \in$ A, i.e., A∈T. So IFS(G) ⊆ T. Hence T is a discrete IFT on G.

Remark (3.6)(i) If there exist atleast one element x in a group G such that $x^{-1} \neq x$, then the GIFTS on G is not a discrete IFTS.

(ii) If |G| ≥ 2, then the GIFTS(G,T) on G is not an indiscrete IFTS, i.e., there exist A∈IFS(G), which is different from $\varnothing_\sim$ and $G_\sim$ such that A∈T.

Theorem (3.7) Every intuitionistic fuzzy subgroup of a group G is T-open set in GIFTS.

Proof. Let A = $\langle \mu_A, \nu_A \rangle$ be any IFSG of the group G. To show that A ∈T.

Let $x_{(p,q)} \in A \Rightarrow \mu_A(x) \geq p$ and $\nu_A(x) \leq q$, i.e., $\mu_A(x^{-1}) \geq p$ and $\nu_A(x^{-1}) \leq q \Rightarrow (x^{-1})_{(p,q)} \in A$. So $A \in T$.

Remark (3.8) Converse of theorem (3.7) is not true i.e. T-open set need not be IFSG of G.

Example (3.9) Consider the group G = { e , a , b , ab }, where $a^2 = b^2 = e$ and $ab = ba$ be the Klein four group. Define an IFS A = $\langle \mu_A, \nu_A \rangle$ of G by

$$\mu_A(x) = \begin{cases} 1 & if\ x = e \\ 0.3 & if\ x = a \\ 0.4 & if\ x = b \\ 0.2 & if\ x = ab \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & if\ x = e \\ 0.4 & if\ x = a \\ 0.3 & if\ x = b \\ 0.5 & if\ x = ab \end{cases}$$

Clearly, A is not an IFSG of G. But A is T-open set, where T is a GIFT on G.

Now, the question arises, when a T-open set of G is an IFSG of G. In this direction, we first prove the following lemma

Lemma (3.10) Let A is a intuitionistic fuzzy subgroup of G, then for all $x_{(p,q)}$, $y_{(r,s)} \in A$ we have $(xy)_{(u,v)} \in A$, where $u = \min\{p, r\}$ and $v = \max\{q, s\}$

Proof. Let A = $\langle \mu_A, \nu_A \rangle$ be a IFSG of G and $x_{(p,q)}$, $y_{(r,s)} \in A$, then we have $\mu_A(x) \geq p$; $\nu_A(x) \leq q$ and $\mu_A(y) \geq r$; $\nu_A(y) \leq s$. Therefore

$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{p, r\} = u$ and $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \min\{q, s\} = v$

implies $\mu_A(xy) \geq u$ and $\nu_A(xy) \leq v$, i.e., $(xy)_{(u,v)} \in A$.

Theorem (3.11) Let A be a T-open set in GIFTS (G,T), then A is an IFSG of G if and only if for all $x_{(p,q)}$, $y_{(r,s)} \in A$ we have $(xy)_{(u,v)} \in A$, where $u = \min\{p, r\}$ and $v = \max\{q, s\}$

Proof. When $A \in T$ and is also an IFSG of G, then the result follows by lemma (3.10).

Conversely, let A = $\langle \mu_A, \nu_A \rangle$ be T-open set such that $\forall\ x_{(p,q)}$, $y_{(r,s)} \in A$ we have $(xy)_{(u,v)} \in A$, where $u = \min\{p, r\}$ and $v = \max\{q, s\}$

To show that A is an IFSG of G.

Let x , $y \in$ G be any element and suppose that $\mu_A(x) = p$, $\nu_A(x) = q$ and $\mu_B(y) = r$, $\nu_A(y) = s$

Case(i) when (p, q) ≠ (r, s) , then either p ≠ r or q ≠ s or both

Subcase (i) when p ≠ r and q = s, wlog let p < r and q = s, then min{p, r} = u.

Now, $\mu_A(x) = p$ and $\mu_B(y) = r > p \Rightarrow x_{(p,q)}$, $y_{(p,q)} \in A$. As $A \in T$, so $(y^{-1})_{(p,q)} \in A$.

Thus $x_{(p,q)}$, $(y^{-1})_{(p,q)} \in A \Rightarrow (xy^{-1})_{(p,q)} \in A$ i.e. $\mu_A(xy^{-1}) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \leq q = \max\{q, s\} = \max\{\nu_A(x), \nu_A(y)\}$.

Subcase (ii) when p = r and q ≠ s, wlog let q < s, then max{q, s}= s.

