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Analysis of Queuing System Consisting of Multiple Parallel Channels in Series Connected to Non-Serial Servers with Finite Waiting Space and Balking, Reneging

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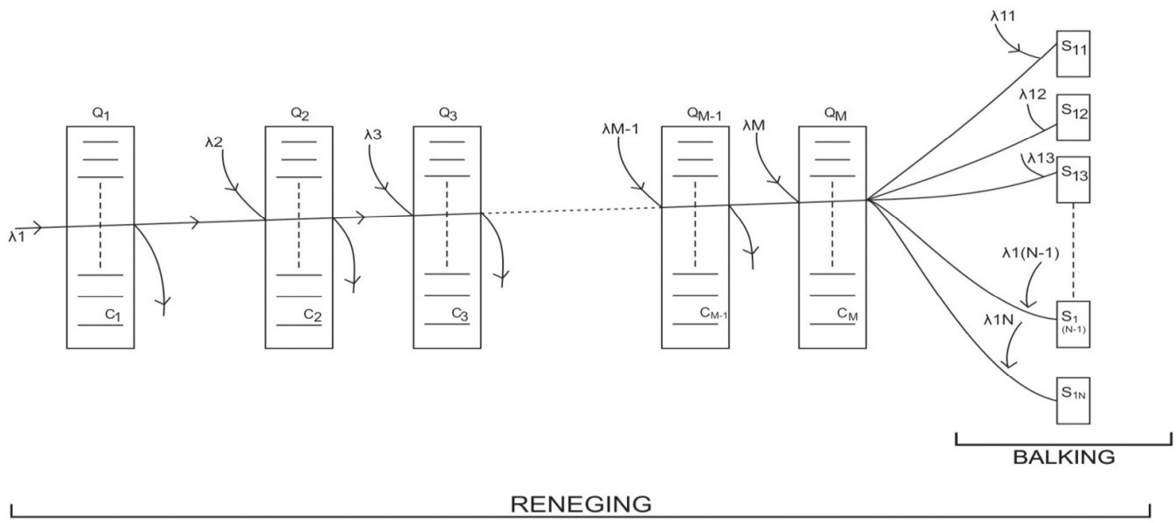
Abstract : This chapter provides the steady-state analysis of more general queuing system in the sense that

- M service channels in series are linked with N non-serial channels having reneging and balking phenomenon where each of M service channels has identical multiple parallel channels.
- The input process is Poisson and the service time distribution is exponential.
- The service discipline follows SIRO-rule (Service in random order) instead of FIFO-rule (first in first out).
- The customer becomes impatient in queue after sometime and may leave the system without getting service.
- The input process depends upon the queue size in non-serial channels.
- Waiting space is finite.

The practical situations where such a model finds application are of common occurrence. For example, consider the administration of a particular state at the level of district head quarter consisting of Patwaris, Kanoongoes, Tehsildars, Sub-divisional magistrates, district commisioner etc. These officers correspond to the servers of serial channels. Education department, Health department, Irritation department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that the officers call the people for hearing randomly .Further District commisioner may send the customers to different departments such as education, health, irrigation etc. if their problems are related to such departments.



1. Introduction

The problem of serial queues studied by [1], [2], [3], [5], [6] in steady-state with Poisson assumptions with the restriction that the customer must go through each service channel before leaving the system. [7] studied the problem of serial queues introducing the concept of reneging. The steady-state solutions of multiple parallel channels in series with impatient customers are obtained by [8]. The solutions of serial and non-serial queuing processes with reneging and balking phenomenon have been studied by [10]. The steady-state solution of serial and non-serial queuing processes with reneging and balking due to long queue and some urgent message and feedback phenomenon is obtained by [9].This chapter provides the steady-state analysis of more general queuing system in the sense that

- M service channels in series are linked with N non-serial channels having reneging and balking phenomenon where each of M service channels has identical multiple parallel channels.

