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PRIMITIVITY OF PERFECT RESIDUUM OF PERMUTATION GROUPS

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Abstract

In this paper, the construction of permutation groups which include symmetric groups, alternating groups, dihedral groups and the groups generated by the semidirect product (wreath products) of two permutation groups. The perfect residuum of the constructed groups has been obtained and their primitivity status has been investigated which enable us to formulate some results on such properties concerning the groups. A standard computer program, Groups algorithm and programming (GAP) has been employed in enhancing and validating the result obtained.

Keywords: Primitive groups, Perfect Residuum, permutation groups.

Introduction

The concept of residuum is very important in the theory of permutation groups more especially on investigating the solvability status of groups. The research adopted the concept of M. Bello et all (2017), work on a numerical search for polycyclic and locally nilpotent permutation groups.

Definition 1.1

A smallest normal subgroup of a group that has solvable factor group is called a perfect residuum of the group.

Definition 1.2

A subgroup N of a group G is normal in G if the left and right cosets are the same, that is if $gH = Hg \ \forall g \in G$ and a subgroup H of G.

Definition 1.3

A group G is said to act on a set X when there is a map $\emptyset: G \times X \rightarrow X$ such that the following conditions holds for all elements $x \in X$.

i. $\emptyset(e, x) = x$ where e is the identity element of G
ii. $\emptyset(g, \emptyset(h, x)) = \emptyset(gh, x) \ \forall \ g, h \in G$

Definition 1.4
A group action is transitive if it possess only a single group orbit. That is for every pair of elements x and y, there is a group element g $\ni gx = y$. A group is said to be intransitive if it is not transitive.
If for every two pairs of points x_1, x_2 and y_1, y_2 there is a group element $\ni gx_i = y_i$, then the group action is called doubly transitive. Similarly, a group action can be triply transitive and in general, a group action is k-transitive if every set $\{x_1, x_2, \dots, x_k\}$ of 2k distinct elements has a group element $g \ni gx_i = y_i$
An action is k-fold transitive if for any k-tuples of distinct elements $\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ there is $g \in G \ni y_i = (x_i, g),\ i = 1,2, \dots, k$
Definition 1.5
A group action is primitiveif there is no non-trivial partition of X preserved by the group G. A doubly transitive group action is primitive and a primitive action is transitive, but neither the, converse holds.
Definition 1.6
Let G be a transitive group. A subset X of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g \epsilon G$ either $Xg = X$ or Xg and X are disjoint. In particular, 1- element subsets of $\Omega$and the empty set are obviously sets of imprimitivity of every group G on Ω; these are called trivial sets of imprimitivity. We say that G is primitive on Ω if the only sets of imprimitivity are the trivial ones; otherwise G is imprimitive on Ω
Definition 1.7
The factor group of the normal subgroup N in a group G written as G/N is the set of cosets of N in G.
Definition 1.8
A composition series for a group G is a finite chain of subgroups
$G = G_0 > G_1 > G_2 > \dots > G_n = (1)$
such that, for $i = 0, 1, \dots, n-1, G_{i+1}$ is a normal subgroup of G_i and the quotient group G_i / G_{i+1} is simple.
The quotient groups $G_0 / G_1, G_1 / G_2, \dots, G_{-1} / G_n$
are called the composition factors of G.
Example
Let $G = S_4$, and consider the following chain of subgroups:
$S_4 > A_4 > V_4 > \langle(1\ 2)(3\ 4)\rangle > 1$ . . . . (4.1)
We know that $A_4 \le S_4$and $V_4 \le A_4$. Since V_4 is an abelian group,
$\langle(1\ 2)(3\ 4)\rangle \trianglelefteq V_4$. Certainly$1 \trianglelefteq \langle(1\ 2)(3\ 4)\rangle$. Hence (4.1) is a series of subgroups, each normal in the previous one. We can calculate the order of each subgroup, and hence calculate the order of the quotient groups:
$|S_4/A_4| = 2$
$|A_4/V_4| = 3$
$|V_4/\langle(1\ 2)(3\ 4)\rangle| = 2$
$|\langle(1\ 2)(3\ 4)\rangle| = 2.$
Thus the quotients are all of prime order. We now make use of the fact that a group G of prime order p is both cyclic and simple (see Example 3.6), to see that the factors for the series (4.1) are cyclic simple groups. Thus (4.1) is composition series for S4, with composition factors
C_2, C_3, C_2, C_2
Definition 1.9
Let G be a group. A subnormal series of G is a finite chain of subgroups
$G = G_0 > G_1 > G_2 > \dots > G_n = (1)$
such that G_{i+1} is a normal subgroup of G_i for $i = 0, 1, \dots, n-1$. The collection of quotient groups $G_0/G_1, G_1/G_2, \dots, G_{-1}/G_n$ are the factors of the series, and the length of the series is n.
Note that we do not require each subgroup in the subnormal series to be normal in the whole group, only that it is normal in the previous subgroup in the chain.
A normal series is a series where G_i is a normal subgroup of G for all i. Note also that the length n is also the number of factors occurring.
We will be interested in three different types of subnormal series in this research, and for all three we will require special properties of the factors. The first case is where the factors are all required to be simple groups.
Definition 1.10
The series of subgroups $G_0, G_1, G_2, \dots, G_n$ Such that $G = G_n \supset G_{n-1} \supset G_{n-2} \supset \dots \supset G_1 \supset G_0 = \{1\}$ where G_i/G_{i+1} is abelianis called a solvable series.

