Journal of Natural Sciences Research www.iiste.org
ISSN 2224-3186 (Paper) ISSN 2225-0921 (Online)
Vol.8, No.8, 2018

An Application of Second Derivative Backward Differentiation Formula Hybrid Block Method on Stiff Ordinary Differential Equations

I.A.Bakari[1*] Y.Skwame[2] Kumleng .G.M[3]
1.Department of Mathematics, Federal University, Dutse, Nigeria
2.Department of Mathematics, Adamawa State University, Mubi, Nigeria
3.Department of Mathematics, University of Jos, P.M.B. 2084,Plateau, State, Nigeria

Abstract
In this paper, we developed a continuous scheme of four and five step with one off-grid point at collocation which provides the approximate solution of both linear and nonlinear stiff ordinary differential equations with constant step size. The continuous scheme is evaluated at both interpolation and collocation where necessary to give continuous hybrid block scheme and high order of accuracy with low error constants. Numerical results of the schemes are presented to compare with exact solutions and the results have shown that the (SDHBBDF) performed favorably when compare with existing methods.
Keywords: Collocation and interpolation, Hybrid block methods, Second Derivative and Stiff systems

1. Introduction

A first –order differential equation is an equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

In which $f(x, y)$ is function two variables defined on a region in the $xy - plane$ with initial condition $y(x_0) = y_0$ $a \leq x \leq b$ which is called initial value problems. The solution of equation (1) has been discussed by various researchers among them are Onumayi *et al* (1994) propose a linear multistep method of order 5 that is self-starting for the direct solution of the general second-order initial value problem (IVP). The method is derived by the interpolation and collocation of the assumed approximate solution and its second derivative. Some researchers have attempted the solution of directly using linear multistep methods without reduction to system of first order ordinary differential equations by Mohammed *et al* (2010). The block method produced numerical solutions with less than computational effort as to compare to non-block method by Majid (2004). Ebadi *et al*. (2010) worked on the hybrid BDF with one additional off-grid point introduced in the first derivative of the solution to improve the absolute stability region of the method. Block methods have been considered by various authors among who are A-stable implicit one block methods with higher orders by Shampine and Watts (1972). Hybrid blocks method of order six for the numerical solution of first order initial value problems. The method is based on collocation of the differential system and interpolation of the approximate at the grid and off-grid points by Areo and Adeniyi (2013).

In this paper, we developed four and five step of Second Derivative Hybrid Block Backward Differentiation Formula (SDHBBDF) for Numerical Solution of Stiff Ordinary Differential Equations.

2. The Derivation of the Method

We define a k-step SDHBBDF to numerical solution of (1) as

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j}(x) = h\beta_k(x) y_{n+k}(x) \tag{2}$$

Where $\alpha_j(x)$, $\beta_k(x)$ are the continuous coefficients and $k \in \mathbb{Z}^+$ is the step number of the method. And

$$\alpha_j(x) = \sum_{j=0}^{k} \alpha_j x^j, \qquad j \in \{0,1 \dots \dots k-1\} \tag{3}$$

$$\beta_k(x) = \sum_{j=0}^{k} \beta_k x^j, \qquad j \in \{0,1 \dots \dots k-1\} \tag{4}$$

$x_{n+j}: j = 0,1,2, \dots .t-1$ in (2) are ($0 \leq t \leq k$ arbitrary chosen interpolation points taken from $[x_n, \dots .., x_{n+k}]$ and $x_j: j = 0,1, \dots, m-1$ are m collocation points belonging to $[x_n, \dots .., x_{n+k}]$. To get $\alpha_j(x)$, and $\beta_k(x)$, Sirisena *et al*(2004) arrived at a matrix equation of the form

