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On Sylow Subgroups of Permutation Groups

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Abstract
A research on Sylow Subgroups of permutation groups is carried out in this paper. The research investigates the nature of Sylow subgroups of permutation groups and examines its behaviors.

Introduction
The concept of Sylow subgroups originate from Sylow theorems which are collection of theorems that give detailed information about the number of subgroups of fixed order that a given finite group contains. The Sylow theorems assert a partial converse to Langrange's theorem.

The research construct different types of permutation groups, which includes symmetric groups, dihedral groups and groups generated by wreath products of two permutation groups and investigates their behavior in terms of normality in the parent group, primitivity and transitivity of the sylow subgroups of these groups. The research adopt the concept of M. Bello et all (2017), work on a numerical search for polycyclic and locally nilpotent permutation groups.

Definition 1.1
Let G be a group, and let p be a prime number
- A group of order p^k for some $k \geq 1$ is called a p-group.
- A subgroup of $p^k$for some $k \geq 1$ is called a p-subgroup.
- If $|G| = p^\alpha m$ where p does not divide m, then a subgroup of order p^α is called a Sylow p- subgroup of G.

The illustrations of the above are given below:
(i) $G = \{(1),(34),(12),(12)(34),(13)(24),(1324),(1423),(14)(23)\}$
G is a p-group since, $|G| = 2^3$.
Some ofthe subgroups of G are as follows:

$$K_0 = (1)$$
$$K_1 = \{(1),(12)\}$$
$$K_2 = \{(1),(34)\}$$
$$K_3 = \{(1),(12)(34)\}$$
$$K_4 = \{(1),(13)(24)\}$$
$$K_5 = \{(1),(14)(23)\}$$
$$K_6 = \{(1),(12),(34),(12)(34)\}$$

The subgroups K_1, K_2, K_3, K_4, K_5 are p-subgroups, since they have order p and K_6 whose order is p-power.
(ii) $G = \{(1),(465),(456),(132),(132)(465),(132)(456),(123),(123)(465),$
$(123)(456),(14)(25)(36),(143625),(142536),(163524),(162435),$
$(16)(24)(35),(152634),(15)(26)(34),(153426)\}$
$|G| = 2 \times 3^2$.
The subgroups of G are as follows:

$$T_0 = (1)$$
$$T_1 = \{(1),(14)(25)(36)\}$$
$$T_2 = \{(1),(15)(26)(34)\}$$
$$T_3 = \{(1),(16)(24)(35)\}$$
$$T_4 = \{(1),(465),(456)\}$$
$$T_5 = \{(1),(123),(132)\}$$
$$T_6 = \{(1),(132)(465),(123)(456)\}$$
$$T_7 = \{(1),(123)(465),(132)(456)\}$$
$$T_8 = \{(1),(14)(25)(36),(132)(465),(162435),(123)(456),(153426)\}$$
$$T_9 = \{(1),(15)(26)(34),(132)(465),(142536),(123)(456),(163524)\}$$
$$T_{10} = \{(1),(16)(24)(35),(132)(465),(152634),(123)(456),(143625)\}$$
$$T_{11} = \{(1),(132)(456),(123)(465),(14)(25)(36),(16)(24)(35),(15)(26)(34)\}$$
$$T_{12} = \{(1),(465),(456),(123)(456),(123),(123)(465),(132)(465),(132)(456),(132)\}$$
$$T_{13} = \{(1),(465),(456),(132),(132)(465),(132)(456),(123),(123)(465),(123)(456),$$

$(14)(25)(36), (143625), (142536), (163524), (162435), (16)(24)(35), (152634),$
$(15)(26)(34), (153426)\}$

The Sylow 2-subgroups are T_1, T_2, T_3 and the Sylow 3-subgroupT_{12}

Definition 1.2

A subgroup N of a group G is normal in G if the left and right cosets are the same, that is if $gH = Hg \ \forall g \in G$ and a subgroup H of G.

Definition 1.3

A group G is said to act on a set X when there is a map $\emptyset: G \times X \to X$ such that the following conditions holds for all elements $x \in X$.

i. $\emptyset(e, x) = x$ where e is the identity element of G
ii. $\emptyset(g, \emptyset(h, x)) = \emptyset(gh, x) \ \forall \ g, h \ \in G$

Definition 1.4

A group action is transitive if it possess only a single group orbit. That is for every pair of elements x and y, there is a group element g $\ni gx = y$. A group is said to be intransitive if it is not transitive.