Definition 1.11 (Milne, J.S, 2009)

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A group *G* is solvable if there is a finite collection of groups $G_0 G_1 \ldots G_n$ such that $(1) = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ where $G_i \trianglelefteq G$ and G_{i+1}/G_i is abelian.If $|G| = 1$ then *G*is considered as solvable group.

Theorem 1.1 (Audu M.S, 2003)
Let C and D be permutation groups on Γand Δrespectively. Let C^{Δ}be the set of all maps of Δ into the permutation group C. That is $C^{\Delta} = \{f: \Delta \longrightarrow C\} \forall f_1 f_2 \in C$in _. Let $f_1 f_2$ in C^{Δ}be defined $\forall\ \delta$ in$\Delta$by
$(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$
With respect to this operation of multiplicationC^{Δ}acquire a structure of a group.

Proof

(i) C^{Δ} is non-empty and is closed with respect to multiplication. If $f_1 f_2 \in C^{\Delta}$then $f_1(\delta) f_2(\delta) \in c$. Hence $f_1(\delta). f_2(\delta) \in C$. This implies that $(f_1 f_2)(\delta) \in C$and so $f_1 f_2 \in C^{\Delta}$
(ii) Since multiplication is associative so also is the multiplication in C^{Δ}.
(iii) The identity element in C^{Δ} is the map e : Δ → C given by e(δ)=1 for all δ ∈ Δ and 1 ∈ C.
(iv) Every element f ∈$C^{\Delta}$is defined for all δ ∈ Δ by $f^{-1}(\delta) = f(\delta)^{-1}$. Thus CΔ is a group with respect to the multiplication defined above. (We denote this group by P).

Lemma 1.2 (Audu M.S, 2003)
Assume that *D* acts on *P* as follows: $f^d(\delta) = f(\delta d^{-1}) forall \delta \in \Delta d \in D$. Then D acts on P as a group.
Proof
Take $f f_1 f_2 \in P$and $d d_1 d_2 \in D$ then
(i) $(f^{d_1})^{d_2}(\delta) = f^{d_1}(\delta d - 1\ 2) = f(\delta d_2^{-1}) = f(\delta d_2^{-1} d_1^{-1}) = f^{d_1 d_2}(\delta)$
(ii) $f^1(\delta) = f(\delta 1^{-1}) = f(\delta)$
(iii) $(f_1 f_2)^d(\delta) = f_1 f_2(\delta d^{-1}) = f_1(\delta d^{-1}) f_2(\delta d^{-1}) = f_1{}^d(\delta) f_2{}^d(\delta)$. Thus D acts on P as a group

Theorem 1.3 (Audu M.S, 2003)
Let *D* act on *P* as a group. Then the set of all ordered pairs $(f d)$ with $f \in P$ and $d \in D$ acquires the structure of a group when we define for all $f_1\ f_2 \in P and\ d_1 d_2 \in D (f_1 d_1)(f_2 d_2) = (f_1 f_2{}^{d_1^{-1}}\ d_1 d_2)$

