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Numerical solution of Boundary Value Problems by Piecewise Analysis Method

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Abstract
In this paper, we use an efficient numerical algorithm for solving two point fourth-order linear and nonlinear boundary value problems, which is based on the homotopy analysis method (HAM), namely, the piecewise – homotopy analysis method (P-HAM).The method contains an auxiliary parameter that provides a powerful tool to analysis strongly linear and nonlinear (without linearization) problems directly. Numerical examples are presented to compare the results obtained with some existing results found in literatures. Results obtained by the RHAM performed better in terms of accuracy achieved.
Keywords: Piecewise-homotopy analysis, perturbation, Adomain decomposition method, Variational Iteration, Boundary Value Problems.

1. Introduction.

In recent years, much attention has been given to develop some analytical methods for solving boundary value problems including the perturbation methods, decomposition method and variational iteration method etc. It is well known that perturbation method [1, 2] provide the most versatile tools available in nonlinear analysis of engineering problems. The major drawback in the traditional perturbation techniques is the over dependence on the existence of small parameter. This condition over strict and greatly affects the application of the perturbation techniques because most of the nonlinear problems
(especially those having strong nonlinearity) do not even contain the so called small parameter; moreover, the determination of the small parameter is a complicated process and requires special techniques. These facts have motivated to suggest alternative techniques such as decomposition method, variational iteration method and homotopy analysis method.

The homotopy analysis method (HAM) proposed by [9, 10] is a general analytical approach to solve various types of linear and non-linear equations, including Partial differential equations, Ordinary differential equations, difference equations and algebraic equations. More importantly different from all perturbation and traditional non-perturbation methods. The homotopy analysis method provides simple way to ensure the convergence of solution in series form and therefore, the HAM is even for strong nonlinear problems. The homotopy analysis method (HAM), is based on homotopy, a fundamental concept in topology and differential geometry. In homotopy analysis method, one constructs a continuous mapping of an initial guess approximation to the exact solution of problems to be considered. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure convergence of solution series. The method enjoys great freedom in choosing initial approximation and auxiliary linear operator. By means of this kind of freedom, a complicated non-linear problems can be transformed into an infinite numbers of simpler linear sub-problems.

In this paper, we consider the general fourth –order boundary value problems of the type:

$$y^{iv}(x) = f(x, y, y', y'', y''') \qquad (1)$$

With the boundary conditions:

$$y(a) = \alpha_{1;} \qquad y'(a) = \alpha_2$$

$$y(b) = \beta_1 \qquad y'(b) = \beta_2 \qquad (2)$$

Or,

$$y(a) = \alpha_{1;} \qquad y''(a) = \alpha_2$$

$$y(b) = \beta_1 \qquad y''(b) = \beta_2 \tag{3}$$

Where f is continuous function on [a, b] and the parameters α_i and β_i , $i = 1, 2$ are real constants. Such type of boundary value problems arise in the mathematical modeling of the viscoelastic flows, deformation of beams and plate deflection theory and other branches of mathematical, physical and engineering ([3, 7, 8, 13]). Several numerical methods including Finite Difference, B – Spline were developed for solving fourth order boundary value problems [3]. In this paper, we used Piecewise Homotopy Analysis Method to obtain the numerical solution of fourth order boundary value problems.

2. Description of the new homotopy analysis method.

In this section, the basic idea of the new homotopy analysis method is introduced. We start by considering the following differential equation:

$$N[y(x)] = 0 \tag{4}$$

Where N is an operator, $y(x)$ is unknown function and x, the independent variable. Let $y_0(x)$ denotes an initial guess of the exact solution $y(x), \ h \neq 0$

an auxiliary parameter,

$$H(x) \neq 0$$

an auxiliary function and L, the auxiliary linear operator with the property $L[y(x)] = 0$ when $y(x) = 0$. Then, using $q \ \varepsilon \ [0,1]$ as an embedded parameter, we construct such a homotopy

$$(1-q)L[\phi(x,q) - y_0(x)] - q\,h\,H(x)\,N[\phi(x,q)] = \hat{H}[\phi(x,q); y_0(x)H(x), h, q] \tag{5}$$

