On Reduction of Some Differential Equations using Symmetry Methods

Manjit Singh
Yadavindra College of Engineering,Punjabi University Guru Kashi Campus,Talwandi Sabo
E-mail of the corresponding author: manjitcsir@gmail.com

Abstract
In this paper Lie group theory is used to reduce the order of ordinary differential equations. For an ordinary differential equation admitting one parameter Lie group symmetry, order of differential equation, in principle, can always be reduce by one. Ordinary differential equation admitting mutli parameter Solvable Lie symmetry, the order can be reduced by cardinality of admitted symmetry group.Seperation of variables by canonical coordinates of admitted is also given.
Keywords: Lie group, infinitesimal transformations, vector algebra, admitted symmetry

1. Introduction

Lie Group Theory was originally idea of Sophus Lie who was inspired by lecture of his fellow Norwegian Sylow on Galois Theory of solving algebraic equations, he was curious about how he can develop a similar theory for solving ordinary differential equations(ODE).Historically work of Sophus Lie faded into obscurity until it was re-discovered by his successors Vessiot, E. Cartan, E. and Birkhoff, G. who exploited the applications of Lie Group Theory to ordinary differential equations(ODE).The success of this theory is basically due to the perfection of the necessary tools of analysis and algebra, especially the availability with sufficiently useful hypotheses of the Implicit Function Theorem and the Existence-Uniqueness of Ordinary Differential Equations. In Lie Group Theory the invariance of ordinary differential equations (ODE)is studied under group of transformations called Lie Group transformations which are precisely characterised by their infinitesimals. Once admitted symmetry group for ordinary differential equation is recognized an algorithm can be developed for reducing order of differential equation plus quadratures thus leading to general solution. The beauty of Lie Group Theory Lies in the fact that the complicated non-linear conditions under continuous group action can be reduced to far simpler linear conditions.

2. Definitions

- In Lie group theory the independent and dependent variables of differential equation are transformed to new set of variable by definition
$$\mathbf{x}^* = \boldsymbol{F}(\mathbf{x}, \epsilon)$$
where ϵ being parameter of transformation, variable $\mathbf{x}$ represents both dependent and independent variables of differential equation. The Lie group theory requires that under these transformations, the given differential equation remains invariant, under such case of invariance, we call these transformations as admitted symmetry of differential equation
- When above equation is expanded using Taylor' s series, we finds
$$\mathbf{x}^* = \mathrm{x} + \epsilon \mathrm{X} + O(\epsilon^2)$$
Where X is called infinitesimal of the transformation and it is basic functional character of admitted symmetry for given differential equation, cardinality of infinitesimals is exactly equal to total number of dependent and independent variables in the differential equation.
- When all infinitesimals X_i's for differential equation are obtained, they can be taken as components of vector field given as
$$\boldsymbol{V} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \cdots + X_n \frac{\partial}{\partial x_n} = \sum X_i \frac{\partial}{\partial x_i}.$$
Using this vector field, an invariance criteria can be developed, which leads to complete determination of symmetry group of given differential equation.
- In order to incorporate higher order derivatives, so as to formulate invariance criteria for given differential equation, prolongation of vector field can be defined as under
$$V^{(\boldsymbol{k})} = \sum \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \eta_i^{(1)} \frac{\partial}{\partial u_i} + \ldots . \eta_{i_1 i_2 \ldots . i_k}^{(k)} \frac{\partial}{\partial u_{i_1 i_2 \ldots . i_k}}, \text{ for } k \geq 1.$$
- The Function $H(\boldsymbol{x})$ is said to be invariant under vector field $\boldsymbol{V}$ if following condition is met
- $\mathrm{H}(\mathbf{x}^*) = \mathrm{H}(\mathbf{x})$ consequently $\mathbf{V}.\mathrm{H}(\mathbf{x}) = 0$, in case of differential equation of nth order, like $y_n = f(\boldsymbol{x}, y_{\mathbf{1},\ldots.} y_{n-1})$, for $\mathbf{x} = (x_{\mathbf{1}}, \ldots . , x_m)$ being m independent variables, then this nth differential is said to be invariant under vector field $\mathbf{V}$, if we have $\mathrm{V}^{(\mathbf{k})}\left(\mathrm{y_n} - \mathrm{f}(\mathbf{x}, \mathrm{y}_{\mathbf{1},\ldots.}\mathrm{y_{n-1}})\right) = 0$, when this is expanded

using definition of prolongation will give overdetermined system of partial differential equation for ξ_i and η.