$$DC = I \tag{5}$$

Where I is the identity matrix of dimension $(t+m) \times (t+m)$while $D$and C are matrices defined as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & . & . & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & . & . & x_{n+1}^{t+m-1} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & x_{n+t-1}^3 & x_{n+t-1}^4 & . & . & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & . & . & (t+m-1)x_n^{t+m-2} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & . & . & (t+m-1)x_{n+1}^{t+m-2} \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 2x_{n+t-1} & 3x_{n+t-1}^2 & 4x_{n+t-1}^3 & . & . & (t+m-1)x_{n+t-1}^{t+m-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & . & . & (t+m-1)(t+m-2)x_n^{t+m-3} \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & . & . & (t+m-1)(t+m-2)x_{n+1}^{t+m-3} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 2 & 6x_{n+t-1} & 12x_{n+t-1}^2 & . & . & (t+m-1)(t+m-2)x_{n+t-1}^{t+m-3} \end{pmatrix} \quad (6)$$

The above matrix (6) is the multistep collocation matrix of dimension $(t+m) \times (t+m)$ and

$$C = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & . & \alpha_{t-1,1} & h\beta_{0,1} & . & h\beta_{m-1,1} & h^2\gamma_{0,1} & . & h^2\gamma_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & . & \alpha_{t-1,2} & h\beta_{0,2} & . & h\beta_{m-1,2} & h^2\gamma_{0,2} & . & h^2\gamma_{m-1,2} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,j+m} & \alpha_{1,j+m} & . & \alpha_{t-1,j+m} & h\beta_{0,j+m} & . & h\beta_{m-1,j+m} & h^2\gamma_{0,j+m} & . & h^2\gamma_{m-1,j+m} \end{pmatrix} \quad (7)$$

Where t and m are defined as the number of interpolation and collocation points used respectively.
The columns of the matrix $C = D^{-1}$ give the continuous coefficients
$\alpha_j(x)$; $j = 0,1, \ldots \ldots k-1$, $\beta_j(x)$; $j = 0,1, \ldots \ldots k-1$ and $\gamma_j(x)$; $j = 0,1, \ldots \ldots k-1$
The proposed four and five step of second derivative hybrid block backward differentiation formula (SDHBBDF) was developed subjected to the following conditions for matrix D:
For $k = 4$, evaluating (6) at $x_n = 0, h, 2h, \frac{7}{2}h \; and \; 4h$ and (2) becomes

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + h\left[\beta_3(x)f_{n+3} + \beta_{n+\frac{7}{2}}(x)f_{n+\frac{7}{2}} + \beta_4(x)f_{n+4}\right]$$
$$+h^2[+\gamma_3(x)g_{n+3} + \gamma_4(x)g_{n+4}] \quad (8)$$

Thus the matrix Din (6) becomes

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^3 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 \\ 0 & 1 & 2x_{n+7/2} & 3x_{n+7/2}^2 & 4x_{n+7/2}^3 & 5x_{n+7/2}^3 & 6x_{n+7/2}^5 & 7x_{n+7/2}^6 & 8x_{n+7/2}^7 \\ 0 & 1 & 6x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{pmatrix} \quad (9)$$

Using Maple software, the inverse of the matrix in (9) is obtained and this yields the elements of the matrix D. The element of the matrix D substituted into (8) yields the continuous formulation of the method as:

$$y_n-\frac{43889121}{5338519}y_{n+1}+\frac{364359465}{5338519}y_{n+2}-\frac{325808863}{5338519}y_{n+3}$$
$$=\frac{6h}{5338519}[43986619f_{n+3}-131818725f_{n+\frac{7}{2}}+39066003f_{n+4}$$
$$+\frac{6h^2}{5338519}[100933g_n+5360461g_{n+1}-7124706g_{n+4}]$$

$$y_n+\frac{3870639}{528139}y_{n+1}-\frac{49623435}{528139}y_{n+2}+\frac{45224657}{528139}y_{n+3}$$
$$=-\frac{6h}{528139}\left[4800101f_{n+3}-16213248f_{n+\frac{7}{2}}+4696857f_{n+4}\right]$$
$$+\frac{h^2}{528139}[1211195g_{n+1}-41449734g_{n+3}+5086122g_{n+4}]$$