It should be emphasized that we have – function to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non – zero auxiliary parameter h, and the auxiliary function $H(x)$. Enforcing the homotopy (5) to be zero, that is,

$$\hat{H}[\phi(x,q); \ y_0(x), \ H(x), \ h, \ q] = 0 \tag{6}$$

We have the so called zero order deformation equation

$$(1-q)L[\phi(x,q) - y_0(x)] = q\,h\,H(x)\,N[\phi(x,q)] \tag{7}$$

When $q = 0$ the zero order deformation equation (7) becomes

$$\phi(x,0) = y_0(x) \tag{8}$$

And when $q = 1$, $h \neq 0$ *and* $H(x) \neq 0$, the zero order deformation equation (7) is equivalent to:

$$\phi(x,1) = y(x) \tag{9}$$

Thus, according to equations (8) and (9), as the embedding parameter q increases from 0 to 1, $\phi(x,q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$, such a kind of continuous variation is called deformation in homotopy. The power series of q is as follows:

$$\phi(x,q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m \tag{10}$$

Where,

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x,q)}{\partial q^m}\Big|_{q=0} \tag{11}$$

If the initial guess $y_0(x)$, the auxiliary linear operator L, the non- zero auxiliary parameter h and the auxiliary function $H(x)$ are properly chosen so that the power series (10) of $\phi(x,q)$ converges at $q = 1$. Then, we have under these assumptions that the solution series

$$y(x) = \phi(x,1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) \tag{12}$$

For brevity, define the vector

$$\vec{y}_m(x) = \{y_0(x),\ y_1(x),\ \ldots\ldots\ y_m(x)\} \tag{13}$$

According to the definition (11), the governing equation of $y_m(x)$ can be derived from the zero – order deformation (7) . Differentiating the zero – order deformation equation (7) m times with respect to q and then dividing by m! and finally setting $q = 0$, we have the so called modified mth order deformation equation

$$L\left[y_m(x) - (1 - X_m)\ y_{m-1}(x) \right] = hH(x)R_m(\vec{y}_{m-1}(x)) \qquad (14)$$

$$y_m^{(k)}(0) = 0 \ , \qquad k = 0,1,2,............m-1,$$

where,

$$R_m(\vec{y}_m(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,q)]}{\partial q^{m-1}}|q = 0$$

And ,

$$X_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

According to (1) the auxiliary linear operator L can be chosen as

$$L[\phi(x,q)] = \frac{\partial^4 \phi(x,q)}{\partial x^4} \qquad (15)$$

And the non linear operator N can be chosen

$$N[\phi(x,q)] = \frac{\partial^4 \phi}{\partial x^4} - f\left(x,\ \phi,\ \frac{\partial \phi}{\partial x},\ \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^3 \phi}{\partial x^3}\right) \qquad (16)$$

The initial guess $y_0(x)$ of the solution $y(x)$ can be determined as follows:

$$y_0(x) = \sum_{k=0}^{n-1} y^{(k)}(a)\frac{(x-a)^k}{k!}, \qquad k = 0,\ 1,\ \ 2,\ \ 3,\ ......$$

$$\textit{which gives,} \qquad (17)$$

$$y_0(x) = y(a) + y'(a)(x-a) + \frac{1}{2!}y''(a)(x-a)^2 + \frac{1}{3!}y'''(a)(x-a)^3$$

Also, let the expression,

$$S_n(x) = \sum_{i=0}^{n-1} y_i(x) \qquad (18)$$

denotes the n-term approximation to $y(x)$. Our aim is to determine the constants $y''(a)$ *and* $y'''(a)$. This can be achieved by imposing the remaining two boundary conditions (2) at $x = b$ *to the* S_n. Thus we have,

$$S_n(b) = y(b), \qquad S'_n(b) = y'(b)$$

Solving for $y'(a)$ *and* $y''(a)$ yields

$$y''(a) = \frac{2}{(a-b)^2}\left[y'(b)(a-b) + 2y'(a)(a-b) + 3(y(b) - y(a))\right]$$

And,

$$y'''(a) = \frac{6}{(a-b)^3}\left[y'(a)(a-b) + y'(b)(a-b) + 2(y(b) - y(a))\right]$$