3. Lie Fundamental Theorem

Lie's first fundamental theorem states that, there exist parameterization$\tau(\epsilon)$, such that Lie group transformation is equivalent to solution of initial value problem

$$\frac{d\mathbf{x}^*}{d\tau} = X(\mathbf{x}^*), \text{with intial condition } x^* = \mathbf{x} \text{ when } \tau = 0.$$

For sake of simplicity, it is possible to choose $\tau = \epsilon$, this theorem is useful for deriving infinitesimal transformations from infinitesimals.From now on we consider differential equation containing only one dependent and one independent variable namely x and y respectively.

4. Canonical Coordinates

For Lie group of transformations on n variables $\boldsymbol{x} = (x_1, \dots., x_n)$, into new set of coordinates $(y_1, \dots. y_n)$, such that only translation is seen in last coordinates, then $y_i's$, for $1 \le i \le i-1$ are solution of first order PDE $\boldsymbol{V}u(\boldsymbol{x}) = 0$ and y_n is solution of nonhomogenous PDE $\boldsymbol{V}v(\boldsymbol{x}) = 1$, following example shows how canonical coordinates can be used separate or reduce the given ordinary differential equation admitting symmetry group [1]

4.1 Example

Consider first order ordinary differential equation

$$\frac{dy}{dx} = f(x,y)$$

admitting Lie group of transformation in variables (x,y), such that canonical coordinates are (r,s), then under these canonical coordinates above differential equation reduced into much simpler form as under

$$\frac{ds}{dr} = G(r)$$

which upon integration gives solution as

$$s = \int G(r)dr + c$$

on integration we obtain general solution to differential equation, it is evidently clear that the above conversion is only feasible when canonical are simple and are easily obtained.

4.2 Example

Consider Ricaati's equation

$$y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}$$

admitting one parameter scaling symmetry $\mathbf{x}^* = e^{\epsilon}x$ and $\mathbf{y}^* = e^{-2\epsilon}y$ such that $\boldsymbol{V} = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$, then canonical coordinates (r,s) are solutions of characteristics equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dr}{0} \text{ and } \frac{dx}{x} = \frac{dy}{-2y} = \frac{ds}{1}$$

We find on solving that

$$r = x^2y \text{ and } s = \ln|x|$$

thus

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'} = \frac{1}{r^2 - 1}$$

Which on solving gives general solution to differential equation

$$y = \frac{c + x^2}{x^2(c - x^2)}$$

4.4 Example

Consider another case of Ricaati's equation[2]

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

This equation on contrary to traditional methods of solving differential equations, can be easily solved by symmetry methods, it is easily seen that this Ricaati equation admits one parameter inversion symmetry

$$\mathbf{x}^* = \frac{x}{1 - \epsilon x} \text{ and } \mathbf{y}^* = \frac{y}{1 - \epsilon x}$$

Corresponding canonical coordinates are $r = \frac{y}{x}$ and $s = -\frac{1}{x}$, and thus

$$\frac{ds}{dr}=\frac{1}{1+r^2}$$

Which can be easily solved by simple integration method, giving general solution as

$$y=-x\tan\left(\frac{1}{x}+c\right)$$

in general there are many such examples of first order ordinary differential equation which can not be solved using routine methods of integration, however, when symmetry for differential equation is known, it is always possible in principle to solve or simplify such differential equation with much ease.

5. Reduction using Differential Invariants

In contrast to canonical coordinates, reduction of ODE using differential invariants is quite comfortable, for infinitesimals $\xi(x,y)$ and $\eta(x,y)$ of ODE, differential invariants of ODE are written as [3]

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}=\frac{dy_1}{\eta_x+\left(\eta_y-\xi_x\right)-\xi_y f^2}\text{ where }\frac{dy}{dx}=f(x,y)$$

Solving these characteristics equation we obtained differential invariants of ODE as

$$u(x,y)=c_1 \text{ and } v(x,y)=c_2$$

5.1 Example

Take second order ODE[3]

$$x^2\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\frac{dy}{dx}$$

Admitting scaling symmetry $\mathbf{x}^*=\alpha x$ and $\mathbf{y}^*=\alpha y$, such that $\xi(x,y)=x, \eta(x,y)=y$ and $\eta_1=0$, so that the characteristics equations becomes

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dy_1}{0}$$

And differential invariants are

$$u(x,y)=\frac{y}{x}, v(x,y)=y_1 \text{ which gives } \frac{dv}{du}=\frac{x^2y_2}{xy_1-y}=\frac{xy_2}{v-u}$$

thus

$$\frac{d^2y}{dx^2}=\frac{1}{x}(v-u)\frac{dv}{du}$$

Under these changes order of given ODE is reduced by one as

$$(v-u)\frac{dv}{du}+v^2=uv$$

This reduced differential equation can be solved using existing integration methods and even by symmetry methods if its symmetry is known.