Now, $\nu_A(x) = q < s$ and $\nu_B(y) = s \Rightarrow x_{(p,s)}$, $y_{(p,s)} \in A$. As $A \in T$, so $(y^{-1})_{(p,s)} \in A$

Thus $x_{(p,s)}$, $(y^{-1})_{(p,s)} \in A \Rightarrow (xy^{-1})_{(p,s)} \in A$ i.e. $\mu_A(xy^{-1}) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \leq s = \max\{q, s\} = \max\{\nu_A(x), \nu_A(y)\}$

Subcase(iii) when p ≠ r and q ≠ s, wlog let p < r , q < s then min{p, r}= p and max{q, s} = s

Now, $\mu_A(x) = p$ and $\mu_B(y) = r > p$ and $\nu_A(x) = q < s$ and $\nu_B(y) = s$

implies $x_{(p,s)}$, $y_{(p,s)} \in A$. As $A \in T$, so $(y^{-1})_{(p,s)} \in A$

Thus $x_{(p,s)}$, $(y^{-1})_{(p,s)} \in A \Rightarrow (xy^{-1})_{(p,s)} \in A$ i.e. $\mu_A(xy^{-1}) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \leq s = \max\{q, s\} = \max\{\nu_A(x), \nu_A(y)\}$.

Case(ii) when (p, q) = (r, s) ⇒ p = r and q = s, then min{p, r}= p and max{q, s} = s

Thus $x_{(p,s)}$, $(y^{-1})_{(p,s)} \in A \Rightarrow (xy^{-1})_{(p,s)} \in A$ i.e. $\mu_A(xy^{-1}) \geq p = \min\{p, r\} = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy^{-1}) \leq s = \max\{q, s\} = \max\{\nu_A(x), \nu_A(y)\}$.

Thus in all the cases, we see that A is an IFSG of G.

Remark (3.12) The GIFTS on the group G is fully stratified intuitionistic fuzzy topological space by remark (2.11)(ii) and Theorem (3.7).

Lemma(3.13) Let (G,T) be an GFTS on the group G and A, $B \in T$ such that $x_{(p,q)} \in A$ and $y_{(r,s)} \in B$, then $(xy)_{(u,v)} \in AB$, where $u = \min\{p, r\}$ and $v = \max\{q, s\}$.

Proof. Since $x_{(p,q)} \in A$ and $y_{(r,s)} \in B \Rightarrow \mu_A(x) \geq p$, $\nu_A(x) \leq q$ and $\mu_B(y) \geq r$, $\nu_B(y) \leq s$.

$\Rightarrow \min\{\mu_A(x), \mu_B(y)\} \geq \min\{p, r\} = u$ and $\max\{\nu_A(x) , \nu_B(y)\} \leq \max\{q, s\} = v$.

Now, $\mu_{AB}(xy) = \text{Sup}\{\min\{\mu_A(x), \mu_B(y)\}\} \geq u \Rightarrow \mu_{AB}(xy) \geq u$ and
$\nu_{AB}(xy) = \text{Inf}\{\max\{\nu_A(x), \nu_B(y)\}\} \leq v \Rightarrow \nu_{AB}(xy) \leq v$.
Thus, $\mu_{AB}(xy) \geq u$ and $\nu_{AB}(xy) \leq v \Rightarrow (xy)_{(u,v)} \in AB$.

Theorem (3.14) If A and B are T-open sets in G, then AB is also T-open set provided G is an abelian group.

Proof. Let $x_{(p,q)} \in A$ and $y_{(r,s)} \in B$, then by lemma (3.13), we have
$(xy)_{(u,v)} \in AB$, where $u = \min\{p, r\}$ and $v = \max\{q, s\}$.
Also, because $A, B \in T$ $\therefore$ $(x^{-1})_{(p,q)} \in A$ and $(y^{-1})_{(r,s)} \in B$. Again by using lemma (3.13), we have
$(x^{-1}y^{-1})_{(u,v)} \in AB \Rightarrow ((yx)^{-1})_{(u,v)} \in AB$, i.e., $((xy)^{-1})_{(u,v)} \in AB$ [As G is abelian]
Thus, $(xy)_{(u,v)} \in AB \Rightarrow ((xy)^{-1})_{(u,v)} \in AB \Rightarrow AB \in T$.

Theorem (3.15) Let H be a subgroup of a GIFTS (G, T), then the relative IFT T_H on H is a GIFT on H.