- The input process is Poisson and the service time distribution is exponential.
- The service discipline follows SIRO-rule (Service in random order) instead of FIFO-rule (first in first out).
- The customer becomes impatient in queue after sometime and may leave the system without getting service.
- The input process depends upon the queue size in non-serial channels.
- Waiting space is finite.

2. Formulation of Model

The system consists of Q_i (i =1,2,….M) service phases where each service phase Q_i has c_i (i=1,2,….M) identical parallel service facilities and Q_{1j} channels (j=1,2,….N) with respective servers S_i (i=1,2,……M) and S_{1j} (j=1,2,……N).Customers demanding different types of service arrive from outside the system in Poisson distribution with parameters λ_i(i =1,2,….M) at Q_i service phase and λ_{1j}(j=1,2,…..N) at Q_{1j} service phase respectively . But the sight of long queue at Q_{1j} , may discourage the fresh customers from joining it and may decide not to enter the service channel Q_{1j} (j=1,2,….N) then the Poisson input rate λ_{1j} would be $\frac{\lambda_{1j}}{m_j+1}$ where m_j is the queue size of Q_{1j} . Further, the impatient customers joining any service channel may leave the queue without getting service after a wait of certain time. The service time distribution for the servers S_i (i=1,2……,M) and S_{1j} (j=1,2…..,N) are mutually independent negative exponential distribution with service rates $\mu_i(i=1,2,........,M)$ and $\mu_{1j}(j=1,2,....,N)$ respectively. After the completion of service at Q_i (i=1,2,……..M), the customer either leaves the system with probability p_i or joins the next phase with probability q_i such that $p_i+q_i=1$ (i= 1,2…..M-1). After completion of service at Q_M , the customer either leaves the system with probability p_M or joins any of the Q_{1j} (j=1,2,…….,N) with probability $\frac{q_{Mj}}{m_j+1}$ (j=1,2,….,N) such that $p_M+\sum_{j=1}^{N}\frac{q_{Mj}}{m_j+1}$ =1.

If the customers are more than c_i in the Q_i service phase ,all the c_i servers will remain busy and each is putting out the service at mean rate μ_i and thus the mean service rate at Q_i is $c_i\mu_i$,on the other hand if the number of customers is less than c_i in the Q_i service phase ,only n_i out of the c_i servers will be busy and thus the mean service rate at Q_i is $n_i\mu_i$ (i=1,2,…….M).It is assumed that the service commences instantaneously when the customer arrives at an empty service channel.

Here we assume that if at any instant, there are K customers in the system $\left(\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K\right)$, then the customers arriving at that instant will not be allowed to join the system and it is considered as a forced balking and lost for the system.

3. Formulation of Equations

Define $P(n_1, n_2, ..., n_M; m_1, m_2, m_3, ..., m_N; t)$ as the probability that at time 't' , there are n_i customers (which may renege or after being serviced by the Q_i phase either leave the system or join the next service phase) waiting in the Q_i service phase (i=1,2,……..,M) , m_j customers (which may balk or renege or after being serviced leave the system) waiting before the servers S_{1j} (j=1,2,…….N) .

We define the operators T_{i}□ , T□$_{i}$ and T□$_{i,\,i+1}$□ to act upon the vector $\tilde{n}=(n_1,n_2,............,n_M)$ and T_j□ and T□$_j$ and T□$_{j,\,j+1}$□ to act upon the vector $\tilde{m}=(m_1,m_2,............,m_N)$ as follows:

T_{i}□ $(\tilde{n})=(n_1, n_2, ...n_i-1...n_M)$

T□$_{i}$ $(\tilde{n})=(n_1, n_2, ...n_i+1...n_M)$

T□$_{i,\,i+1}$□ $(\tilde{n})=(n_1, n_2, ..., n_i+1, n_{i+1}-1, ..., n_M)$

T_{j}□ $(\tilde{m})=(m_1, m_2, ...m_j-1...m_N)$

$T_{.j}\ (\tilde{m}) = (m_1, m_2, \ldots\ldots\ldots m_j+1 \ldots m_N)$

$T_{\square j\ ,\ j+1\ \square}\ (\tilde{m}) = (m_1, m_2, \ldots\ldots, m_j+1, m_{j+1.}-1 \ldots m_N)$