6. Multi Parameter Lie Group Transformation

In general an ODE may admit multi parameter Lie group tranformation,for example second order ODE admits upto eight parameter Lie group transformation and when order of ODE is n>2 then it may admit upto (n+4) parameter Lie group tranformation[1]. As we have learnt that when an ODE of order n admits one parameter Lie group of tranformation then in principle it possible to reduce the order of ODE by one plus one quadrature using canonical coordinates or by use of differential invariants,intuitively it is natural for one to believe that if an ODE admits r parameter Lie group transformation ,then order of ODE may be reduced to (n-r) plus r quadrature, but in actual practice it is not possible to constructively reduce order of ODE by r ,the reason behind this is that when order is reduced using one symmetry then it is not necessary that reduced ODE would still admit remaining symmetries.The process of reduction of order can be done successively only if the underlying r-Lie algebra is Solvable[4].In order to define multiparameter Lie group of transformations we write

$$\mathbf{x}^*=\boldsymbol{F}(\mathbf{x},\epsilon)$$

Where $x^*=\left(x_1,x_2,\dots,x_n\right)$ and $\epsilon=\left(\epsilon_1,\epsilon_2,\dots,\epsilon_r\right)$, thus this group can be regarded as n parameter Lie group.One of interesting thing about multiparameter Lie group is that associated infinitesimal generators form vector space with respect to commutation relation defined as under

$$\left[X_i,X_j\right]=X_iX_j-X_jX_i$$

Vector space of r infinitesimal generators is so generated is called Lie algebra.In next theorem we prove that, this Lie algebra is Solvable, then order of differential equation can constructively be reduced by r, that is, nth order differential equation can be reduced to (n-r)th order differential equation.

6.1 Theorem :(Solvable Lie Algebra)

Order of nth order differential equation admitting r parameter Lie group of transformations can be contructively reduced to (n-r).[3,4,5]
Proof : For sake of convenience we take example second order differential equation

$$y'' = f(x, y, y')$$

Admitting two parameter Lie group of transformation, such that X_1, X_2 being basis for underlying Lie algebra, being two dimensional algebra so is solvable, so it is possible to write

$$[X_1, X_2] = \lambda X_1$$

Let $u(x,t), v(x,y,y')$ be differential invariants of $X_1^{(2)}$ such that

$$X_1 u = 0, X_1^{(1)} v = 0$$

Then corresponding differential invariant of dv/du which satisfies the equation

$$X_1^{(2)} \frac{dv}{du} = 0$$

Hence ODE reduces to

$$\frac{dv}{du} = H(u, v)$$

For some function H of u and v, from commutation relation,

$$X_1 X_2 u = X_2 X_1 u + \lambda X_1 u = 0$$

$$X_1^{(1)} X_2^{(1)} v = 0, X_1^{(2)} X_2^{(2)} \frac{dv}{du} = 0$$

Hence we obtained

$$X_2 u = \alpha(u),\ X_2^{(1)} v = \beta(u, v)$$

Next step is to use second generator for further reduction, thus we have

$$X_2^{(1)} v = \alpha(u) \frac{\partial}{\partial u} + \beta(u, v)\ \frac{\partial}{\partial v}$$

At this stage second generator is transformed into generator with new coordinates u and v which are basically differential invariants of first generator X_1, in this way it is useful in further reduction of differential equation as it is being admitted by reduced differential equation, let $R(u,v)$ and $S(u,v)$ be canonical coordinates associated with $X_2^{(1)}$, then

$$X_2^{(1)} R = 0 \text{ and } X_2^{(1)} S = 1$$

Such that, differential equation involving differential invariants $u(x,t), v(x,y,y')$ is transformed into separated form as

$$\frac{dS}{dR} = G(R)$$

Which on integration gives

$$S(u(x,y), v(x,y,y')) = \int G(R) dR + C$$

This process can be repeated for third differential equation admitting three parameter solvable Lie algebra.