If for every two pairs of points x_1, x_2 and y_1, y_2 there is a group element $\ni gx_i = y_i$, then the group action is called doubly transitive. Similarly, a group action can be triply transitive and in general, a group action is k-transitive if every set $\{x_1, x_2, \dots, x_k\}$ of 2k distinct elements has a group element $g \ \ni gx_i = y_i$

An action is k-fold transitive if for any k-tuples of distinct elements $\{x_1, x_2, \dots, x_k\}$ *and* $\{y_1, y_2, \dots, y_k\}$ there is $g \in G \ \ni \ y_i = (x_i, g), \ i = 1,2, \dots, k$

Definition 1.5

A group action is primitiveif there is no non-trivial partition of X preserved by the group G. A doubly transitive group action is primitive and a primitive action is transitive, but neither the, converse holds.

Definition 1.6

Let G be a transitive group. A subset X of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g\epsilon G$ either $Xg = X$ *or* Xg *and* X *are disjoint*. In particular, 1- element subsets of Ωand the empty set are obviously sets of imprimitivity of every group G on Ω; these are called trivial sets of imprimitivity. We say that G is primitive on Ω if the only sets of imprimitivity are the trivial ones; otherwise G is imprimitive on Ω

Theorem 1.1 (Fundamental Theorem of Finite Abelian Groups) (Milne, J.S, 2009)

Every finite abelian group G is isomorphic to a direct product of cyclic groups of prime power order.
Furthermore any two such decompositions have the same number of factors of each order.
In other words every finite Abelian group G is isomorphic to a group with the form

$$\mathbb{Z}_{p_1{}^{n_1}} \oplus \mathbb{Z}_{p_2{}^{n_2}} \dots \oplus \mathbb{Z}_{p_k{}^{n_k}}$$

such that the $p'_i s$ do not have to be distinct primes and the prime-powers

$$p_1{}^{n_1} p_2{}^{n_2} \ \dots . p_k{}^{n_k}$$

Theorem 1.2 (Lagrange's Theorem)

Let G be a finite group and $H \leq G$. Then $|H|$divides $|G|$ and $|G: H| = \frac{|G|}{|H|}$

Example

If $|G|$ = 14 then the only possible orders for a subgroup are 1, 2, 7, and 14

Theorem 1.3 (Milne, J.S, 2009)

Let G be a finite group. The following conditions on G areequivalent:
(i) G is nilpotent;
(ii) everySylow subgroup of G is normal;
(iii) G is a direct product of p-groups (for various primes p)

Lemma 1.4 (Hall. M. Jr., 1959)

If G is a group then the Frattini subgroup $\Phi(G)$ is a characteristic subgroup of G

Theorem 1.5 (Cauchy's theorem for abelian groups)

Let A be a finite abelian group. If p is a prime number that divides its order, then A must have an element of order p.

Theorem 1.6 (Sylow's first theorem)

Let G be a finite group and p be a prime number such that its power by $\propto$ is the largest power that will divide $|G|$Then there exist at least one subgroup of order p^α. Such groups are called syllow p-subgroups.

Proof

We divide the proof into two cases.

Case one: p divides the order of the centre $Z(G)$. By Cauchy's theorem for abelian groups,$Z(G)$must have an element of order p, say,a. By induction, the quotient group $G/\langle a \rangle$ must have a subgroup p_0 of order $p^{\alpha-1}$. Then the pre-image of p_0 *in the* $Z(G)$ is the desired subgroup of order p^α. (Note that: in general, if S is any subset of a quotient group G/H then the order of the pre-image of S is the product of its order with the order of the subgroup).

Case two: Assume that p does not divide the order of the centre of G. Write $|G|$ in terms of the class equation
$|G| = |Z(G)| + \sum|Conj(a)|$
Where the sum is over all the distinct non-central conjugacy class of G, that is conjugence class of with more than one element. Since p fails to divide the order of the centre, there must be at least one non-central conjugacy class, say conj(b) whose order is not divisible by p. Recall that $|Conj(b)| = [G: C_G(b)] = |G|/|C_G(b)|$. We observe immediately that p^{α} must divide the order of the subgroup $C_G(b)$. Again by induction G will have a Sylow p-subgroup. This ends the proof.

Theorem 1.7 (Sylow's second theorem)

Let n_p be the number of sylow p-subgroups of a finite group G. Then $n_p \equiv 1 mode\ p$

Proof

We begin with a claim

Claim: Let P be any Sylow p-subgroup. If $g \in G$ is a p-element and $gPg^{-1} = P$, then $g \in P$. To see this, consider the subgroup R generated by g and P. By assumption, $g \in N_G(P)$, so $R \in N_G(P)$. Hence P is a normal subgroup of R. We find the $|R| = |R/P|.|P|\ but\ |R/P|$ is a cyclic group generated by the coset gP. Then gP is a p-element since g is. Hence $|R|$ is a power of p since all its elements are p- elements.