Proof
(i) Closure property follows from the definition of multiplication.
(ii) Take $f_1 f_2 f_3 \in P and\ d_1 d_2 d_3 \in D$. Then

$$[(f_1 d_1)(f_2 d_2)](f_3 d_3) = \left(f_1 f_2{}^{d_1^{-1}}\ d_1 d_2\right)(f_3 d_3)$$
$$= \left(f_1 f_2{}^{d_1^{-1}}\ f_3{}^{(d_1 d_2)^{-1}}\ d_1 d_2 d_3\right)$$
$$= \left(f_1 f_2{}^{d_1^{-1}}\ f_3{}^{d_2^{-1} d_1^{-1}}\ d_1 d_2 d_3\right)$$

Also we have in the same manner that

$$[(f_1 d_1)(f_2 d_2)](f_3 d_3) = (f_1 d_1)\left(f_2 f_3{}^{d_2^{-1}}\ d_2 d_3\right)$$
$$= \left(f_1 (f_2 f_3{}^{d_2^{-1}}) d_1{}^{-1} d_1 d_2 d_3\right)$$

$$= \left(f_1 f_2{}^{d_1^{-1}}\ f_3{}^{d_2^{-1} d_1^{-1}}\ d_1 d_2 d_3\right).$$

hence multiplication is associative.

(iii) We know that for every $f \in P f^{-1} = f$. Now for every $d \in D$ the map $f \rightarrow f^d$ is an automorphism of P. Also if e is the identity element in P then $e^d = e$. Also $(f^{-1})d = (f^d)^{-1}$. Now $(fd)(e\,1) = (f e^{d-1} d1) = (f e^{d-1} d) = (f(e^{-1})\ d) = (fd)$. Also using the reverse order we have that
$(e\,1)(fd) = (e f^{1^{-1}}\ 1d) = (e f d) = (fd)$ Thus identity element exists.
(iv) $(fd)((f^{-1})^d d^{-1}) = (f(f^{-1})^d)^{-1} d d^{-1}) = (f(f^{-1})^{d d^{-1}}\ d d^{-1})$
$= (f(f^{-1})^1 d d^{-1}) = (e\,1)$

Also

$$((f^{-1})^d d^{-1})(fd) = ((f^{-1})^d f^d d^{-1} d)$$
$$= (ff^{-1})^d d^{-1} d) = (\theta^d 1) = (\theta 1)$$

Thus when D acts on P the set of all ordered pairs $(fd)\ with f \in D d \in D$ is a group if we define $(f_1 d_1)(f_2 d_2) = f_1 f_2^{d_1^{-1}} (d_1 d_2)$.In what follows we supply a formal definition of Wreath Product of permutation groups.

WREATH PRODUCT (Audu M.S, 2003)

The Wreath product of C by D denoted by W = C wr D is the semidirect product of P by D so that $W = \{(fd) \mid f \in P d \in D\}$ with multiplication in W defined as $(f_1 d_1)(f_2 d_2) = f_1 f_2^{d_1^{-1}} (d_1 d_2)$ for all $f_1 f_2 \in P and\ d_1 d_2 \in D$. Henceforth we write f d instead of (fd) for elements of W.

Theorem 1.4 (Audu M.S, 2003)

Let D act on P as $f^d(\delta) = f(\delta d^{-1})$ where $f \in P d \in D and \delta \in \Delta$. Let W be the group of all juxtaposed symbols f d with $f \in P d \in D$ and multiplication given by $(f_1 d_1)(f_2 d_2) = f_1 f_2^{d_1^{-1}} (d_1 d_2)$. Then W is a group called the semi-direct product of P by D with the defined action.

Based on the forgoing we note the following:

- If C and D are finite groups then the wreath product W determined by an action of D on a finite set is a finite group of order $|W| = |C|^{|\Delta|} . |D|$.
- P is a normal subgroup of W and D is a subgroup of W.
- The action of W on Γ × Δ is given by $(\alpha\beta) fd = (\alpha f(\beta)\beta d) where \alpha \in \Gamma and \beta \in \Delta$.

We shall at this point identify the conditions under which a sup group will be soluble or nilpotent and study them for further investigation.

Theorem 1.5 (Thanos G., 2006)

G is solvable if and only if $G^{(n)} = 1$, for some n.

Proposition 2.1

Let *G* be solvable and $H \leq G$. Then

1. *H* is solvable.
2. If $H \lhd G$, then G/H is solvable.

Proof

Start from a series with abelian slices. $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$. Then $H = H \cap G_0 \trianglerighteq H \cap G_1 \trianglerighteq \cdots \trianglerighteq H \cap G_n = \{1\}$. When *H* is normal, we use the canonical projection $\pi : G \to G/H$ to get $G/H = \pi(G_0) \trianglerighteq \cdots \trianglerighteq \pi(G_n) = \{1\}$; the quotients are abelian as well, so G/H is still solvable.