By substituting $y''(a)$ *and* $y'''(a)$ into (17), we have

$$y_0(x) = y(a) + y'(a)(x-a) + [r(x)]^2[y'(b)(a-b) + 2y'(a)(a-b) + 3(y(b) - y(a))] - [r(x)]^3[y'(a)(a-b) + y'(b)(a-b) + 2(y(b) - y(a))] \quad (19)$$

Where,

$$r(x) = \frac{(x-a)}{(b-a)}$$

Note that the higher order deformation equation (14) is governing by the linear operator L, and the term $R(\vec{y}_{m-1}(x)$ can be expressed simply by (15) for any nonlinear operator N. Therefore, $y_m(x)$ can be easily gained, especially by means of computational software such as MATLAB. The solution $y(x)$ given by the above approach is dependent of $L,\ h,\ H(x)$ *and* $y_0(x)$.

3. The Basic idea of Piecewise – Homotopy Analysis Method (PHAM).

It is a remarkable property of homotopy analysis method (HAM) that the value of the auxiliary parameter h can be freely chosen to increase the convergence rate of the solution series. But the freedom of choosing h is subject to the so called valid region of h. It was discovered that the solution series h does not always give a good approximation even when h is chosen within the valid region. Nevertheless by chance, there are cases where a valid h value chosen this way does give good approximation for a large range of x. Therefore in approximation for a large x range, we propose that many different h values should be used and each h value offers good approximation only about certain small range of x. the segments of approximation correspond to their range of x are then be combined and joined together piece by piece to form the large x range approximation desire. T his method is named as "Piecewise - Homotopy Analysis Method (P-HAM), which means that there most be many h – values, each for a different value of x. The horizontal segment of each h – curve will give a valid h – region, each corresponds to one value of x. It is noteworthy that P- HAM adopts only one approximation series $y(x,h)$ all along, but different h-values are substituted for different interval of x such that a piecewise approximation series is formed.

That is,

$$y(x) = \begin{cases} y(x, h_1), & x < x_1 \\ y(x, h_2), & x_1 \le x < x_2 \\ y(x, h_3), & x_2 \le x < x_3 \\ : & \\ : & \\ y(x, h_{n-1}), & x_{n-2} \le x < x_{n-1} \\ y(x, h_n) & x_{n-1} \le x < x_2 \end{cases}$$

4. Numerical Examples.

In this section, we demonstrate the effectiveness of the P-HAM with illustrative examples. All numerical results obtained by P – HAM are compared with the results obtained by various numerical methods.

Example 1.

Consider the following linear problem:

$$y''''(x) = (1+c)\,y''(x) - cy(x) + \frac{1}{2}cx^2 - 1, \qquad 0 < x < 1,$$

$$\textit{Subject to},$$

$$y(0) = 1, \qquad y'(0) = 1$$

$$y(1) = \frac{1}{2} + Sinh(1) \qquad y'(1) = \frac{3}{2} + Cosh(1)$$

The exact solution for this problem is $y(x) = 1 + \frac{1}{2}x^2 + Sinh(x)$

For each test point, the error between the exact solution and the results obtained by the HPM, ADM and HAM and compared in tables 1 and 2.

Also Table 3 shows different values of x and their corresponding values of h for cases c=10, 100 and 1000. Our approximate solution obtained using P-HAM are in d agreement with the exact solutions in all cases. With only two iterations, that is, $y_0 + y_1$, a better approximation has been obtained than the results by HPM, ADM and HAM.

Using (14) and (19), we obtain:

$$y_0(x) = 1 + x + 0.482522945x^2$$

$$y_1(x) = 0.192678247\,x^3 - h\int_0^x\int_0^\tau \int_0^{\tau_2}\int_0^{\tau_1}\left[(1+c)y''_0(x) - cy_0(I) + \frac{1}{2}c\tau^2 - 1\right]d\tau\, d\tau_1\, d\tau_2\, d\tau_3$$

$$y_m(x) = -h\int_0^x\int_0^\tau \int_0^{\tau_2}\int_0^{\tau_1}\left[((1+c)y'_{m-1}(\tau) - cy_{m-1}(\tau))X_m + (1-X_m)(\frac{1}{2}c\tau^2 - 1)\right]d\tau\, d\tau_1\, d\tau_2\, d\tau_3$$

In all cases, we have defined our error as

Error = $\left|Exact\ \ solution\ - Approximate\ \ solution\right|,\ a \le x \le b$

Table 1: Error of Example 1: c = 10.