Let S_p be the set of all sylow p-subgroups of G, then G act on this set by conjugation. Let $P, Q \in S_p$ be two distinct subgroups. Then Q cannot be fixed under under conjugation by all the elements of P because of the claim. Let O be the O-orbit of Q under conjugation. Then the size of the orbit must be divisible by P because of the order-stabilizer equation

$$|O| = \frac{|P|}{|StabP(Q)|}$$

Since $|P|$is a power of P, the size of any orbit must be a power of p. the case $|O| = P^0 = 1$ is ruled out since Q cannot be fixed by all the elements of P. We find that the set of all Sylow p-subgroups is union of p-orbits. There is only one orbit of order one, $\{p\}$ while the other orbits must have orders a positive power of P. we conclude that $n_p = |S_p| = 1 mode\ p$

Theorem 1.8 (Sylow's Third theorem)

Any two Sylow p-subgroups are conjugate

Proof

Let P be any Sylow p-subgroup. Let S_0 be the set of all G-conjugates of p. Then S_0 is p-invariant and $p \in S_0$. By the above observation, $|S_0| = 1\ mode\ p$. If S_0does not exhaust the set of all Sylow p-subgroups, chose one, say $\emptyset$, not in S_0.

Let S_1 be the set of all G-conjugates of $\emptyset$. By the same reasoning as for S_0 with $\emptyset$ playing the role of p, we must have $|S_1| = 1\ mode\ p$. On the other hand, S_1 is p-invariant and $p \notin S_1$. By the above observation, $|S_1| = 0\ mod\ p$ which is a contradiction.

Theorem 1.9 (Sylow's Fourth theorem)

Any p-subgroup B is contained in a Sylow p-subgroup.

Proof

Let B act on the space S_p by conjugation. Then the size of any B-orbit O must be a power of p, since the $|o| = [G: N_G(B)]$. Since the size of S_p is not a power of p then there must be at least one B-orbit with one element, say p. But B must be a subgroup of p since, the subgroup generated by by B and p is a power of p, by Sylow's first theorem and the fact that n_p of Sylow p-subgroups must divide $|G|/|p^{\alpha}|$

2.1 Consider the symmetric group acting on $\Omega_1 = \{1,2,3,4,5\}$

$$S_5 = \{(1), (45), (35), (354), (345), (34), (25), (254), (253), (2543), (2534), (25)(34), (235), (2354), (23), (23)(45), (234), (2345), (245), (24), (2453), (243), (24)(35), (2435), (15), (154), (153), (1543), (1534), (15)(34), (152), (1542), (1532), (15432), (15342), (152)(34), (1523), (15423), (15)(23), (154)(23), (15)(234), (15234), (1524), (15)(24), (15324), (15)(243), (153)(24), (15243), (125), (1254), (1253), (12543), (12534), (125)(34), (12), (12)(45), (12)(35), (12)(354), (12)(345), (12)(34), (123), (123)(45), (1235), (12354), (12345), (1234), (124), (1245), (124)(35), (12435), (12453), (1243), (135), (1354), (13), (13)(45), (134), (1345), (1352), (13542), (132), (132)(45), (1342), (13452), (13)(25), (13)(254), (1325), (13254), (13425), (134)(25), (13524), (135)(24), (1324), (13245), (13)(24), (13)(245), (145), (14), (1453), (143), (14)(35), (1435), (1452), (142), (14532), (1432), (142)(35), (14352), (14523), (1423), (145)(23), (14)(23), (14235), (14)(235), (14)(25), (1425), (14)(253), (14325), (14253), (143)(25)\}$$

$|S_5| = 2^3 \times 3 \times 5$

The Sylow subgroups of S_5 are as follows;

$H_1 = \{(1),(34),(12),(12)(34),(13)(24),(1324),(1423),(14)(23)\}$
$|H_1| = 2^3$
$H_2 = \{(1),(123),(132)\}$
$|H_2| = 3$
$H_3 = \{(1),(15432),(14253),(13524),(12345)\}$
$|H_3| = 5$

The Sylow 2-subgroup (H_1) of S_5 is normal, solvable, nilpotent, transitive, not simple and not primitive.
The Sylow 3-subgroup (H_2) of S_5 is normal, solvable, nilpotent, transitive, simple and primitive.
The Sylow 5-subgroup (H_3) of S_5 is normal, solvable, nilpotent, transitive, simple and primitive.