Proof. The subset H of G has an intuitionistic fuzzy characteristic function χ_H defined as:

$$\chi_H(x) = \left\langle \mu_{\chi_H(x)},\ \nu_{\chi_H(x)} \right\rangle = \begin{cases} (1,0) & \text{if } x \in H \\ (0,1) & \text{if } x \notin H \end{cases}.$$

Clearly, $\chi_H(x) = (1, 0)$, $\forall x \in H$ is a constant intuitionistic fuzzy set on H.
Let $T_H = \{ A \cap \chi_H : \forall A \in T \}$. Then we claim that T_H is a GIFT on H.
For, choose $\underset{\sim}{0} \in T$, then as $(\underset{\sim}{0} \cap \chi_H)(x) = (\min\{0,1\}, \max\{1,1\}) = (0,1) = \underset{\sim}{0}(x)$, so we have $\underset{\sim}{0} \in T_H$. To Show that $H_{\sim} \in T_H$, choose χ_G the characteristic function for G, then $(\chi_G \cap \chi_H)(x) = \chi_H(x)$, so we have $H_{\sim} \in T_H$.
Further, let $\{A_i \cap \chi_H : i \in \Lambda\} \subseteq T_H$. Then, $\bigcup_{i\in\Lambda}(A_i \cap \chi_H) = \left(\bigcup_{i\in\Lambda} A_i\right) \cap \chi_H \in T_H$.
Also, let $A_1 \cap \chi_H$ and $A_2 \cap \chi_H$ be any two members of T_H. Then
$(A_1 \cap \chi_H) \cap (A_2 \cap \chi_H) = (A_1 \cap A_2) \cap \chi_H \in T_H$. Hence T_H is an IFT on H.
Further, let T_1 is a GIFT on H, then we show that $T_1 = T_H$.
Let $A \cap \chi_H \in T_H$, let $x_{(p,q)} \in (A \cap \chi_H) \Rightarrow \mu_{(A \cap \chi H)}(x) \geq p$ and $\nu_{(A \cap \chi H)}(x) \leq q$.
$\Rightarrow \min\{\mu_A(x), \mu\chi_H(x)\} \geq p$ and $\max\{\nu_A(x), \nu\chi_H(x)\} \leq q$.
As $x \in H$ therefore, $\mu\chi_H(x) = 1$ and $\nu\chi_H(x) = 0$, so $\mu_A(x) = \min\{\mu_A(x), \mu\chi_H(x)\} \geq p$
and $\nu_A(x) = \max\{\nu_A(x), \nu\chi_H(x)\} \leq q$ $\therefore x_{(p,q)} \in A$. As $A \in T \Rightarrow (x^{-1})_{(p,q)} \in A$.
Also, $\mu\chi_H(x^{-1}) = 1$ and $\nu\chi_H(x^{-1}) = 0$, as such $\min\{\mu_A(x^{-1}), \mu\chi_H(x^{-1})\} \geq p$ and $\max\{\nu_A(x^{-1}), \nu\chi_H(x^{-1})\} \leq q$ implies that $(x^{-1})_{(p,q)} \in (A \cap \chi_H)$.
Thus, $x_{(p,q)} \in (A \cap \chi_H) \Rightarrow (x^{-1})_{(p,q)} \in (A \cap \chi_H)$ and so, $A \cap \chi_H \in T_1$, i.e., $T_H \subseteq T_1$.
Further, let $B \in T_1$ and so if $x_{(r,s)} \in B \Rightarrow (x^{-1})_{(r,s)} \in B$.
Also, $B = B \cap \chi_H \in T_H$. Therefore, $T_1 \subseteq T_H$ and hence $T_1 = T_H$.
Hence relative intuitionistic fuzzy topology on H is the GIFT on H.

4. Functions on Group Intuitionistic Fuzzy Topological Spaces

Theorem (4.1) Let $f: G \to G'$ is a group isomorphism and T, T′ be GIFTs on groups G and G′ respectively. Then f is intuitionistic fuzzy homeomorphism between GIFTS (G,T) and (G, T′)

Proof. Let $f: G \to G'$ be an group isomorphism, where G and G′ are groups. We show that f is a intuitionistic fuzzy homeomorphism.
For intuitionistic fuzzy continuity: let B be any T′- open set and let $x_{(p,q)} \in f^{-1}(B)$. Then

$\mu_{f^{-1}(B)}(x) \geq p$ and $\nu_{f^{-1}(B)}(x) \leq q \Rightarrow \mu_B(f(x)) \geq p$ and $\nu_B(f(x)) \leq q$

i.e., $(f(x))_{(p,q)} \in B$. As $B \in T'$ we have $\left((f(x))^{-1}\right)_{(p,q)} \in B$, i.e., $(f(x^{-1}))_{(p,q)} \in B$

$\Rightarrow \mu_B(f(x^{-1})) \geq p$ and $\nu_B(f(x^{-1})) \leq q \Rightarrow \mu_{f^{-1}(B)}(x^{-1}) \geq p$ and $\nu_{f^{-1}(B)}(x^{-1}) \leq q$,

i.e., $(x^{-1})_{(p,q)} \in f^{-1}(B)$. Thus $x_{(p,q)} \in f^{-1}(B) \Rightarrow (x^{-1})_{(p,q)} \in f^{-1}(B)$. So, $f^{-1}(B) \in T$.