Proposition 1.6

Let $N \trianglelefteq G$. Then G is solvable if and only if N and G/N are solvable.

Proof

(⇒) This is obvious by Proposition 2.1.

(⇐) Stick together a series for N with abelian slices with the lift to *G* of a series for G/N, using the fourth isomorphism law.

RESULT

2.1 Symmetric groups

2.1.1 Consider the symmetric group acting on $\Omega_1 = \{1,2,3\}$

$S_3 = \{(1), (23), (13), (132), (123), (12)\}$ with primitive perfect residuum (1)

2.1.2 Consider the symmetric group acting on $\Omega_2 = \{1,2,3,4\}$

$S_4 = \{(1), (34), (24), (243), (234), (23), (14), (143), (142), (1432), (1423), (14)(23), (124), (1243), (12), (12)(34), (123), (1234), (134), (13), (1342), (132), (13)(24), (1324)\}$

with primitive perfect residuum (1)

2.1.3 Consider the symmetric group acting on $\Omega_3 = \{1,2,3,4,5\}$

$S_5 = \{(1), (45), (35), (354), (345), (34), (25), (254), (253), (2543), (2534), (25), (34), (235), (2354), (23), (23)(45), (234), (2345), (245), (24), (2453), (243), (24)(35), (2435), (15), (154), (153), (1543), (1534), (15)(34), (152), (1542), (1532), (15432), (15342), (152)(34), (1523), (15423), (15)(23), (154)(23), (15)$

(234), (15234), (1524), (15)(24), (15324), (15)(243), (153)(24), (15243), (125), (1254), (1253), (12543), (12534), (125)(34), (12), (12)(45), (12)(35), (12)(354), (12)(345), (12)(34), (123), (123)(45), (1235), (12354), (12345), (1234), (124), (1245), (124)(35), (12435), (12453), (1243)(135), (1354), (13), (13)(45), (134), (1345), (1352), (13542), (132), (132)(45), (1342), (13452), (13)(25), (13)(254), (1325), (13254), (13425), (134)(25), (13524), (135)(24), (1324), (13245), (13)(24), (13)(245), (145), (14), (1453), (143), (14)(35), (1435), (1452), (142), (14532), (1432), (142)(35), (14352), (14523), (1423), (145)(23), (14)(23), (14235), (14)(235), (14)(25), (1425), (14)(253), (14325), (14253), (143)(25)}

with primitive perfect residuum A_5

2.2 Alternating group

2.2.1 Consider the alternating group acting on $\Omega_4 = \{1,2,3\}$

$A_3 = \{(1), (123), (132)\}$ with primitive perfect residuum (1)

2.2.2 Consider the alternating group acting on $\Omega_4 = \{1,2,3,4\}$

$A_4 = \{(1), (243), (234), (143), (14)(23), (142), (134), (132), (13)(24), (124), (12)(34), (123)\}$

with primitive perfect residuum (1)

2.2.3 Consider the alternating group acting on $\Omega_4 = \{1,2,3,4,5\}$

$A_5 = \{$(1), (354), (345), (254), (25)(34), (253), (245), (243), (24)(35), (235), (23)(45), (234), (154), (15)(34), (153), (15)(24), (15243), (15324), (152), (15432), (15342), (15234), (15)(23), (15423), (145), (143),

(14)(35), (142), (14352), (14532), (14)(25), (14325), (14253), (14523), (14)(23), (14235), (125), (12543), (12534), (12)(45), (12)(34), (12)(35), (124), (12435), (12453), (123), (12354), (12345), (135), (13)(45), (134), (13542), (13452), (132), (13524), (13245), (13)(24), (13)(25), (13254), (13425)}

with primitive perfect residuum A_5

2.3 Dihedral group

2.3.1 Consider the dihedral group acting on $\Omega_4 = \{1,2,3\}$

$D_6 = \{(1), (23), (132), (13), (123), (12)\}$ with primitive perfect residuum (1)

2.3.2 Consider the dihedral group acting on $\Omega_4 = \{1,2,3,4\}$

$D_8 = \{(1), (24), (13)(24), (13), (1432), (14)(23), (1234), (12)(34)\}$ with primitive perfect residuum (1)

2.3.3 Consider the dihedral group acting on $\Omega_4 = \{1,2,3,4,5,6,7\}$

$D_{14} = \{$(1), (27)(36)(45), (1765432), (17)(26)(35), (1642753), (16)(25)(34), (1526374), (15)(24)(67), (1473625), (14)(23)(57), (1357246), (13)(47)(56), (1234567), (12)(37)(46)}

with primitive perfect residuum (1)

2.4 Wreath product

2.4.1 Consider the permutation groups M_1 and L_1

$M_1 = \{(1), (123), (132)\}$, $L_1 = \{(1), (45)\}$ acting on the sets $S_1 = \{1,2,3\}$ *and* $\Delta_1 = \{4,5\}$ respectively.