2.2 Consider the symmetric group acting on $\Omega_2 = \{1,2,3,4,5,6\}$

$S_6 = \{(1),(56)(46)(465)(456)(45)(36)(365)(364)(3654)(3645)(36)(45)(346)(3465)(34)(34)$
$(56)(345)(3456)(356)(35)(3564)(354)(35)(46)(3546)(26)(265)(264)(2654)(2645)(26)(45)$
$(263)(2653)(2643)(26543)(26453)(263)(45)(2634)(26534)(26)(34)(265)(34)(26)$
$(345)(26345)(2635)(26)(35)(26435)(26)(354)(264)(35)(26354)(236)(2365)(2364)$
$(23654)(23645)(236)(45)(23)(23)(56)(23)(46)(23)(465)(23)(456)(23)(45)(234)$
$(234)(56)(2346)(23465)(23456)(2345)(235)(2356)(235)(46)(23546)(23564)(2354)(246)$
$(2465)(24)(24)(56)(245)(2456)(2463)(24653)(243)(243)(56)(2453)(24563)(24)$
$(36)(24)(365)(2436)(24365)(24536)(245)(36)(24635)(246)(35)(2435)(24356)(24)(35)(24)$
$(356)(256)(25)(2564)(254)(25)(46)(2546)(2563)(253)(25643)(2543)(253)(46)$
$(25463)(25634)(2534)(256)(34)(25)(34)(25346)(25)(346)(25)(36)(2536)(25)(364)(25436)$
$(25364)(254)(36)(16)(165)(164)(1654)(1645)(16)(45)(163)(1653)(1643)(16543)$
$(16453)(163)(45)(1634)(16534)(16)(34)(165)(34)(16)(345)(16345)(1635)(16)(35)(16435)$
$(16)(354)(164)(35)(16354)(162)(1652)(1642)(16542)(16452)(162)(45)(1632)$
$(16532)(16432)(165432)(164532)(1632)(45)(16342)(165342)(162)(34)(1652)(34)$
$(162)(345)(163452)(16352)(162)(35)(164352)(162)(354)(1642)(35)(163542)$
$(1623)(16523)(16423)(165423)(164523)(1623)(45)(16)(23)(165)(23)(164)(23)(1654)$
$(23)(1645)(23)(16)(23)(45)(16)(234)(165)(234)(16234)(165234)(162345)(16)(2345)$
$(16)(235)(16235)(164)(235)(162354)(164235)(16)(2354)(1624)(16524)(16)(24)$
$(165)(24)(16)(245)(16245)(16324)(165324)(16)(243)(165)(243)(16)(2453)(163245)$
$(163)(24)(1653)(24)(16243)(165243)(162453)(163)(245)(163524)(1624)(35)(16)$
$(2435)(162435)(16)(24)(35)(1635)(24)(1625)(16)(25)(16425)(16)(254)(164)(25)(16254)$
$(16325)(16)(253)(164325)(16)(2543)(164)(253)(163254)(163425)(16)(2534)$
$(1625)(34)(16)(25)(34)(162534)(1634)(25)(163)(25)(16253)(1643)(25)(162543)$
$(164253)(163)(254)(126)(1265)(1264)(12654)(12645)(126)(45)(1263)(12653)$
$(12643)(126543)(126453)(1263)(45)(12634)(126534)(126)(34)(1265)(34)(126)(345)$
$(126345)(12635)(126)(35)(126435)(126)(354)(1264)(35)(126354)(12)(12)(56)$
$(12)(46)(12)(465)(12)(456)(12)(45)(12)(36)(12)(365)(12)(364)(12)(3654)(12)(3645)(12)(36)$
$(45)(12)(346)(12)(3465)(12)(34)(12)(34)(56)(12)(345)(12)(3456)(12)(356)(12)(35)$
$(12)(3564)(12)(354)(12)(35)(46)(12)(3546)(123)(123)(56)(123)(46)(123)(465)(123)(456)$
$(123)(45)(1236)(12365)(12364)(123654)(123645)(1236)(45)(12346)(123465)$
$(1234)(1234)(56)(12345)(123456)(12356)(1235)(123564)(12354)(1235)(46)(123546)$
$(124)(124)(56)(1246)(12465)(12456)(1245)(124)(36)(124)(365)(12436)(124365)$
$(124536)(1245)(36)(12463)(124653)(1243)(1243)(56)(12453)(124563)(124)(356)$
$(124)(35)(124356)(12435)(1246)(35)(124635)(125)(1256)(125)(46)(12546)(12564)$
$(1254)(125)(36)(12536)(125)(364)(125436)(125364)(1254)(36)(125)(346)(125346)$
$(125)(34)(1256)(34)(12534)(125634)(12563)(1253)(125643)(12543)(1253)(46)$
$(125463)(136)(1365)(1364)(13654)(13645)(136)(45)(13)(13)(56)(13)(46)(13)(465)(13)$
$(456)(13)(45)(134)(134)(56)(1346)(13465)(13456)(1345)(135)(1356)(135)(46)$
$(13546)(13564)(1354)(1362)(13652)(13642)(136542)(136452)(1362)(45)(132)(132)$
$(56)(132)(46)(132)(465)(132)(456)(132)(45)(1342)(1342)(56)(13462)(134652)$
$(134562)(13452)(1352)(13562)(1352)(46)(135462)(135642)(13542)(13)(26)(13)(265)$
$(13)(264)(13)(2654)(13)(2645)(13)(26)(45)(1326)(13265)(13264)(132654)(132645)$
$(1326)(45)(13426)(134265)(134)(26)(134)(265)(1345)(26)(134526)(13526)(135)$
$(26)(135264)(1354)(26)(135)(264)(135426)(13624)(136524)(136)(24)(1365)(24)$
$(136)(245)(136245)(1324)(1324)(56)(13246)(132465)(132456)(13245)(13)(24)(13)$
$(24)(56)(13)(246)(13)(2465)(13)(2456)(13)(245)(13524)(135624)(135246)(135)$
$(246)(1356)(24)(135)(24)(13625)(136)(25)(136425)(136)(254)(1364)(25)(136254)$