The following difference-differential equations hold:

$$\frac{d}{dt}P(\tilde{n},\tilde{m},t) = -\left[\sum_{i=1}^{M}\lambda_i + \sum_{j=1}^{N}\frac{\lambda_{ij}}{m_j+1} + \sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i} + \delta_{n_i-c_i}r_{in_i}\right) + \sum_{j=1}^{N}\delta(m_j)\left(\mu_{ij} + R_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t)$$
$$+\sum_{i=1}^{M}\lambda_i P(T_{i\ \square}(\tilde{n}),\tilde{m};t) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\ \square}(\tilde{m});t)$$
$$+\sum_{i=1}^{M}\delta_{n_i-c_i}r_{in_i+1}P(T_{\square i}(\tilde{n}),\tilde{m};t) + \sum_{i=1}^{M-1}q_i\mu_{in_i+1}P(T_{\square\ i\ ,i+1\ \square}(\tilde{n}),\tilde{m};t)$$
$$+\sum_{i=1}^{M}p_i\mu_{in_i+1}P(T_{\square\ i}(\tilde{n}),\tilde{m};t) + \sum_{j=1}^{N}\mu_{Mn_M+1}\frac{q_{Mj}}{m_j}P(n_1,n_2,\ldots,n_M+1,T_{j\ \square}(\tilde{m});t)$$
$$+\sum_{j=1}^{N}\left(\mu_{1j} + R_{jm_j+1}\right)P(\tilde{n},T_{\square\ j}(\tilde{m});t)$$

--(1)

for $n_i \geq 0$, $m_j \geq 0$ and $\left(\sum_{i=1}^{M}n_i + \sum_{j=1}^{N}m_j\right)$ < K; (i = 1,2,3,…,M; j=1,2,3,…,N), and

$$\frac{d}{dt}P(\tilde{n},\tilde{m};t) = -\left[\sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i} + \delta_{n_i-c_i}r_{in_i}\right) + \sum_{j=1}^{N}\delta(m_j)\left(\mu_{ij} + R_{jm_j}\right)\right]P(\tilde{n},\tilde{m};t) +$$
$$+\sum_{i=1}^{M}\lambda_i P(T_{i\ \square}(\tilde{n}),\tilde{m};t) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\ \square}(\tilde{m});t)$$
$$+\sum_{i=1}^{M-1}q_i\mu_{in_i+1}P(T_{\square\ i\ ,i+1\ \square}(\tilde{n}),\tilde{m};t) + \sum_{j=1}^{N}\mu_{Mn_M+1}\frac{q_{Mj}}{m_j}P(n_1,n_2,\ldots,n_M+1,T_{j\ \square}(\tilde{m});t)$$

-----------------(2)

for $n_i \geq 0$; $m_j \geq 0$ and $\left(\sum_{i=1}^{M}n_i + \sum_{j=1}^{N}m_j\right) = K$; (i= 1, 2, 3, …, M; j=1, 2, 3, …, N

Where

$$\delta(x) = \begin{bmatrix} 1 & when & x \neq 0 \\ 0 & when & x = 0 \end{bmatrix}$$

$$\delta_{(n_i - c_i)} = \begin{bmatrix} 0 & when & n_i < c_i \\ 1 & when & n_i \geq c_i \end{bmatrix}$$

$$\mu_{in_i} = \begin{bmatrix} n_i \mu_i & when & 1 \leq n_i < c_i \\ c_i \mu_i & when & n_i \geq c_i \end{bmatrix}$$

$$r_{in_i} = \frac{\mu_i e^{\frac{-\mu_i T_{0i}}{n_i}}}{(1 - e^{\frac{-\mu_i T_{0i}}{n_i}})} \quad ; i = 1, 2, \ldots\ldots M$$

$$R_{jm_j} = \frac{\mu_{1j} e^{\frac{-\mu_{1j} T_{0j}}{m_j}}}{(1 - e^{\frac{\mu_{1j} T_{0j}}{m_j}})} \quad ; j = 1, 2, \ldots\ldots N$$