$$y_n-\frac{916731}{24929}y_{n+1}-\frac{6327585}{24929}y_{n+2}+\frac{28091}{97}y_{n+3}$$
$$=-\frac{6h}{24929}[685831f_{n+3}-2803968f_{n+\frac{7}{2}}+770427f_{n+4}$$
$$-\frac{18h^2}{24929}[86514g_{n+2}+440719g_{n+3}-45321g_{n+4}]$$

$$y_{n+\frac{7}{2}}=-\frac{4675}{354361344}y_n+\frac{35973}{118120448}y_{n+1}-\frac{865375}{118120448}y_{n+2}+\frac{356854225}{354361344}y_{n+3}$$
$$+\frac{35h}{59060224}[510455f_{n+3}+350976f_{n+\frac{7}{2}}-29115f_{n+4}+\frac{525h^2}{59060224}[4363g_{n+3}+291g_{n+4}] \quad (10)$$

$$y_{n+4}=\frac{1}{100933}y_n-\frac{20}{100933}y_{n+1}+\frac{324}{100933}y_{n+2}+\frac{100628}{100933}y_{n+3}+\frac{12h}{100933}[1906f_{n+3}+$$
$$4608f_{n+\frac{7}{2}}+1921f_{n+4}+\frac{72h^2}{100933}[17g_n-22g_{n+4}] \quad (11)$$

For $k=5$, evaluating (6) at $x_n=0,h,2h,3h,\frac{9}{2}h\ and\ 5h$ and (2) becomes

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)y_{n+1}+\alpha_2(x)y_{n+2}+\alpha_3(x)y_{n+3}+\alpha_4(x)y_{n+4}$$
$$+h\left[\beta_4(x)f_{n+4}+\beta_{n+\frac{9}{2}}(x)f_{n+\frac{9}{2}}+\beta_5(x)f_{n+5}\right]$$
$$+h^2[+\gamma_4(x)g_{n+4}+\gamma_5(x)g_{n+5}] \quad (12)$$

Thus the matrix Din (6) becomes

$$D=\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^5 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\
1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+2}^9 \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^8 \\
0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^3 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & 9x_{n+4}^8 \\
0 & 1 & 2x_{n+9/2} & 3x_{n+9/2}^2 & 4x_{n+9/2}^3 & 5x_{n+9/2}^3 & 6x_{n+9/2}^5 & 7x_{n+9/2}^6 & 8x_{n+9/2}^7 & 9x_{n+9/2}^8 \\
0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & 9x_{n+5}^8 \\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 & 72x_{n+4}^7 \\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 & 56x_{n+5}^6 & 72x_{n+5}^7
\end{pmatrix}$$

(13)

Similarly, we generate the continuous formulation of the new method as:

$$y_n-\frac{58246497728}{8106149141}y_{n+1}+\frac{287889150672}{8106149141}y_{n+2}-\frac{80444045248}{8106149141}y_{n+3}+\frac{1612464238619}{8106149141}y_{n+4}+\frac{12h}{8106149141}[60796088575f_{n+4}-247880384512f_{n+\frac{9}{2}}+69021826272f_{n+5}]+\frac{72h^2}{40530745705}[66359537g_n-93948417580g_{n+4}+10159234608g_{n+5}]$$

$$y_n+\frac{3679563944}{743599935}y_{n+1}-\frac{3572959464}{82622215}y_{n+2}+\frac{4537703608}{16524443}y_{n+3}-\frac{176463191063}{743599935}y_{n+4}=\frac{4h}{24766645}\left[5486226727f_{n+4}-25244983296f_{n+\frac{9}{2}}+6934049604f_{n+5}\right]+\frac{8h^2}{24766645}[66359537g_{n+1}+5898605983g_{n+4}-608582079g_{n+5}]$$