(1325)(13256)(1325)(46)(132546)(132564)(13254)(13425)(134256)(134625)
(1346)(25)(134)(256)(134)(25)(13)(25)(13)(256)(13)(25)(46)(13)(2546)(13)(2564)(13)
(254)(146)(1465)(14)(14)(56)(145)(1456)(1463)(14653)(143)(143)(56)(1453)(14563)
(14)(36)(14)(365)(1436)(14365)(14536)(145)(36)(14635)(146)(35)(1435)(14356)
(14)(35)(14)(356)(1462)(14652)(142)(142)(56)(1452)(14562)(14632)(146532)(1432)
(1432)(56)(14532)(145632)(142)(36)(142)(365)(14362)(143652)(145362)(1452)
(36)(146352)(1462)(35)(14352)(143562)(142)(35)(142)(356)(14623)(146523)(1423)
(1423)(56)(14523)(145623)(146)(23)(1465)(23)(14)(23)(14)(23)(56)(145)(23)(1456)
(23)(14236)(142365)(14)(236)(14)(2365)(145)(236)(145236)(146)(235)(146235)
(14)(235)(14)(2356)(14235)(142356)(14)(26)(14)(265)(1426)(14265)(14526)(145)
(26)(14)(263)(14)(2653)(14326)(143265)(145326)(145)(263)(14263)(142653)(143)
(26)(143)(265)(1453)(26)(145263)(14)(2635)(14)(26)(35)(143526)(1435)(26)(1426)
(35)(142635)(14625)(146)(25)(1425)(14256)(14)(25)(14)(256)(146325)(146)(253)
(14325)(143256)(14)(253)(14)(2563)(1425)(36)(142536)(143625)(1436)(25)(14)
(2536)(14)(25)(36)(1463)(25)(146253)(143)(25)(143)(256)(14253)(142563)(156)(15)
(1564)(154)(15)(46)(1546)(1563)(153)(15643)(1543)(153)(46)(15463)(15634)
(1534)(156)(34)(15)(34)(15346)(15)(346)(15)(36)(1536)(15)(364)(15436)(15364)(154)
(36)(1562)(152)(15642)(1542)(152)(46)(15462)(15632)(1532)(156432)(15432)
(1532)(46)(154632)(156342)(15342)(1562)(34)(152)(34)(153462)(152)(346)(152)
(36)(15362)(152)(364)(154362)(153642)(1542)(36)(15623)(1523)(156423)(15423)
(1523)(46)(154623)(156)(23)(15)(23)(1564)(23)(154)(23)(15)(23)(46)(1546)(23)
(156)(234)(15)(234)(156234)(15234)(15)(2346)(152346)(15236)(15)(236)(152364)
(154)(236)(15)(2364)(154236)(15624)(1524)(156)(24)(15)(24)(15246)(15)(246)
(156324)(15324)(156)(243)(15)(243)(153246)(15)(2463)(1563)(24)(153)(24)(156243)
(15243)(153)(246)(152463)(1524)(36)(153624)(152436)(15)(2436)(1536)(24)
(15)(24)(36)(15)(26)(1526)(15)(264)(15426)(15264)(154)(26)(15)(263)(15326)(15)(2643)
(154326)(153264)(154)(263)(15)(2634)(153426)(15)(26)(34)(1526)(34)(1534)
(26)(152634)(15263)(153)(26)(152643)(1543)(26)(153)(264)(154263)}

$|S_6| = 720$

The Sylow subgroups of S_6 are as follows;