Let Let $P = L_1^{\Delta_1} = \{f: \Delta_1 \rightarrow L_1\}$ *then* $|P| = |M_1|^{\Delta_1} = 3^2 = 9$

We can easily verify that $G_1$is a group with respect to the operations

$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1)$ *where* $\delta_1 \in \Delta_1$.

The wreath product of P_1 and Q_1 is given by W_1, where

$W_1 = \{$(1), (465), (456), (132), (132)(465), (132)(456), (123), (123)(465), (123)(456), (14)(25)(36), (143625), (142536), (163524), (162435), (16)(24)(35), (152634), (15)(26)(34), (153426)}

with imprimitive perfect residuum (1)

2.4.2 Consider the permutation groups M_1 and L_1

$M_1 = \{(1), (15432), (14253), (13524), (12345)\}$

, $L_1 = \{(1), (678), (687)\}$ acting on the sets $S_2 = \{1,2,3,4,5\}$ *and* $\Delta_2 = \{6,7,8\}$ respectively.

Let Let $P = L_2^{\Delta_2} = \{f: \Delta_2 \rightarrow L_2\}$ *then* $|P| = |M_2|^{\Delta_2} = 5^3 = 125$

We can easily verify that $G_1$is a group with respect to the operations

$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1)$ *where* $\delta_1 \in \Delta_1$.

The wreath product of M_2 and L_2 is given by W_2, where

$W_2 = \{$(1), (1115141312), (1114121513)(1113151214)(1112131415)(610 9 8 7)(610 9 8 7)(1115141312)(610 9 8 7)(1114121513)(610 9 8 7)(1113151214)(610 9 8 7)(1112131415)(6 9 710 8)(6 9 710 8)(1115141312)(6 9 710 8)(1114121513)(6 9 710 8)(1113151214)(6 9 710 8)(1112131415)(6 810 7 9)(6 810 7 9)(1115141312)(6 810 7 9)(1114121513)(6 810 7 9)(1113151214)(6 810 7 9)(1112131415)(6 7 8 910)(6 7 8 910)(1115141312)(6 7 8 910)