Hence f is intuitionistic fuzzy continuous.

For intuitionistic fuzzy open mapping: let $A \in T$ and $y_{(r,s)} \in f(A) \Rightarrow \mu_{f(A)}(y) \geq r$ and $\nu_{f(A)}(y) \leq s$
$\Rightarrow \mu_A(x) \geq r$ and $\nu_A(x) \leq s$

[$\because f$ is isomorphism $\therefore \left(\mu_{f(A)}(y), \nu_{f(A)}(y)\right) = \left(\mu_{f(A)}(f(x)), \nu_{f(A)}(f(x))\right) = \left(\mu_A(x),\ \nu_A(x)\right)$]

$\Rightarrow x_{(r,s)} \in A$. As A is T-open set so $(x^{-1})_{(r,s)} \in A$ i.e. $\mu_A(x^{-1}) \geq r$ and $\nu_A(x^{-1}) \leq s$

Again, as f is isomorphism, we have $f(x^{-1}) = y^{-1}$ and $\mu_A(x^{-1}) \geq r$ and $\nu_A(x^{-1}) \leq s$

Thus, $\mu_{f(A)}(y^{-1}) = \mu_{f(A)}(f(x^{-1})) = \mu_A(x^{-1}) \geq r$ and $\nu_{f(A)}(y^{-1}) = \nu_{f(A)}(f(x^{-1})) = \nu_A(x^{-1}) \leq s$

$\Rightarrow (y^{-1})_{(r,s)} \in f(A)$. So, f is an intuitionistic fuzzy open mapping.

Hence f is a intuitionistic fuzzy homeomorphism.

Theorem (4.2) Let f: (G,T) → (G′,T′) be a mapping. Then f is intuitionistic fuzzy continuous if and only if $f(x^{-1}) = (f(x))^{-1}$, $\forall\, x \in$ G.

Proof. Firstly, let $f(x^{-1}) = (f(x))^{-1}$, $\forall\, x \in$ G hold.

Let B be any T′- open set and let $x_{(p,q)} \in f^{-1}$(B). Then

$\mu_{f^{-1}(B)}(x) \geq p$ and $\nu_{f^{-1}(B)}(x) \leq q \Rightarrow \mu_B(f(x)) \geq p$ and $\nu_B(f(x)) \leq q$

i.e., $(f(x))_{(p,q)} \in B$. As B $\in T'$ we have $\left((f(x))^{-1}\right)_{(p,q)} \in B$, i.e., $(f(x^{-1}))_{(p,q)} \in B$

$\Rightarrow \mu_B(f(x^{-1})) \geq p$ and $\nu_B(f(x^{-1})) \leq q \Rightarrow \mu_{f^{-1}(B)}(x^{-1}) \geq p$ and $\nu_{f^{-1}(B)}(x^{-1}) \leq q$,

i.e., $(x^{-1})_{(p,q)} \in f^{-1}(B)$. *Thus* $x_{(p,q)} \in f^{-1}(B) \Rightarrow (x^{-1})_{(p,q)} \in f^{-1}(B)$. So, $f^{-1}(B) \in T$.

Hence f is intuitionistic fuzzy continuous.

Conversely, let f: (G,T) → (G′,T′) be intuitionistic fuzzy continuous mapping.

To show $f(x^{-1}) = (f(x))^{-1}$, $\forall\, x \in$ G.

Let $x \in$ G $\Rightarrow f(x) \in$ G′. Suppose that B be an T′- open set such that $\{f(x)\}_{(r,s)} \in$ B

$\mu_B(f(x)) \geq r$ and $\nu_B(f(x)) \leq s$, i.e., $\mu_{f^{-1}(B)}(x) \geq r$ and $\nu_{f^{-1}(B)}(x) \leq s$, *i.e.*, $x_{(r,s)} \in f^{-1}(B)$.

As the map f is intuitionistic fuzzy continuous $\therefore f^{-1}(B) \in T$, so $\left(x^{-1}\right)_{(r,s)} \in f^{-1}(B)$

$\Rightarrow \mu_{f^{-1}(B)}(x^{-1}) \geq r$ and $\nu_{f^{-1}(B)}(x^{-1}) \leq s$, i.e., $\mu_B(f(x^{-1})) \geq r$ and $\nu_B(f(x^{-1})) \leq s$

So, $(f(x^{-1}))_{(r,s)} \in B$. Thus, we see that if $(f(x))_{(r,s)} \in B \Rightarrow (f(x^{-1}))_{(r,s)} \in B$.