X	Exact solution	Error of ADM	Error of HPM	Error of HAM	Error of P-HAM
0.0	1.000000000	0.000000	0.0000000	0.0000000	0.0000000
0.1	1.105166750	1.7×10^{-4}	1.7×10^{-4}	1.5×10^{-4}	9.3×10^{-8}
0.2	1.221336002	5.7×10^{-4}	5.7×10^{-4}	4.9×10^{-4}	1.9×10^{-8}
0.3	1.349520293	1.0×10^{-3}	1.0×10^{-3}	8.9×10^{-4}	6.0×10^{-8}
0.4	1.400752325	1.4×10^{-3}	1.4×10^{-3}	1.2×10^{-3}	5.1×10^{-8}
0.5	1.646095305	1.6×10^{-3}	1.6×10^{-3}	1.4×10^{-3}	6.0×10^{-9}
0.6	1.816653582	1.6×10^{-3}	1.6×10^{-3}	1.3×10^{-3}	2.7×10^{-7}
0.7	2.003583701	1.2×10^{-3}	1.2×10^{-3}	1.7×10^{-3}	1.0×10^{-9}
0.8	2.208105982	7.6×10^{-3}	7.6×10^{-3}	6.4×10^{-4}	2.3×10^{-6}
0.9	2.431516725	2.5×10^{-3}	2.5×10^{-3}	2.7×10^{-4}	2.6×10^{-6}
1.0	2.675201193	0.0000000	0.0000000	0.0000000	0.0000000

Table 2 : Numerical values when C=100

x	Error of ADM	Error of HPM	Error of HAM	Error of P-HAM
0.0	0.0000	0.0000	0.0000	0.0000
0.1	6.5×10^{-4}	6.5×10^{-4}	1.5×10^{-4}	8.3×10^{-6}
0.2	2.6×10^{-3}	2.6×10^{-3}	4.9×10^{-4}	4.7×10^{-7}
0.3	6.3×10^{-3}	6.3×10^{-3}	8.9×10^{-4}	9.3×10^{-7}
0.4	1.2×10^{-2}	1.2×10^{-2}	1.2×10^{-3}	1.6×10^{-6}
0.5	1.8×10^{-2}	1.8×10^{-2}	1.4×10^{-3}	8.6×10^{-6}
0.6	2.4×10^{-2}	2.4×10^{-2}	1.4×10^{-3}	5.3×10^{-6}
0.7	2.6×10^{-2}	2.6×10^{-2}	1.2×10^{-3}	9.9×10^{-6}
0.8	2.0×10^{-2}	2.0×10^{-2}	9.7×10^{-4}	9.1×10^{-6}
0.9	8.6×10^{-3}	8.6×10^{-3}	7.9×10^{-4}	5.6×10^{-6}
1.0	0.0000	0.0000	0.0000	0.0000

Table 3 : Numerical values when C=1000

x	Error of ADM	Error of HPM	Error of HAM	Error of P-HAM
0.0	0.0000	0.0000	0.0000	0.0000
0.1	1.3×10^{-2}	1.3×10^{-2}	1.5×10^{-4}	1.8×10^{-7}
0.2	1.1×10^{-1}	1.1×10^{-1}	4.9×10^{-4}	3.7×10^{-8}
0.3	3.9×10^{-1}	3.9×10^{-1}	9.2×10^{-4}	6.6×10^{-7}
0.4	9.4×10^{-1}	9.4×10^{-1}	1.3×10^{-3}	4.9×10^{-6}
0.5	1.6×10^{0}	1.6×10^{0}	1.7×10^{-3}	1.2×10^{-6}
0.6	2.3×10^{0}	2.3×10^{0}	2.2×10^{-3}	8.0×10^{-6}
0.7	2.6×10^{0}	2.6×10^{0}	2.8×10^{-3}	1.2×10^{-5}

0.8	2.1 x 10^{0}	2.1 x 10^{0}	3.9 x 10^{-3}	2.2 x 10^{-7}
0.9	9.1 x 10^{-1}	9.1 x 10^{-1}	6.1 x 10^{-3}	4.5 x 10^{-8}
1.0	0.0000	0.0000	0.0000	0.0000

Table 4: Different h-values chosen within the horizontal segments of the h-curves for all x range when c=10, 100 and 1000.