(1114121513)(6 7 8 910)(1113151214)(6 7 8 910)(1112131415)(15432)(1 5 4 3 2)
(1115141312)(1 5 4 3 2)(1114121513)(1 5 4 3 2)(1113151214)(1 5 4 3 2)(1112131415)
(1 5 4 3 2)(610 9 8 7)(1 5 4 3 2)(610 9 8 7)(1115141312)(1 5 4 3 2)(610 9 8 7)
(1114121513)(1 5 4 3 2)(610 9 8 7)(1113151214)(1 5 4 3 2)(610 9 8 7)(1112131415)
(1 5 4 3 2)(6 9 710 8)(1 5 4 3 2)(6 9 710 8)(1115141312)(1 5 4 3 2)(6 9 710 8)
(1114121513)(1 5 4 3 2)(6 9 710 8)(1113151214)(1 5 4 3 2)(6 9 710 8)(1112131415)
(1 5 4 3 2)(6 810 7 9)(1 5 4 3 2)(6 810 7 9)(1115141312)(1 5 4 3 2)(6 810 7 9)
(1114121513)(1 5 4 3 2)(6 810 7 9)(1113151214)(1 5 4 3 2)(6 810 7 9)(1112131415)
(1 5 4 3 2)(6 7 8 910)(1 5 4 3 2)(6 7 8 910)(1115141312)(1 5 4 3 2)(6 7 8 910)
(1114121513)(1 5 4 3 2)(6 7 8 910)(1113151214)(1 5 4 3 2)(6 7 8 910)(1112131415)
(14253)(1 4 2 5 3)(1115141312)(1 4 2 5 3)(1114121513)(1 4 2 5 3)(1113151214)
(1 4 2 5 3)(1112131415)(1 4 2 5 3)(610 9 8 7)(1 4 2 5 3)(610 9 8 7)(1115141312)
(1 4 2 5 3)(610 9 8 7)(1114121513)(1 4 2 5 3)(610 9 8 7)(1113151214)(1 4 2 5 3)
(610 9 8 7)(1112131415)(1 4 2 5 3)(6 9 710 8)(1 4 2 5 3)(6 9 710 8)(1115141312)
(1 4 2 5 3)(6 9 710 8)(1114121513)(1 4 2 5 3)(6 9 710 8)(1113151214)(1 4 2 5 3)
(6 9 710 8)(1112131415)(1 4 2 5 3)(6 810 7 9)(1 4 2 5 3)(6 810 7 9)(1115141312)
(1 4 2 5 3)(6 810 7 9)(1114121513)(1 4 2 5 3)(6 810 7 9)(1113151214)(1 4 2 5 3)
(6 810 7 9)(1112131415)(1 4 2 5 3)(6 7 8 910)(1 4 2 5 3)(6 7 8 910)(1115141312)
(1 4 2 5 3)(6 7 8 910)(1114121513)(1 4 2 5 3)(6 7 8 910)(1113151214)(1 4 2 5 3)
(6 7 8 910)(1112131415)(13524)(1 3 5 2 4)(1115141312)(1 3 5 2 4)(1114121513)
(1 3 5 2 4)(1113151214)(1 3 5 2 4)(1112131415)(1 3 5 2 4)(610 9 8 7)(1 3 5 2 4)
(610 9 8 7)(1115141312)(1 3 5 2 4)(610 9 8 7)(1114121513)(1 3 5 2 4)(610 9 8 7)
(1113151214)(1 3 5 2 4)(610 9 8 7)(1112131415)(1 3 5 2 4)(6 9 710 8)(1 3 5 2 4)
(6 9 710 8)(1115141312)(1 3 5 2 4)(6 9 710 8)(1114121513)(1 3 5 2 4)(6 9 710 8)
(1113151214)(1 3 5 2 4)(6 9 710 8)(1112131415)(1 3 5 2 4)(6 810 7 9)(1 3 5 2 4)
(6 810 7 9)(1115141312)(1 3 5 2 4)(6 810 7 9)(1114121513)(1 3 5 2 4)(6 810 7 9)
(1113151214)(1 3 5 2 4)(6 810 7 9)(1112131415)(1 3 5 2 4)(6 7 8 910)(1 3 5 2 4)
(6 7 8 910)(1115141312)(1 3 5 2 4)(6 7 8 910)(1114121513)(1 3 5 2 4)(6 7 8 910)
(1113151214)(1 3 5 2 4)(6 7 8 910)(1112131415)(12345)(1 2 3 4 5)(1115141312)
(1 2 3 4 5)(1114121513)(1 2 3 4 5)(1113151214)(1 2 3 4 5)(1112131415)(1 2 3 4 5)
(610 9 8 7)(1 2 3 4 5)(610 9 8 7)(1115141312)(1 2 3 4 5)(610 9 8 7)(1114121513)
(1 2 3 4 5)(610 9 8 7)(1113151214)(1 2 3 4 5)(610 9 8 7)(1112131415)(1 2 3 4 5)
(6 9 710 8)(1 2 3 4 5)(6 9 710 8)(1115141312)(1 2 3 4 5)(6 9 710 8)(1114121513)
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13 6)(111 8 31310 515 7 212 9 414 6)(111 7 212 8 313 9 41410 515 6)(111 6 51510
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21210)(111 8 414 6 212 9 515 7 31310)(111 7 313 9 515 6 212 8 41410)(11510 514
9 413 8 312 7 211 6)(115 9 413 7 21110 514 8 312 6)(115 8 31210 514 7 211 9 4
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(11510 413 8 211 6 514 9 312 7)(115 9 312 6 514 8 21110 413 7)(115 8 211 9 312
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7 513 9 215 6 412 8)(114 9)(21510)(311 6)(412 7)(513 8)(114 8 513 7 412 6 31110
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6 311 7 412 8 513 9)(114 9 513 8 412 7 311 6 21510)(114 8 412 6 215 9 513 7 3
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11 6)(112 6)(213 7)(314 8)(415 9)(51110)(11210 511 9 415 8 314 7 213 6)
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(314 9)(41510)(511 6)(112 6 51110 415 9 314 8 213 7)(11210 415 8 213 6 511 9 3
14 7)(112 9 314 6 511 8 21310 415 7)(112 8 213 9 31410 415 6 511 7)(112 7 511 6
41510 314 9 213 8)(112 6 415 9 213 7 51110 314 8)(11210 314 7 511 9 213 6 415 8
)(112 9 21310 314 6 415 7 511 8)(112 8)(213 9)(31410)(415 6)(511 7)(112 7 41510
213 8 511 6 314 9)(112 6 314 8 51110 213 7 415 9)(11210 213 6 314 7 415 8 511
9)(112 9)(21310)(314 6)(415 7)(511 8)(112 8 511 7 415 6 31410 213 9)(112 7 314
9 511 6 213 8 41510)(112 6 213 7 314 8 415 9 51110)(11210)(213 6)(314 7)(415 8)
(511 9)(112 9 511 8 415 7 314 6 21310)(112 8 415 6 213 9 511 7 31410)(1 611)
(2 712)(3 813)(4 914)(51015)(1 611 51015 4 914 3 813 2 712)(1 611 4 914 2 712 5