Let H = { f (x), { f (x)}$^{-1}$}. Clearly, H $\subseteq$ G′. Define the IFS C on G′ as follow:

$$\mu_C(y) = \begin{cases} r & \text{if } y \in H \\ 0 & \text{if } y \notin H \end{cases} \text{ and } \nu_C(y) = \begin{cases} s & \text{if } y \in H \\ 0 & \text{if } y \notin H \end{cases} \text{; where } r, s \in (0,1] \text{ such that } r + s \leq 1.$$

Obviously, C $\in$ T′. As proved earlier, $\{f(x)\}_{(r,s)} \in$ C $\Rightarrow \{f(x^{-1})\}_{(r,s)} \in$ C

This means that $f(x^{-1}) = \{f(x)\}^{-1}$. Hence proved.

Theorem (4.3) Group intuitionistic fuzzy topological space is an intuitionistic fuzzy topological property.

Proof. Let (G,T) be a GIFTS and f:(G,T)→ (X,T′) be a intuitionistic fuzzy homeomorphism. We will show that (X, T′) is also a GIFTS. To prove this, we first show that X is a group under the operation:

$y_1y_2 = f(x_1)f(x_2) = f(x_1x_2)$ i.e. $y_1y_2 = f(f^{-1}(y_1)f^{-1}(y_2)) = f(x_1x_2)$; $\forall\ y_1, y_2 \in X$

Closure property: let $y_1, y_2 \in X$, then $y_1y_2 = f(f^{-1}(y_1)f^{-1}(y_2)) = f(x_1x_2) \in X$.

Associativity: let $y_1, y_2, y_3 \in X$, then

$y_1(y_2y_3) = y_1(f(x_2x_3)) = f(x_1)(f(x_2x_3)) = f(x_1(x_2x_3)) = f((x_1x_2)x_3))$
$= f((x_1x_2)x_3) = f(x_1x_2)f(x_3) = (f(x_1)f(x_2))(f(x_3))$
$= (y_1y_2)y_3$.

Existence of identity element: let e be the identity element of G.

We show that f(e) is the identity element of X.

Let y$\in$X be any element. Then $y f(e) = f(f^{-1}(y) f^{-1}(f(e)) = f(f^{-1}(y)\, e) = f(f^{-1}(y)) = y$

Similarly, we can show that $f(e)y = y$.

Existence of inverse: let y$\in$X. Then $\exists\, x \in$ G such that y = $f(x)$. We show that $y^{-1} = f(x^{-1})$ is the inverse of y. Now,
$y\,y^{-1} = f[f^{-1}(y) f^{-1}(y^{-1})] = f[f^{-1}(f(x)) f^{-1}(f(x^{-1}))] = f(xx^{-1}) = f(e)$

Similarly, we can show that $y^{-1}y = f(e)$. Hence X is a group.

Next, we show that the topology T′ is the GIFT on X.
Let T″ be the GIFT on X. Then we show that T′ = T″.

Let y$\in$X be any element then $\exists$ a unique $x \in$ G such that $f(x)$ = y

Let B $\in T'$. As f is intuitionistic fuzzy continuous $\therefore$ $f^{-1}(\mathrm{B}) \in$ T. As T is a GIFT on G

Therefore $x_{(p,q)} \in f^{-1}(\mathrm{B}) \;\Rightarrow\; (x^{-1})_{(p,q)} \in f^{-1}(\mathrm{B})$

i.e. $\mu_{f^{-1}(\mathrm{B})}(x) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x) \leq q \;\Rightarrow\; \mu_{f^{-1}(\mathrm{B})}(x^{-1}) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x^{-1}) \leq q$

i.e. $\mu_B(f(x)) \geq p$ and $\nu_B(f(x)) \leq q \;\Rightarrow\; \mu_B(f(x^{-1})) \geq p$ and $\nu_B(f(x^{-1})) \leq q$

i.e. $\mu_B(\mathrm{y}) \geq p$ and $\nu_B(\mathrm{y}) \leq q \;\Rightarrow\; \mu_B(y^{-1}) \geq p$ and $\nu_B(y^{-1}) \leq q$

i.e. $\mathrm{y}_{(p,q)} \in B \;\Rightarrow\; (\mathrm{y}^{-1})_{(p,q)} \in B$. So B $\in$ T″ [$\because$ T″ is a GIFT on X]

Thus B $\in T'$ $\Rightarrow$ B $\in$ T″ $\therefore$ T′ $\subseteq$ T″.