Where r_{in_i} and R_{jm_j} are the average rates at which the customers renege after a wait of certain time T_{0i} and T_{0j} whenever there are n_i and m_j customers in the Q_i and Q_{1j} service phases respectively and $P(\tilde{m},\tilde{n};t) = 0$ if any of the arguments is negative.

4. Steady-State Equations:

We write the following Steady-State equations of the queuing model by equating the time derivatives to zero in the equations (1) and (2)

$$\left[\sum_{i=1}^{M}\lambda_i + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i} + \delta_{n_i-c_i} r_{in_i}\right) + \sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j} + R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$= \sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\cdot}(\tilde{m}))$$
$$+\sum_{i=1}^{M}\delta_{n_i-c_i} r_{in_i+1}P(T_{\cdot i}(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M-1} q_i\mu_{in_i+1}P(T_{\cdot i,\, i+1\,\cdot}(\tilde{n}),\tilde{m})$$
$$+\sum_{i=1}^{M} p_i\mu_{in_i+1}P(T_{\cdot i}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\mu_{Mn_M+1}P(n_1,n_2,\ldots,n_M+1,T_{j\cdot}(\tilde{m}))$$
$$+\sum_{j=1}^{N}\left(\mu_{1j} + R_{jm_j+1}\right)P(\tilde{n},T_{\cdot j}(\tilde{m})) \qquad \text{------(3)}$$

for $n_i \geq 0$; $m_j \geq 0$ and $\left(\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j\right) < K;$

and

$$\left[\sum_{i=1}^{M}\delta(n_i)\left(\mu_{in_i}+\delta_{n_i-c_i}r_{in_i}\right)+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\lambda_i P(T_{i\bullet}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\bullet}(\tilde{m}))$$
$$+\sum_{i=1}^{M-1}q_i\mu_{in_i+1}P(T_{\bullet i,i+1\bullet}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_{Mn_M+1}\frac{q_{Mj}}{m_j}P(n_1,n_2,\ldots,n_M+1,T_{j\bullet}(\tilde{m}))$$ ----------------------(4)

for $n_i\geq 0$; $m_j\geq 0$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K;$

Two cases arise depending upon the number of customers n_i and number of channels c_i at Q_i phase (i= 1, 2, 3, …, M)

5. CASE(1)

When the number of customers n_i before Q_i phase is less than the number of identical service channels c_i (i.e. $n_i < c_i$;i = 1, 2, 3,…, M) , then there is no reneging in Q_i and the service is immediately available to the customers on arrival. Then under such situation $\delta_{n_i-c_i}=0$ and $\mu_{in_i}=n_i\mu_i$

Steady State Equations:

The equations (3) and (4) reduce to

$$\left[\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}n_i\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\lambda_i P(T_{i\bullet}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1i}}{m_j}P(\tilde{n},T_{j\bullet}(\tilde{m}))$$
$$+\sum_{i=1}^{M-1}q_i\mu_i(n_i+1)P(T_{\bullet i,i+1\bullet}(\tilde{n}),\tilde{m})$$
$$+\sum_{i=1}^{M}p_i\mu_i(n_i+1)P(T_{\bullet i}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_M(n_M+1)P(n_1,n_2,\ldots,n_M+1,T_{j\bullet}(\tilde{m}))$$
$$+\sum_{j=1}^{N}\left(\mu_{1j}+R_{jm_j+1}\right)P(\tilde{n},T_{\bullet j}(\tilde{m}))$$ -- (5)

for $n_i\geq 0$; $m_j\geq 0$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j<K;$

and

$$\left[\sum_{i=1}^{M}n_i\mu_i+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\lambda_i P(T_{i\bullet}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\bullet}(\tilde{m}))$$
$$+\sum_{i=1}^{M-1}q_i\mu_i(n_i+1)P(T_{\bullet i,i+1\bullet}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_M(n_M+1)P(n_1,n_2,\ldots,n_M+1,T_{j\bullet}(\tilde{m}))$$ -----------(6)

for $n_i\geq 0$; $m_j\geq 0$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K;$