1015 3 813)(1 611 3 813 51015 2 712 4 914)(1 611 2 712 3 813 4 914 51015)
(1 615 51014 4 913 3 812 2 711)(1 615 4 913 2 711 51014 3 812)(1 615 3 812 510
14 2 711 4 913)(1 615 2 711 3 812 4 913 51014)(1 615)(2 711)(3 812)(4 913)(51014)
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(1 613 3 815 51012 2 714 4 911)(1 613 2 714 3 815 4 911 51012)(1 613)(2 714)
(3 815)(4 911)(51012)(1 613 51012 4 911 3 815 2 714)(1 613 4 911 2 714 51012 3
815)(1 612 2 713 3 814 4 915 51011)(1 612)(2 713)(3 814)(4 915)(51011)
(1 612 51011 4 915 3 814 2 713)(1 612 4 915 2 713 51011 3 814)(1 612 3 814 510
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(3 712)(4 813)(5 914)(11014 4 812 2 615 5 913 3 711)(11014 3 711 5 913 2 615 4
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(11013 4 811 2 614 5 912 3 715)(11012 2 613 3 714 4 815 5 911)(11012)(2 613)
(3 714)(4 815)(5 911)(11012 5 911 4 815 3 714 2 613)(11012 4 815 2 613 5 911 3
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(11011 5 915 4 814 3 713 2 612)(11011 4 814 2 612 5 915 3 713)(11011 3 713 5 9
15 2 612 4 814)(11011 2 612 3 713 4 814 5 915)(1 914 4 712 21015 5 813 3 611)

(1 914 3 611 5 813 21015 4 712)(1 914 21015 3 611 4 712 5 813)(1 914)(21015)
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711)(1 913 21014 3 615 4 711 5 812)(1 913)(21014)(3 615)(4 711)(5 812)
(1 913 5 812 4 711 3 615 21014)(1 913 4 711 21014 5 812 3 615)(1 912 21013 3 6
14 4 715 5 811)(1 912)(21013)(3 614)(4 715)(5 811)(1 912 5 811 4 715 3 614 21013)
(1 912 4 715 21013 5 811 3 614)(1 912 3 614 5 811 21013 4 715)(1 911)(21012)
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613)(1 911 3 613 5 815 21012 4 714)(1 911 21012 3 613 4 714 5 815)(1 915 5 814
4 713 3 612 21011)(1 915 4 713 21011 5 814 3 612)(1 915 3 612 5 814 21011 4 713
)(1 915 21011 3 612 4 713 5 814)(1 915)(21011)(3 612)(4 713)(5 814)(1 813 31015
5 712 2 914 4 611)(1 813 2 914 31015 4 611 5 712)(1 813)(2 914)(31015)(4 611)
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715 31013)(1 811 31013 5 715 2 912 4 614)(1 811 2 912 31013 4 614 5 715)
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(1 712 2 813 3 914 41015 5 611)(1 712)(2 813)(3 914)(41015)(5 611)(1 712 5 611 4
1015 3 914 2 813)(1 712 41015 2 813 5 611 3 914)(1 712 3 914 5 611 2 813 41015)
(1 711)(2 812)(3 913)(41014)(5 615)(1 711 5 615 41014 3 913 2 812)(1 711 41014 2
812 5 615 3 913)(1 711 3 913 5 615 2 812 41014)(1 711 2 812 3 913 41014 5 615)
(1 715 5 614 41013 3 912 2 811)(1 715 41013 2 811 5 614 3 912)(1 715 3 912 5 6
14 2 811 41013)(1 715 2 811 3 912 41013 5 614)(1 715)(2 811)(3 912)(41013)(5 614)
(1 714 41012 2 815 5 613 3 911)(1 714 3 911 5 613 2 815 41012)(1 714 2 815 3 9
11 41012 5 613)(1 714)(2 815)(3 911)(41012)(5 613)(1 714 5 613 41012 3 911 2 815)
(1 713 3 915 5 612 2 814 41011)(1 713 2 814 3 915 41011 5 612)(1 713)(2 814)
(3 915)(41011)(5 612)(1 713 5 612 41011 3 915 2 814)(1 713 41011 2 814 5 612 3
915)}