Conversely, let B $\in$ T″. Then for any element y $\in$ X, $\exists$ unique $x \in G$ such that $f(x)$ = y

Now, $\mathrm{y}_{(p,q)} \in B \Rightarrow (\mathrm{y}^{-1})_{(p,q)} \in B$ i.e. $\mu_B(y) \geq p$ and $\nu_B(y) \leq q \;\Rightarrow\; \mu_B(y^{-1}) \geq p$ and $\nu_B(y^{-1}) \leq q$

i.e. $\mu_B(f(x)) \geq p$ and $\nu_B(f(x)) \leq q \;\Rightarrow\; \mu_B(f(x^{-1})) \geq p$ and $\nu_B(f(x^{-1})) \leq q$

i.e. $\mu_{f^{-1}(\mathrm{B})}(x) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x) \leq q \;\Rightarrow\; \mu_{f^{-1}(\mathrm{B})}(x^{-1}) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x^{-1}) \leq q$

i.e. $x_{(p,q)} \in f^{-1}(\mathrm{B}) \;\Rightarrow\; (x^{-1})_{(p,q)} \in f^{-1}(\mathrm{B})$ i.e. $f^{-1}(\mathrm{B}) \in T$.

As f is open mapping. Therefore $f(f^{-1}(\mathrm{B})) \in T'$ i.e. B $\in$ T′

Thus B $\in$ T″ $\Rightarrow$ B $\in$ T′ so T″ $\subseteq$ T′. Hence T′ = T″.

Corollary (4.4) Let f : (G,T) → (G′,T′) be a intuitionistic fuzzy continuous mapping and K ={ $x \in$ G : $f(x)$ = e′}. Then the topology induced on the set K is a GIFT, where e′ is the identity element of G′.

Proof. Let T_K = { A $\cap$ χ_K : $\forall$ A$\in$T }. Then we claim that T_K is a GIFT on K.

Let $x \in$ K be any element, then $f(x)$ = e′. Now $f(x^{-1}) = \{f(x)\}^{-1} = \{e'\}^{-1} = e' \Rightarrow x^{-1} \in$ K.

Let $x_{(p,q)} \in (A \cap \chi_K) \Rightarrow \mu_{(A\cap\chi_K)}(x) \geq p$ and $\nu_{(A\cap\chi_K)}(x) \leq q$.

$\Rightarrow$ min{ $\mu_A(x)$, $\mu_{\chi K}(x)$} $\geq$ p and max {$\nu_A(x)$, $\nu_{\chi K}(x)$}$\leq$ q

$\Rightarrow \mu_A(x) \geq p$ and $\nu_A(x) \leq q$ [As $\mu_{\chi K}(x)$ =1 and $\nu_{\chi K}(x)$ = 0].

So $x_{(p,q)} \in$A. As A $\in$T $\Rightarrow (x^{-1})_{(p,q)} \in$A, i.e., $\mu_A(x^{-1}) \geq p$ and $\nu_A(x^{-1}) \leq q$, also $\mu_{\chi K}(x)$ =1 and $\nu_{\chi K}(x)$ = 0. Therefore min{$\mu_A(x^{-1})$, $\mu_{\chi K}(x^{-1})$} $\geq$ p and max{$\nu_A(x^{-1})$, $\nu_{\chi K}(x)$}$\leq$ q

i.e., $\mu_{(A\cap\chi K)}(x^{-1}) \geq p$ and $\nu_{(A\cap\chi K)}(x^{-1}) \leq q$ i.e. $(x^{-1})_{(p,q)} \in (A \cap \chi_K)$.

Thus $x_{(p,q)} \in (A \cap \chi_K) \Rightarrow (x^{-1})_{(p,q)} \in (A \cap \chi_K)$. Hence T_K is a GIFT on K.

Theorem (4.5) Let f : (G,T) → (G,T) be a mapping from a GIFTS (G,T) onto itself defined by $f(x) = x^{-1}$; $\forall x \in$G. Then f is bijective and intuitionistic fuzzy continuous.

Proof. Clearly, f is one-one and onto. Let B$\in$T be any open set. Then we show that f^{-1}(B)$\in$T.

Let $x_{(p,q)} \in f^{-1}(B) \;\Rightarrow\; \mu_{f^{-1}(\mathrm{B})}(x) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x) \leq q$

$\Rightarrow \mu_{\mathrm{B}}(f(x)) \geq p$ and $\nu_{\mathrm{B}}(f(x)) \leq q \;\Rightarrow\; \mu_{\mathrm{B}}\left(\{f(x)\}^{-1}\right) \geq p$ and $\nu_{\mathrm{B}}\left(\{f(x)\}^{-1}\right) \leq q$

i.e., $\mu_{\mathrm{B}}(f(x^{-1})) \geq p$ and $\nu_{\mathrm{B}}(f(x^{-1})) \leq q \;\Rightarrow\; \mu_{f^{-1}(\mathrm{B})}(x^{-1}) \geq p$ and $\nu_{f^{-1}(\mathrm{B})}(x^{-1}) \leq q$,

i.e., $\left(x^{-1}\right)_{(p,q)} \in f^{-1}(B)$. *Thus,* $x_{(p,q)} \in f^{-1}(B) \;\Rightarrow\; \left(x^{-1}\right)_{(p,q)} \in f^{-1}(B)$.