Steady State Solutions:

The Steady State solutions of the above equations (5) and (6) can be verified to be

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left[\frac{1}{\lfloor n_1}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\right]\left[\frac{1}{\lfloor n_2}\left(\frac{\lambda_2+q_1\alpha_1^{'}}{\mu_2}\right)^{n_2}\right]$$

$$\left[\frac{1}{\lfloor n_3}\left(\frac{\lambda_3+q_2\alpha_2^{'}}{\mu_3}\right)^{n_3}\right]\cdots\left[\frac{1}{\lfloor n_M}\left(\frac{\lambda_M+q_{M-1}\alpha_{M-1}^{'}}{\mu_M}\right)^{n_M}\right]$$

$$\left[\frac{1}{\lfloor m_1}\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)^{m_1}}{\prod_{j=1}^{m_1}(\mu_{11}+R_{1j})}\right]\left[\frac{1}{\lfloor m_2}\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)^{m_2}}{\prod_{j=1}^{m_2}(\mu_{12}+R_{2j})}\right]$$

$$\cdots\left[\frac{1}{\lfloor m_N}\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)^{m_N}}{\prod_{j=1}^{m_N}(\mu_{1N}+R_{Nj})}\right] \qquad \text{---(7)}$$

Where $\alpha_1^{'},\alpha_2^{'},\alpha_3^{'},\ldots\alpha_{M-1}^{'}$ and ρ_M are the same as mentioned below.
where

$$\rho_M=\frac{\lambda_M+q_{M-1}\alpha_{M-1}^{'}}{\mu_M}$$

$$\alpha_1^{'}=\lambda_1$$

$$\alpha_k^{'}=\lambda_k+q_{k-1}\alpha_{k-1}^{'} \quad ; \quad k=2,3,\ldots\ldots,M-1$$

6. CASE (II)

When the number of customers before Q_i phase is more than or equal to the number of identical service channels c_i (i.e. $n_i \geq c_i$), then there is reneging in Q_i (i=1,2,…,M). So under such situation $\delta_{n_i-c_i}=1$ and $\mu_{in_i}=c_i\mu_i$ and $\delta(n_i)=1$.

Steady State Equations:
The equations (3) and (4) reduce to

$$\left[\sum_{i=1}^{M}\lambda_i+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{i=1}^{M}\left(c_i\mu_i+r_{in_i}\right)+\sum_{j=1}^{N}\delta\left(m_j\right)\left(\mu_{1j}+R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$

$$=\sum_{i=1}^{M}\lambda_i P(T_{i\square}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j\square}(\tilde{m}))$$

$$+\sum_{i=1}^{M}r_{in_i+1}P(T_{\square i}(\tilde{n}),\tilde{m})+\sum_{i=1}^{M-1}q_i\mu_i c_i\ P(T_{\square i,i+1\square}(\tilde{n}),\tilde{m})$$

$$+\sum_{i=1}^{M}p_i\mu_i c_i\ P(T_{\square i}(\tilde{n}),\tilde{m})+\sum_{j=1}^{N}\mu_M c_M P(n_1,n_2,\ldots,n_M+1,T_{j\square}(\tilde{m}))$$

$$+\sum_{j=1}^{N}\left(\mu_{1j}+R_{jm_j+1}\right)P(\tilde{n},T_{\square j}(\tilde{m})) \qquad \text{--(8)}$$

for $m_j \geq 0$ and $\left(\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j\right)<K;$