X	C=10	C=100	C=1000
0.0	1.000000	1.000000	1.000000
0.1	3.500000	6.100000	0.649000
0.2	9.445000	0.983000	0.098800
0.3	2.684000	0.277000	0.027800
0.4	0.964900	0.099000	0.009900
0.5	0.383500	0.039000	0.003930
0.6	0.155400	0.015800	0.001580
0.7	0.059500	0.006000	0.000600
0.8	0.018900	0.001900	0.000193
0.9	0.003500	0.000350	0.000036
1.0	1 x 10^{-9}	1 x 10^{-9}	1 x 10^{-9}

Example 2: Consider the following non linear boundary –value problem:

$$y^{iv}(x) = \left(\frac{dy}{dx}\right)^2 - y\left(\frac{d^3y}{dx^3}\right) - 4x^2 + e^x(1 + x^2 - 4x); \qquad 0 \le x \le 1$$

With boundary conditions

Y(0) = 1 $y^1(0) = 1$

Y(1) = 1 + e $y^1(1) = 2 + e$

The analytical solution for this problem is $y(x) = x^2 + e^x$

This problem is solve by the same method applied in Example 1, therefore its iteration formulation reads:

$y_0(x) = 1 + x + x^2(2e - 4)$

$$y_1(x) = x^3(3-e) - h\int_0^x\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1}[y_0^1(\tau)y_0^1(\tau) - y_0^1(\tau)y_0^{111}(\tau) - 4\tau^2 + e^\tau(1+\tau^2-4\tau)]d\tau d\tau_1 d\tau_2 d\tau_3$$

And for m = 2, 3,…

$$y_m(x) = -h\int_0^x\int_0^{\tau_3}\int_0^{\tau_2}\int_0^{\tau_1}[\sum_{i=0}^{m-1}(y_i^1(\tau)y^1_{m-1-i}(\tau) - y_{m-1}(\tau)y^{111}_{m-1}(\tau))\aleph_m - (1-\aleph_m)4\tau^2 + (1-\aleph_m)e^\tau(1+\tau^2 - 4\tau)]d\tau d\tau_1 d\tau_2$$

Table 4: Errors of the first order approximate solution obtained by HAM and P-HAM for example 2

X	Analytical solution	Error of HAM	Error of P-HAM
0.0	1.0000000	0.0000	0.0000
0.1	1.115170918	5.2×10^{-4}	5.1×10^{-7}
0.2	1.261402758	1.7×10^{-4}	7.4×10^{-7}
0.3	1.439858808	2.9×10^{-3}	2.3×10^{-7}
0.4	1.651824698	3.7×10^{-3}	5.4×10^{-7}
0.5	1.898721271	4.4×10^{-3}	7.7×10^{-8}
0.6	2.182118800	4.1×10^{-3}	4.6×10^{-7}
0.7	2.503752707	2.9×10^{-3}	1.3×10^{-6}
0.8	2.865540928	1.9×10^{-3}	7.6×10^{-7}
0.9	3.269603111	6.1×10^{-4}	5.1×10^{-7}
1.0	3.718281828	0.0000	0.0000

Table 5: Different h-values for all x

X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
h	-1.0	-61.05	-11.84	-4.035	-1.653	-0.7286	-0.3212	-0.1318	-0.0445	-0.00873	-1×10^{-9}

We conclude that we have achieved a good approximation with the numerical solution of the equation by using the first two terms only of the P-HAM series discussed above. It is evident that the overall errors can be made smaller by adding new terms of the series.

5. DISCUSSION AND CONCLUSION

In this work, the new homotopy analysis method (P-HAM) has been successively developed and used to solve both linear and non-linear boundary value problems. The P-HAM does not required us to calculate the unknown constant which is usually a derivative at the boundary. All numerical approximation by P-HAM are compared with the results in many other methods such as Homotopy perturbation method, Adomian decomposition method and homotopy analysis method. From the results as illustrative examples, it may be concluded that this technique is very powerful and efficient in finding the analytical solutions for a large class of differential equations.

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