As an example we illustrate reduction of Blasius equation[3, 4]

$$y''' + \frac{1}{2} y y'' = 0$$

Admitting two parameter Lie group of transformation group with associated infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}, X_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$$

Since two dimensional Lie algebra is always solvable, therefore we can start reduction with X_1
Differential invariants of X_1 are

$$u(x,y) = y, v(x,y,y') = y' = y_1, v_1 = \frac{dv}{du} = \frac{y_2}{y_1}$$

thus in new coordinates $u(x,y)$ and $v(x,y,y')$, second generator X_2 takes the new form as

$$X_2 u = -u, X_2^{(1)} v = -2v, X_2^{(2)} v_1 = -v_1$$

New set of differential invariants $U(u,v)$ and $V(u,v,v_1)$ of $X_2^{(1)}$ are

$$U(u,v) = \frac{v}{u^2} \text{ and } V(u,v,v_1) = \frac{v_1}{u}$$

In new set of coordinates Blasius equation reduces to first ODE as

$$\frac{dV}{dU} = \frac{V}{U}\left[\frac{\frac{1}{2} + V + U}{2U - V}\right]$$

This is much simpler in comparison to original differential equation. However, if we able to find its symmetry, then in same way it can be simplified using canonical coordinates, leading to its general solution.In case of solvable group one must take care of generator which is being taken at first place[6].

7. Determination of Point Symmetry

If given ODE admits one parameter symmetry group it is possible to develop a procedure to reduce the order of differential equation plus quadrature and order can be reduced more than one time if ODE admit multiparameter solvable symmetry group, as one may desire more than one symmetry admitted by given ODE Sophus Lie proposed a method of finding various symmetries admitted by ODE popularly known as Lie classical method which depends on solving overdetermined system of linear PDE's.

Consider nth order differential equation

$$y^{(n)} = f(x, y, y^{(1)}, y^{(2)}, \dots, y^{(n-1)})$$

Let the admitted vector field be

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

Where infinitesimals $\xi(x, y)$ and $\eta(x, y)$ are obtainable using invariance criterion

$$X^{(n)}[y^{(n)} - f(x, y, y^{(1)}, y^{(2)}, \dots, y^{(n-1)})] \equiv 0$$

On simplification, this invariance criteria gives overdetermined system of partial differential equations for infinitesimals $\xi(x, y)$ and $\eta(x, y)$.

8. Applications

I. When symmetry group act on differential equation or system of differential,then using infinitesimal transformation new solutions to diferential equation can be constructed using known solutions.
II. Symmetry group can be used to reduced total number of dependent and independent variables in partial differential equation there by reducing it into ordinary differential equation.
III. Symmetry group can be used classify differential equation into equivalence classes so that differential equations falling into same equivalence class can be solved by common method.
IV. Symmetry groups can be used to reduce order of ordinary differential equation, for example under admitted symmetry second order differential equation can be reduced to first order and even third order differential equation can be reduced to first order under solvable Lie's algebra.
V. Symmetry gropus and exact solutions can be used to check accuracy and reliability of certain numerical techniques devised for solving physically relevant partial differential equations.

By examining the group invariance properties of nonlinear PDE, it is possible to determine whether the equation is transformable into linear equation by one-one tranformation

9. Conclusion

The Lie classical method is finest of all techniques developed so far, based on systematic algorithm, equipped with Lie algebra and being based on strong principles of group theory, Lie classical method can be applied to any kind of differential equation for symmetry reduction and thereby exact solution, many symbolic programmes are also developed for generating and solving determining equations for this method, symbolic manipulation packages are written in Maple for easy application of symmetry method..

References

[1] P. J. Olver, *Applications of Lie Groups to Differential Equations*: Springer New York, 2000.
[2] P. E. Hydon and P. E. Hydon, *Symmetry Methods for Differential Equations: A Beginner's Guide*: Cambridge University Press, 2000.
[3] G. W. Bluman , *Symmetry and integration methods for differential equations*. New York: Springer, 2002.
[4] G. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations*: Springer, 2002.
[5] G. Bluman, "A reduction algorithm for an ordinary differential equation admitting a solvable lie group," *SIAM J. Appl. Math.,* vol. 50, pp. 1689-1705, 1990.
[6] "Determining Equations and Differential Constraints," *Journal of Nonlinear Mathematical Physics,* vol. 2, p. 283, 1995..