$$y_n-\frac{2380511872}{71172879}y_{n+1}-\frac{736521408}{23724293}y_{n+2}+\frac{19444153216}{23724293}y_{n+3}3-\frac{53813556431}{71172879}y_{n+4}=\frac{4h}{23724293}\left[1300871131f_{n+4}-7145226240f_{n+\frac{9}{2}}+1922981184f_{n+5}\right]-\frac{8h^2}{23724293}[132719074g_{n+2}-1733268759g_{n+4}+167226176g_{n+5}]$$

$$y_n-\frac{414537320}{18774287}y_{n+1}+\frac{110394952}{2682941}y_{n+2}+\frac{19790816104}{18774287}y_{n+3}-\frac{27122653735}{18774287}y_{n+4}=\frac{12h}{18774287}\left[782440151f_{n+4}-4864139264f_{n+\frac{9}{2}}+1241907228f_{n+5}\right]+\frac{72h^2}{18774287}[132719074g_{n+3}+429136117g_{n+4}-35225544g_{n+5}]$$

$$y_{n+\frac{9}{2}} = \frac{6770575}{2174469308416}y_n - \frac{7688475}{135904331776}y_{n+1} + \frac{4798521}{7346180096}y_{n+2} - \frac{1424749725}{135904331776}y_{n+4} + \frac{315h}{543617327104}\left[526591065f_{n+4} - 346107904f_{n+\frac{9}{2}} - 25923240f_{n+5}\right] + \frac{99225h^2}{271808663552}[114319g_{n+4} + 5964g_{n+5}]$$

$$y_{n+5} = -\frac{213}{66359537}y_n + \frac{3625}{66359537}y_{n+1} - \frac{1000}{1793501}y_{n+2} + \frac{429000}{66359537}y_{n+3} + \frac{65964125}{66359537}y_{n+4} + \frac{60h}{66359537}\left[247625f_{n+4} + 614400f_{n+\frac{9}{2}} + 250051f_{n+5}\right] + \frac{1800h}{66359537}[335g_{n+4} - 563g_{n+5}]$$

(14)

3. Analysis of the New Methods

We consider the analysis of the newly constructed methods such as order, error constant ,consistency , convergence and the regions of absolute stability of the methods. Following (Fatunla, 1991) and(Lambert,1973) we defined the local truncation error associated with (11) and (14) to be linear difference operator

$$L[y(t);h] = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k} \quad . \tag{15}$$

Assuming that $y(t)$ is sufficiently differentiable, we can we expand the terms in (15) as a Taylor series and comparing the coefficients of h gives

$$L[y(t);h] = c_0 y(t) + c_1 h y'^{(t)} + c_2 h^2 y''^{(t)} + \cdots + c_p h^p y^p(t) + \cdots \tag{16}$$

Where the constant coefficients $C_p, p = 0,1,2 \ldots, j = 1,2, \ldots k$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

.

.

.

$$C_q = [\tfrac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j]$$

According to (Henrici, 1962), methods (8) and (12) have order p if

$C_0 = C_1 = C_2 \ldots C_p = 0$ and $C_{p+1} \neq 0$. Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(t_n)$ is the principal local truncation error at appoint t_n. It was established from the evaluation that block methods (11) and (14) have order and error constants.

4. CONVERGENCE

The convergence of the new block methods is determined using the approach by Fatunla (1991) and Chollom etal (2007) for linear multistep methods, where the block methods are represented in single block, r point multistep method of the form

$$A^{(0)}y_{m+1} = \sum_{i=1}^{k} A^{(i)}y_{m+1} + h\sum_{i=0}^{k} B^{(i)}f_{m+1} \quad (17)$$

Where h is a fixed mesh size within a block, $A^i, B^i, i = 0,1,2, \ldots . k$ are $r \times r$ identity while $y_m, y_{m-1} and\ y_{m+1}$ are vectors of numerical estimates.