and

$$\left[\sum_{i=1}^{M}\left(c_i\mu_i + r_{in_i}\right) + \sum_{j=1}^{N}\delta\left(m_j\right)\left(\mu_{ij} + R_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$= \sum_{i=1}^{M}\lambda_i P(T_{i}(\tilde{n}),\tilde{m}) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_{j}(\tilde{m}))$$
$$+\sum_{i=1}^{M} r_{in_i+1}P(T_{i}(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M-1} q_i c_i \mu_i P(T_{i,\,i+1}(\tilde{n}),\tilde{m})$$
$$+\sum_{j=1}^{N} c_M\mu_M P(n_1,n_2,\ldots,n_M+1,T_{j}(\tilde{m}))$$ ---(9)

for $m_j \geq 0$ and $\left(\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j\right) = K;$

Steady State Solution:

The Steady state solution of the above equations can be verified to be:

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left[\frac{(\lambda_1)^{n_1}}{\prod_{i=1}^{n_1}(c_1\mu_1 + r_{1i})}\right]\left[\frac{\left\{\lambda_2\left(c_1\mu_1 + r_{1n_1+1}\right) + c_1q_1\mu_1\alpha_1\right\}^{n_2}}{\prod_{i=1}^{n_2}(c_2\mu_2 + r_{2i})\left(c_1\mu_1 + r_{1n_1+1}\right)^{n_2}}\right]$$
$$\left[\frac{\left\{\lambda_3\prod_{i=1}^{2}\left(c_i\mu_i + r_{in_i+1}\right) + c_2q_2\mu_2\alpha_2\right\}^{n_3}}{\prod_{i=1}^{n_3}(c_3\mu_3 + r_{3i})\prod_{i=1}^{2}\left(c_i\mu_i + r_{in_i+1}\right)^{n_3}}\right]\cdots\left[\frac{\left\{\lambda_M\prod_{i=1}^{M-1}\left(c_i\mu_i + r_{in_i+1}\right) + c_{M-1}q_{M-1}\mu_{M-1}\alpha_{M-1}\right\}^{n_M}}{\prod_{i=1}^{n_M}(c_M\mu_M + r_{Mi})\prod_{i=1}^{M-1}\left(c_i\mu_i + r_{in_i+1}\right)^{n_M}}\right]$$
$$\left[\frac{\left\{\lambda_{11} + c_M q_{M1}\mu_M\rho_M'\right\}^{m_1}}{\lfloor m_1\prod_{j=1}^{m_1}\left(\mu_{11} + R_{1j}\right)}\right]\left[\frac{\left\{\lambda_{12} + c_M q_{M2}\mu_M\rho_M'\right\}^{m_2}}{\lfloor m_2\prod_{j=1}^{m_2}\left(\mu_{12} + R_{2j}\right)}\right]\cdots\cdots$$
$$\cdots\cdots\left[\frac{\left\{\lambda_{1M} + c_M q_{MN}\mu_M\rho_M'\right\}^{m_N}}{\lfloor m_N\prod_{j=1}^{m_N}\left(\mu_{1N} + R_{Nj}\right)}\right]$$ --------------(10)

Where $\alpha_1,\alpha_2,\alpha_3,\ldots\alpha_{M-1}$ and ρ_M' are the same as mentioned below:

$$\rho_M' = \frac{\lambda_M\prod_{i=1}^{M-1}(c_i\mu_i + r_{in_i+1}) + c_{M-1}\mu_{M-1}q_{M-1}\alpha_{M-1}}{(c_M\mu_M + r_{Mn_M+1})\prod_{i=1}^{M-1}(c_i\mu_i + r_{in_i+1})}$$

$$\alpha_1 = \lambda_1$$

$$\alpha_k = \lambda_k\prod_{i=1}^{k-1}(c_i\mu_i + r_{in_i+1}) + q_{k-1}\alpha_{k-1}\ \mu_{k-1}c_{k-1};\ \ k = 2,3,\ldots\ldots\ldots,M-1.$$