with imprimitive perfect residuum (1)

SUMMARY OF RESULT

- The perfect residuum of a solvable group is always identity while for unsolvable group is not trivial.
- The perfect residuum of permutation group is prirmitive

3.1 VALIDATION

3.1.1 Algorithm for the result in 2.1.1

```
gap> S3:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> P1:=PerfectResiduum(S1);
Group(())
gap> IsPrimitive(P1);
true
gap>quit;
```

3.1.2 Algorithm for the result in 2.1.2

```
gap> S4:=SymmetricGroup(4);
Sym( [ 1 .. 4 ] )
gap> P2:=PerfectResiduum(S4);
Group(())
gap> IsPrimitive(P2);
true
gap>quit;
```

3.1.3 Algorithm for the result in 2.1.3

```
gap> S5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )
gap> P3:=PerfectResiduum(S5);
Alt( [ 1 .. 5 ] )
gap> IsPrimitive(P3);
true
gap>quit;
```

3.1.4 Algorithm for the result in 2.2.1

```
gap> A3:=AlternatingGroup(3);
Alt( [ 1 .. 3 ] )
gap> P4:=PerfectResiduum(A3);
Group(())
gap> IsPrimitive(P4);
true
gap>quit;
```

3.1.5 Algorithm for the result in 2.2.2

```
gap> A4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> P5:=PerfectResiduum(A4);
Group(())
gap> IsPrimitive(P5);
true
gap>quit;
```

3.1.6 Algorithm for the result in 2.2.3

```
gap> A5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> P6:=PerfectResiduum(A5);
Alt( [ 1 .. 5 ] )
gap> IsPrimitive(P6);
true
gap>quit;
```

3.1.7 Algorithm for the result in 2.3.1

```
gap> D6:=DihedralGroup(IsGroup,6);
Group([ (1,2,3), (2,3) ])
gap> P7:=PerfectResiduum(D3);
Group(())
gap> IsPrimitive(P7);
true
gap>quit;
```

3.1.8 Algorithm for the result in 2.3.2

```
gap> D8:=DihedralGroup(IsGroup,8);
Group([ (1,2,3,4), (2,4) ])
gap> P8:=PerfectResiduum(D2);
Group(())
gap> IsPrimitive(P8);
true
gap>quit;
```

3.1.9 Algorithm for the result in 2.3.3

```
gap> D14:=DihedralGroup(IsGroup,14);
Group([ (1,2,3,4,5,6,7), (2,7)(3,6)(4,5) ])
gap> P9:=PerfectResiduum(D4);
Group(())
gap> IsPrimitive(P9);
true
gap>quit;
```

3.1.10 Algorithm for the result in 2.4.1

```
gap> M1:=Group((1,2,3));
Group([ (1,2,3) ])
gap> L1:=Group((4,5));
Group([ (4,5) ])
gap> W1:=WreathProduct(M1,L1);
Group([ (1,2,3), (4,5,6), (1,4)(2,5)(3,6) ])
gap> P10:=PerfectResiduum(W1);
Group(())
gap> IsPrimitive(P10);
true
gap>quit;
```

3.1.11 Algorithm for the result in 2.4.2

```
gap> M2:=Group((1,2,3,4,5));
Group([ (1,2,3,4,5) ])
gap> L2:=Group((6,7,8));
Group([ (6,7,8) ])
gap> W2:=WreathProduct(M2,L2);
Group([ (1,2,3,4,5), (6,7,8,9,10), (11,12,13,14,15), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15) ])
gap> P11:=PerfectResiduum(W2);
Group(())
gap> IsPrimitive(P11);
true
gap>quit;
```

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