$H_4 = \{(1), (56), (34), (34)(56), (12), (12)(56), (12)(34), (12)(34)(56), (13)(24), (13)(24)(56), (1324), (1324)(56), (1423), (1423)(56), (14)(23), (14)(23)(56)\}$

$|H_4| = 2^4$

$H_5 = \{(1), (465), (456), (132), (132)(465), (132)(456), (123), (123)(465), (123)(456)\}$

$|H_5| = 3^2$

$H_6 = \{(1), (15432), (14253), (13524), (12345)\}$

$|H_6| = 5$

The Sylow 2-subgroup (H_4) of S_6 is normal, solvable, nilpotent, not transitive, not simple and not primitive.
The Sylow 3-subgroup (H_5) of S_6 is normal, solvable, nilpotent, not transitive, not simple and not primitive.
The Sylow 5-subgroup (H_6) of S_6 is normal, solvable, nilpotent, transitive, simple and primitive.

2.3 Consider the alternating group acting on $\Omega_1 = \{1,2,3,4,5\}$

$A_5 = \{(1), (354), (345), (254), (25)(34), (253), (245), (243), (24)(35), (235), (23)(45), (234), (154), (15)(34), (153), (15)(24), (15243), (15324), (152), (15432), (15342), (15234), (15)(23), (15423), (145), (143), (14)(35), (142), (14352), (14532), (14)(25), (14325), (14253), (14523), (14)(23), (14235), (125), (12543), (12534), (12)(45), (12)(34), (12)(35), (124), (12435), (12453), (123), (12354), (12345), (135), (13)(45), (134), (13542), (13452), (132), (13524), (13245), (13)(24), (13)(25), (13254), (13425)\}$

$|A_5| = 60$

The Sylow subgroups of A_5 are as follows;

$H_7 = \{(1), (12)(34), (13)(24), (14)(23)\}$

$|H_7| = 2^2$

$H_8 = \{(1), (123), (132)\}$

$|H_8| = 3$

$H_9 = \{(1), (15432), (14253), (13524), (12345)\}$

$|H_9| = 5$

The Sylow 2-subgroup (H_7) of A_5 is normal, solvable, nilpotent, transitive, not simple and not primitive.
The Sylow 3-subgroup (H_8) of A_5 is normal, solvable, nilpotent, transitive, simple and primitive.
The Sylow 5-subgroup (H_9) of A_5 is normal, solvable, nilpotent, transitive, simple and primitive.

2.4 Consider the Dihedral group acting on $\Omega_3 = \{1,2,3,4,5,6,7\}$

$$D_1 = \{(1),(27)(36)(45),(1765432),(17)(26)(35),(1642753),(16)(25)(34),(1526374),(15)(24)(67),(1473625),(14)(23)(57),(1357246),(13)(47)(56),(1234567),(12)(37)(46)\}$$

$$|D_1| = 2 \times 7$$

The Sylow subgroups of D_1 are as follows;

$$T_1 = \{(1),(27)(36)(45)\}$$

$$|T_1| = 2$$

$$T_2 = \{(1),(1765432),(1642753),(1526374),(1473625),(1357246),(1234567)\}$$

$$|T_2| = 7$$

The Sylow 2-subgroup (T_1) of D_1 is normal, solvable, nilpotent, not transitive, simple and not primitive.
The Sylow 7-subgroup (T_2) of D_1 is normal, solvable, nilpotent, transitive, simple and primitive.

2.5 Consider the Dihedral group acting on $\Omega_4 = \{1,2,3,4,5,6,7,8,9,10\}$

$$D_2 = \{(1),(\ 2\,10)(\ 3\ 9)(\ 4\ 8)(\ 5\ 7),(\ 1\ 9\ 7\ 5\ 3)(\ 2\,10\ 8\ 6\ 4),(\ 1\ 9)(\ 2\ 8)(\ 3\ 7)(\ 4\ 6),(\ 1\ 7\ 3\ 9\ 5)(\ 2\ 8\ 4\,10\ 6),(\ 1\ 7)(\ 2\ 6)(\ 3\ 5)(\ 8\,10),(\ 1\ 5\ 9\ 3\ 7)(\ 2\ 6\,10\ 4\ 8),(\ 1\ 5)(\ 2\ 4)(\ 6\,10)(\ 7\ 9),(\ 1\ 3\ 5\ 7\ 9)(\ 2\ 4\ 6\ 8\,10),(\ 1\ 3)(\ 4\,10)(\ 5\ 9)(\ 6\ 8),(\ 1\,10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2),(\ 1\,10)(\ 2\ 9)(\ 3\ 8)(\ 4\ 7)(\ 5\ 6),(\ 1\ 8\ 5\ 2\ 9\ 6\ 3\,10\ 7\ 4),(\ 1\ 8)(\ 2\ 7)(\ 3\ 6)(\ 4\ 5)(\ 9\,10),(\ 1\ 6)(\ 2\ 7)(\ 3\ 8)(\ 4\ 9)(\ 5\,10),(\ 1\ 6)(\ 2\ 5)(\ 3\ 4)(\ 7\,10)(\ 8\ 9),(\ 1\ 4\ 7\,10\ 3\ 6\ 9\ 2\ 5\ 8),(\ 1\ 4)(\ 2\ 3)(\ 5\,10)(\ 6\ 9)(\ 7\ 8),(\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\,10),(\ 1\ 2)(\ 3\,10)(\ 4\ 9)(\ 5\ 8)(\ 6\ 7)\}$$