4.1 Definition: A numerical method is said to be A-stable if the whole of the left-half plane $\{Z: Re(Z) \leq 0\}$ is contained in the region. $\{Z: Re(Z) \leq 1\}$ Where $R(Z)$ is called the stability polynomial of the method (Lambert,1973)

4.2 Definition: A numerical method is said to be A (α)-stable, $\alpha\epsilon(0,\frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda| -\alpha < \pi - argh\lambda < \alpha\}$ (Lambert,1973),

The block method (11) expressed in the form of (17) gives the characteristic polynomial of the block method

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)}) \quad (18)$$

$$= \lambda^4(\lambda - 1) = 0 \quad (19)$$

Therefore,$\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.The block method (17) by definition is A(α)--stable and by Henrici (1962), the block method is convergent.

Similarly, The block method (14) expressed in the form of (17) gives the characteristic polynomial of the block method (18)

$$= \lambda^5(\lambda - 1) = 0 \quad (20)$$

Therefore,$\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.The block method (17) by definition is A(α)--stable and by Henrici (1962), the block method is convergent.

5. Numerical experiments

Problem 1: Consider the Stiff nonlinear system of two dimensional Kaps problem with corresponding initial conditions $\begin{bmatrix} y'_1(x) \\ y'_2(x) \end{bmatrix} = \begin{bmatrix} -1002y_1(x) + & 1000y_2^2(x) \\ y_1(x) - y_2(x) + & y_2^2(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The analytic solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-2x) \\ \exp(-x) \end{bmatrix}$$

Problem 2: A two-dimensional SODEs is considered (Wu, X.Y and Xiu, J.L 2001)

$y'_1 = -500000.5y_1 + 499999.5y_2 \quad y_1(0) = 0$

$y'_2 = 499999.5y_1 - 500000.5y_2 \quad y_2(0) = 2$

The exact solutions are: $y_1(x) = -e^{\lambda_1 x} + e^{\lambda_2 x}$

$$y_2(x) = e^{\lambda_1 x} + e^{\lambda_2 x} \quad \lambda_1 = -10^6, \lambda_2 = -1$$

6. Conclusion

There are different hybrid block methods for solving stiff ordinary differential equations with components of both grid and off-grid points. In this paper, we proposed high order block hybrid $k - step$ (SDHBBDF) with $k = 4$ and 5 for numerical solution of both linear and nonlinear stiff systems. The propose methods give good approximate solution and reduced computational cost as shown in figure 3,4,5and 6. Moreover, the schemes are A(α)-stable, consistent and convergent.

7. References

Onimanyi, P., Awoyemi, D.O.,Jator, S.N and Sirisena, U.W. (1994), New Linear Multistep Methods with Continuous Coefficients for First Order Initial Value Problem. J.Nig, Math.SOC.
13;37-51.

Mohammed, U. , Jiya, M. and Mohammed, A. A.(2010), A Class of six step block method for Solution of general second order ordinary differential equations. Pacifics Journal of Science and Technology. 11(2); 273-277.

Majid, Z.A. (2004) , Parallel block methods for solving ordinary differential equations. PhD the Thesis, University, Putra Malaysia.

Ebadi, M. and Gokhale, M.Y.(2010). Hybrid BDF methods for the numerical solution ofOrdinary differential equations. Numer. Algor, 253, 1-17.

Watts, H.A .and Shampine, L.F. (1972). A-stable Block Implicit one-step methods BIT Numerical Mathematics Volume 12, pp 252-266.

Areo, E.A and Adeniyi, R.B. (2013) Six-Order Hybrid Block Method for the Numerical Solution Firsr Order Initial Value Problems. Mathematics Theory and Modeling Vol 3, pp113-120.

Serisena, U.W.,Kumleng,G.M and Yahya, Y.A. (2004) A new Butcher type two-step block Hybrid multistep method for accurate and efficient parallel of ODEs. Abacus 31(2A); Math Series.

Fatunla,S.O. (1991), Block Methods for Second Order Differential Equations. International Journal Computer Mathematics. 41; 55-63.

Lambert, J.D. (1973), Computational Methods in Ordinary Differential Equations. John Wiley And Sons, New York, NY.