So, $f^{-1}(B) \in T$. Hence f is intuitionistic fuzzy continuous.

5. Product of Group Intuitionistic Fuzzy Topological Spaces

In this section, we study the product of group intuitionistic fuzzy topological spaces. We also discuss similarities with difference from product of intuitionistic fuzzy topological spaces and also from the intuitionistic fuzzy topological groups.

Theorem (5.1) Let (G_1, T_1) and (G_2, T_2) be two GIFTS. Then product intuitionistic fuzzy topology on $G_1 \times G_2$ is contained in the GIFT on $G_1 \times G_2$. The equality between the two topologies need not hold.

Proof. We know that $G_1 \times G_2$ is a group under the point wise operation defined by

$$(x_1 , y_1)\,(x_2 , y_2) = (x_1x_2 , y_1\, y_2) \quad ; \;\forall\, (x_1 , y_1) , (x_2 , y_2) \in G_1 \times G_2$$

The element $(e_1,e_2)$is the identity of $G_1 \times G_2$, where e_1, e_2 be the identities of G_1,G_2 respectively Let (x, y) be any element of $G_1 \times G_2$. Then (x^{-1}, y^{-1}) is the inverse of (x, y) in $G_1 \times G_2$. Let T′ be the product IFT on $G_1 \times G_2$ and T be the GIFT on $G_1 \times G_2$. We show that T′⊆ T. Let A ∈ T′ and let z = (x, y) be any element of $G_1 \times G_2$. Then

$z_{(p,q)} \in A = B\times C,$ where $B \in T_1$, $C \in T_2$ $\Rightarrow$ $\mu_{B\times C}(x,y) \geq p$ and $\nu_{B\times C}(x,y) \leq q$

$\Rightarrow$ min $\{ \mu_B(x),\ \mu_C(y)\} \geq p$ and max$\{ \nu_B(x),\ \nu_C(y)\} \leq q$

$\Rightarrow \mu_B(x) \geq p$, $\mu_C(y) \geq p$ and $\nu_B(x) \leq q$, $\nu_C(y) \leq q$

$\Rightarrow \mu_B(x) \geq p$, $\nu_B(x) \leq q$ and $\mu_C(y) \geq p$, $\nu_C(y) \leq q$

$\Rightarrow x_{(p,q)} \in B$ and $y_{(p,q)} \in C$ $\Rightarrow$ $(x^{-1})_{(p,q)} \in B$ and $(y^{-1})_{(p,q)} \in C$ [$\because$ $B \in T_1$, $C \in T_2$]

$\Rightarrow \mu_B(x^{-1}) \geq p$, $\nu_B(x^{-1}) \leq q$ and $\mu_C(y^{-1}) \geq p$, $\nu_C(y^{-1}) \leq q$

$\Rightarrow \mu_B(x^{-1}) \geq p$, $\mu_C(y^{-1}) \geq p$ and $\nu_B(x^{-1}) \leq q$, $\nu_C(y^{-1}) \leq q$

$\Rightarrow$ $\Rightarrow$ min $\{ \mu_B(x^{-1}),\ \mu_C(y^{-1})\} \geq p$ and max$\{ \nu_B(x^{-1}),\ \nu_C(y^{-1})\} \leq q$

$\Rightarrow$ $\mu_{B\times C}(x^{-1},y^{-1}) \geq p$ and $\nu_{B\times C}(x^{-1},y^{-1}) \leq q$, i.e., $\mu_{B\times C}(z^{-1}) \geq p$ and $\nu_{B\times C}(z^{-1}) \leq q$

$\Rightarrow \mu_A(z^{-1}) \geq p$ and $\nu_A(z^{-1}) \leq q$, i.e., $(z^{-1})_{(p,q)} \in A$

Thus $z_{(p,q)} \in A \Rightarrow (z^{-1})_{(p,q)} \in A$. So, A∈ T. Therefore, T′ ⊆ T.

Next, we show that the equality between T and T′ need not hold.
Let G_1 = (Z , +) and G_2 = (2Z, +). Let T_1, T_2 be the GIFT on G_1 and G_2 respectively. Again suppose that T′ be the product IFT on $G_1\times G_2$ and T be the GIFT on $G_1 \times G_2$.
Consider the subset H ={(2, 4) , (-2, -4)} of $G_1 \times G_2$. Define the IFS A on $G_1\times G_2$ such that

$$\mu_A(x,y) = \begin{cases} p & ;\ if\ (x,y) \in \mathrm{H} \\ 0 & ;\ otherwise \end{cases} \quad \text{and} \quad \nu_A(x,y) = \begin{cases} q & ;\ if\ (x,y) \in \mathrm{H} \\ 1 & ;\ otherwise \end{cases}$$