$$|D_2| = 2^2 \times 5$$

The Sylow subgroups of D_2 are as follows;

$$T_3 = \{(1),(\ 2\,10)(\ 3\ 9)(\ 4\ 8)(\ 5\ 7),(\ 1\ 6)(\ 2\ 7)(\ 3\ 8)(\ 4\ 9)(\ 5\,10),(\ 1\ 6)(\ 2\ 5)(\ 3\ 4)(\ 7\,10)(\ 8\ 9)\}$$

$$|T_3| = 2^2$$

$$T_4 = \{(1),(\ 1\ 5\ 9\ 3\ 7)(\ 2\ 6\,10\ 4\ 8),(\ 1\ 9\ 7\ 5\ 3)(\ 2\,10\ 8\ 6\ 4),(\ 1\ 3\ 5\ 7\ 9)(\ 2\ 4\ 6\ 8\,10),(\ 1\ 7\ 3\ 9\ 5)(\ 2\ 8\ 4\,10\ 6)\}$$

$$|T_4| = 5$$

The Sylow 2-subgroup (T_3) of D_2 is normal, solvable, nilpotent, not transitive, not simple and not primitive.
The Sylow 7-subgroup (T_4) of D_2 is normal, solvable, nilpotent, not transitive, simple and not primitive.

2.6 Consider the permutation groups P_1 and Q_1

$\boldsymbol{P_1} = \{(1),(12)\}$, $\boldsymbol{Q_1} = \{(1),(345),(354)\}$ acting on the sets $S_1 = \{1,2\}\ and\ \Delta_1 = \{3,4,5\}$ respectively.
Let Let $P = {Q_1}^{\Delta_2} = \{f{:}\,\Delta_2 \longrightarrow Q_1\} then |P| = |Q_1|^{\Delta_2} = 2^3 = 8$
We can easily verify that $G_1$is a group with respect to the operations
$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1) where\ \delta_1 \in \Delta_1$.
The wreath product of P_1 and Q_1 is given by W_1, where

$$W_1 = \{(1),(56),(34),(34)(56),(12),(12)(56),(12)(34),(12)(34)(56),(153)(264),(154263),(153264),(154)(263),(164253),(163)(254),(164)(253),(163254),(135)(246),(135246),(136245),(136)(245),(146235),(146)(235),(145)(236),(145236)\}$$

$|W_1| = 2^3 \times 3$
The Sylow subgroups of W_1 are as follows;
$G_1 = \{(1),(56),(34),(34)(56),(12)(34)(56),(12)(34),(12)(56),(12)\}$
$|G_1| = 2^3$
$G_2 = \{(1),(153)(264),(135)(246)\}$
$|G_2| = 3$
The Sylow 2-subgroup (G_1) of W_1 is normal, solvable, nilpotent, not transitive, not simple and not primitive.
The Sylow 3-subgroup (G_2) of W_1 is normal, solvable, nilpotent, not transitive, simple and not primitive.

2.7 Consider the permutation groups P_2 and Q_2

$P_2 = \{(1),(15432),(14253),(13524),(12345)\}$, $Q_2 = \{(1),(6,7)\}$ acting on the sets $S_1 = \{1,2,3,4,5\}\ and\ \Delta_1 = \{6,7\}$ respectively.
Let Let $P = {Q_2}^{\Delta_2} = \{f{:}\,\Delta_2 \longrightarrow Q_1\} then |P| = |Q_2|^{\Delta_2} = 5^2 = 25$
We can easily verify that $G_1$is a group with respect to the operations
$(f_1 f_2)\delta_1 = f_1(\delta_1) f_2(\delta_1) where\ \delta_1 \in \Delta_1$.
The wreath product of P_2 and Q_2 is given by W_2, where