Here , it is mentioned that the customers leave the system at constant rate as long as there is a line provided that the customers are served in the order in which they arrive. Putting $R_{jm_j} = R_j$ (j=1,2,3,…,N) in equations (5), (6),(8) and (9) and $r_{in_i} = r_i$ in equations (8) and (9), the steady-state solutions (7) and (10) reduce to

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left[\frac{1}{\lfloor n_1}\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\right]\left[\frac{1}{\lfloor n_2}\left(\frac{\lambda_2+q_1\alpha_1'}{\mu_2}\right)^{n_2}\right]$$
$$\left[\frac{1}{\lfloor n_3}\left(\frac{\lambda_3+q_2\alpha_2'}{\mu_3}\right)^{n_3}\right]\cdots\left[\frac{1}{\lfloor n_M}\left(\frac{\lambda_M+q_{M-1}\alpha_{M-1}'}{\mu_M}\right)^{n_M}\right]$$
$$\left[\frac{1}{\lfloor m_1}\left(\frac{\lambda_{11}+\mu_M q_{M1}\rho_M}{\mu_{11}+R_1}\right)^{m_1}\right]\left[\frac{1}{\lfloor m_2}\left(\frac{\lambda_{12}+\mu_M q_{M2}\rho_M}{\mu_{12}+R_2}\right)^{m_2}\right]\cdots\cdots$$
$$\cdots\cdots\left[\frac{1}{\lfloor m_N}\left(\frac{\lambda_{1M}+\mu_M q_{MN}\rho_M}{\mu_{1N}+R_N}\right)^{m_N}\right] \text{------}(11)$$

for $n_i < c_i$, $m_j \geq 0$; (i=1,2,3,……,M) ; j=1,2,3,……..N).

Where $\alpha_1', \alpha_2', \alpha_3', \ldots \alpha_{M-1}'$ and ρ_M are the same as mentioned before.

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left[\frac{\lambda_1}{(c_1\mu_1+r_1)}\right]^{n_1}\left[\frac{\lambda_2(c_1\mu_1+r_1)+c_1q_1\mu_1\alpha_1}{(c_1\mu_1+r_1)(c_2\mu_2+r_2)}\right]^{n_2}$$
$$\left[\frac{\lambda_3\prod_{i=1}^{2}(c_i\mu_i+r_i)+c_2q_2\mu_2\alpha_2}{\prod_{i=1}^{3}(c_i\mu_i+r_i)}\right]^{n_3}\cdots\left[\frac{\lambda_M\prod_{i=1}^{M-1}(c_i\mu_i+r_i)+c_{M-1}q_{M-1}\mu_{M-1}\alpha_{M-1}}{\prod_{i=1}^{M}(c_i\mu_i+r_i)}\right]^{n_M}$$
$$\left[\frac{1}{\lfloor m_1}\left(\frac{\lambda_{11}+c_M q_{M1}\mu_M\rho_M'}{(\mu_{11}+R_1)}\right)^{m_1}\right]\left[\frac{1}{\lfloor m_2}\left(\frac{\lambda_{12}+c_M q_{M2}\mu_M\rho_M'}{(\mu_{12}+R_2)}\right)^{m_2}\right]$$
$$\cdots\cdots\left[\frac{1}{\lfloor m_N}\left(\frac{\lambda_{1N}+c_M q_{MN}\mu_M\rho_M'}{(\mu_{1N}+R_N)}\right)^{m_N}\right] \text{------}(12)$$

for $n_i \geq c_i$, $m_j \geq 0$; (i=1,2,3,……,M ; j=1,2,3,……..N) .

Where $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{M-1}$ and ρ_M' are the same as mentioned before.

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing condition $\sum_{\tilde{n}=0}^{K}\sum_{\tilde{m}=0}^{K}P(\tilde{n},\tilde{m})=1$ and $\sum_{i=1}^{M}n_i+\sum_{l=1}^{N}m_j=K$ and with the restrictions that the traffic intensity of each service channel of the system is less than unity.

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