But the smallest T′-open set containing A is B × C, where B ∈T_1 and C∈T_2 defined as:

$$\mu_B(x) = \begin{cases} p & ;\ if\ x = 2\ ,\ \text{-}2 \\ 0 & ;\ otherwise \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} q & ;\ if\ x = 2\ ,\ \text{-}2 \\ 1 & ;\ otherwise \end{cases}$$

$$\mu_C(x) = \begin{cases} p & ;\ if\ x = 4\ ,\ \text{-}4 \\ 0 & ;\ otherwise \end{cases} \quad \text{and} \quad \nu_C(x) = \begin{cases} q & ;\ if\ x = 4\ ,\ \text{-}4 \\ 1 & ;\ otherwise \end{cases}$$

. Then

$$\left(B\times C\right)(x,y) = \left(\mu_{B\times C}(x,y),\ \nu_{B\times C}(x,y)\right) = \left(\min\{\mu_B(x),\mu_C(y)\},\ \max\{\nu_B(x),\nu_C(y)\}\right)$$

$$= \begin{cases} (p,q) & if\ (x,y) \in \{(2,4),(2,-4),(-2,4),(-2,-4)\} \\ (0,1) & otherwise \end{cases}$$

Thus, A ∉ T′. Hence, T′ ≠ T.

Remark (5.2) If in the above theorem (5.1), we have $x^{-1}= x$ and $y^{-1} = y$ for all $x\in G_1$ and for all $y\in G_2$, then the product IFT on $G_1 \times G_2$ is same as the GIFT on $G_1 \times G_2$.

Next, we show that GIFTS (G, T) is not a intuitionistic fuzzy topological group as defined in (2.13)

Example (5.3) Consider G ={1, w ,w^2}, where w is non-real cube root of unity. Then (G, *) is a group. Consider the IFSs A and B on G as follows:
A ={<1, 0.9, 0.1> , <w, 0.6, 0.3> , <w^2, 0.6, 0.3>} and B = {<1, 0.8, 0.1> , <w, 0.3, 0.5> , <w^2, 0.3, 0.5>}.
Clearly B ⊆ A.
Consider T ={$\varnothing_\sim$, $G_\sim$, A , B}. It is easy to verify that T is a GIFT on G.
Define the mapping f: G → G by $f(a) = a^{-1}$; ∀ a∈G and
g : G × G → G by $g(a, b) = ab$; ∀ a, b∈G.
Clearly, the map f is intuitionistic fuzzy continuous by Theorem (4.5). Further, let A ∈T, then

$$\{g^{-1}(A)\}(a,\ b) = \left(\mu_{g^{-1}(A)}(a,\ b),\ \nu_{g^{-1}(A)}(a,\ b)\right) = \left(\mu_A\{g(a,\ b)\},\ \nu_A\{g(a,\ b)\}\right) = \left(\mu_A(ab),\ \nu_A(ab)\right),\ \forall (a,\ b) \in G\times G.$$

i.e., g^{-1}(A) = { < (1,1), 0.9, 0.1>, < (1,w), 0.6, 0.3>, < (w,1), 0.6, 0.3>, < (1, w^2), 0.6, 0.3>, < (w^2,1), 0.6, 0.3>, < (w, w^2), 0.6, 0.3>, < (w^2, w), 0.6, 0.3>, < (w, w), 0.6, 0.3>, < (w^2, w^2), 0.6, 0.3>}. Now, we show that g^{-1}(A) is not open set in product fuzzy topological space (G,T)×(G,T), for then $g^{-1}(A) = C\times D$, for some C, D∈T, which is not possible in this case.

Now, the question arises, when a GIFTS be a fuzzy topological group. In this direction we have the following result.

Theorem (5.4) Let G be a group such that $x^{-1}= x$, for all $x\in$ G. Then the GIFTS (G,T) is same as the intuitionistic fuzzy topological group.

Proof. It follows from Theorems (3.5) and Example (2.16)(ii)

6. Conclusion

In this paper, the notion of group intuitionistic fuzzy topological spaces is introduced. It has been observed that the group intuitionistic fuzzy topology is different from discrete and indiscrete intuitionistic fuzzy topology, the cases when they behave same have been examined. It is seen that group intuitionistic fuzzy topology is a hereditary property subject to the subgroup of the group G. Also it is noticed that group intuitionistic fuzzy topology is a topological and productive property. Moreover, it has been seen that the notion of group intuitionistic fuzzy topology is different from that of intuitionistic fuzzy topological group as introduced by Hur, Jun and Ryou. The cases when the two are same have also been established.

7. Scope of further study

Many properties like intuitionistic fuzzy connectedness, intuitionistic fuzzy compactness, intuitionistic fuzzy separation axioms, convergence of sequence in group intuitionistic fuzzy topological space (G,T) are yet to be examined. This work is under progress.

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