$$W_2 = \{(1),(\ 6\,10\ 9\ 8\ 7),(\ 6\ 9\ 7\,10\ 8),(\ 6\ 8\,10\ 7\ 9),(\ 6\ 7\ 8\ 9\,10),(15432),(\ 1\ 5\ 4\ 3\ 2)(\ 6\,10\ 9\ 8\ 7),(\ 1\ 5\ 4\ 3\ 2)(\ 6\ 9\ 7\,10\ 8),(\ 1\ 5\ 4\ 3\ 2)(\ 6\ 8\,10\ 7\ 9),(\ 1\ 5\ 4\ 3\ 2)(\ 6\ 7\ 8\ 9\,10),(14253),(\ 1\ 4\ 2\ 5\ 3)(\ 6\,10\ 9\ 8\ 7),(\ 1\ 4\ 2\ 5\ 3)(\ 6\ 9\ 7\,10\ 8),(\ 1\ 4\ 2\ 5\ 3)(\ 6\ 8\,10\ 7\ 9),(\ 1\ 4\ 2\ 5\ 3)(\ 6\ 7\ 8\ 9\,10),(13524),(\ 1\ 3\ 5\ 2\ 4)(\ 6\,10\ 9\ 8\ 7),(\ 1\ 3\ 5\ 2\ 4)(\ 6\ 9\ 7\,10\ 8),(\ 1\ 3\ 5\ 2\ 4)$$

(6 810 7 9), (1 3 5 2 4)(6 7 8 910), (12345), (1 2 3 4 5)(610 9 8 7), (1 2 3 4 5) (6 9 710 8), (1 2 3 4 5)(6 810 7 9), (1 2 3 4 5)(6 7 8 910), (1 6)(2 7)(3 8)(4 9)(510), (1 6 510 4 9 3 8 2 7), (1 6 4 9 2 7 510 3 8), (1 6 3 8 510 2 7 4 9), (1 6 2 7 3 8 4 9 510), (110 5 9 4 8 3 7 2 6), (110 4 8 2 6 5 9 3 7), (110 3 7 5 9 2 6 4 8), (110 2 6 3 7 4 8 5 9), (110)(2 6)(3 7)(4 8)(5 9), (1 9 4 7 210 5 8 3 6), (1 9 3 6 5 8 210 4 7), (1 9 210 3 6 4 7 5 8), (1 9)(210)(3 6)(4 7)(5 8), (1 9 5 8 4 7 3 6 210), (1 8 310 5 7 2 9 4 6), (1 8 2 9 310 4 6 5 7), (1 8)(2 9)(310)(4 6)(5 7), (1 8 5 7 4 6 310 2 9), (1 8 4 6 2 9 5 7 310), (1 7 2 8 3 9 410 5 6), (1 7)(2 8)(3 9)(410)(5 6), (1 7 5 6 410 3 9 2 8), (1 7 410 2 8 5 6 3 9), (1 7 3 9 5 6 2 8 410)}

$|W_2| = 5^2 \times 2$

The Sylow subgroups of W_2 are as follows;

$G_3 = \{(1), (16)(27)(38)(49)(5\ 10)\}$

$|G_3| = 2$

$G_4 = \{(1)$, (6 810 7 9), (610 9 8 7), (6 7 8 910), (6 9 710 8), (1 3 5 2 4)(6 810 7 9), (1 3 5 2 4) (610 9 8 7), (1 3 5 2 4)(6 7 8 910), (1 3 5 2 4)(6 9 710 8), (1 3 5 2 4), (1 5 4 3 2) (610 9 8 7), (1 5 4 3 2)(6 7 8 910), (1 5 4 3 2)(6 9 710 8), (1 5 4 3 2), (1 5 4 3 2) (6 810 7 9), (1 2 3 4 5)(6 7 8 910), (1 2 3 4 5)(6 9 710 8), (1 2 3 4 5), (1 2 3 4 5) (6 810 7 9), (1 2 3 4 5)(610 9 8 7), (1 4 2 5 3)(6 9 710 8), (1 4 2 5 3), (1 4 2 5 3) (6 810 7 9), (1 4 2 5 3)(610 9 8 7), (1 4 2 5 3)(6 7 8 910)}

$|G_4| = 5^2 = 25$

The Sylow 2-subgroup (G_3) of W_2 is normal, solvable, nilpotent, not transitive, simple and not primitive.
The Sylow 5-subgroup (G_4) of W_2 is normal, solvable, nilpotent, not transitive, not simple and not primitive.

DISCUSSION ON THE RESULT

The following observations are made from the above;

- All the Sylow subgroups of the groups under investigation are normal in the respective groups.
- The Sylow subgroups of the dihedral groups and the groups generated by semidirect products of two permutations groups under investigation are imprimitive and intransitive, except for for the q-Sylow subgroup of the dihedral group whose order is pq where p,q are primes which is both primitive and transitive.

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