Henrici, p. (1962), Discrete Variable Methods in Ordinary Differential Equations. John Wiley, New York, P 407.
Chollom, J.P., Ndam, J.N and Kumleng, G.M. (2007), On Some properties of the Block Explicit Methods for the solution of ODEs. Research Joural of Mathematics and Statistics, 4(2); 52-56.
Wu, X.Y. and Xiu, J.L. (2001), Two Low Accuracy Methods for Stiff Systems. Appl. Math. Comput. 123(2001), 141-153.

Table 1: Order and Error Constants for the block methods (SDHBBDF Case $k = 4$)

Order	Error constants
8	$\frac{16524443}{2391656512}$
8	$-\frac{23724293}{3549094080}$
8	$-\frac{2682041}{167522880}$
8	$-\frac{27635}{68037378048}$
8	$-\frac{71}{169567440}$

Table 2: Order and Error Constants for the block methods (SDHBBDF Case $k = 5$)

Order	Error Constants
8	$-\frac{5946222278}{607961185575}$
8	$\frac{1173242432}{130129988625}$
8	$\frac{40962556}{1966619025}$
8	$\frac{341646346}{9856500675}$
8	$-\frac{3026919}{17395754467328}$
8	$\frac{310}{1393550277}$

The stability polynomial of the methods which is plotted in MATLAB environment to produce the required absolute stability region of theof the methods as shown in figures 1and 2.

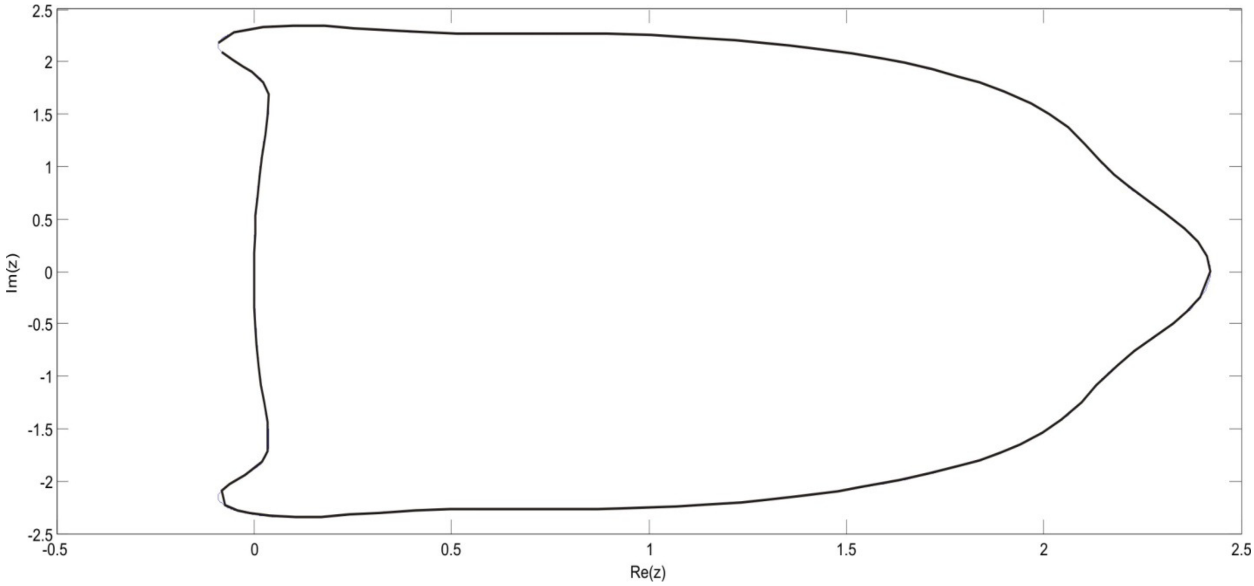

Fig.1 Absolute Stability Regions for the hybrid block methods (SDHBBDF Case$k = 4$)

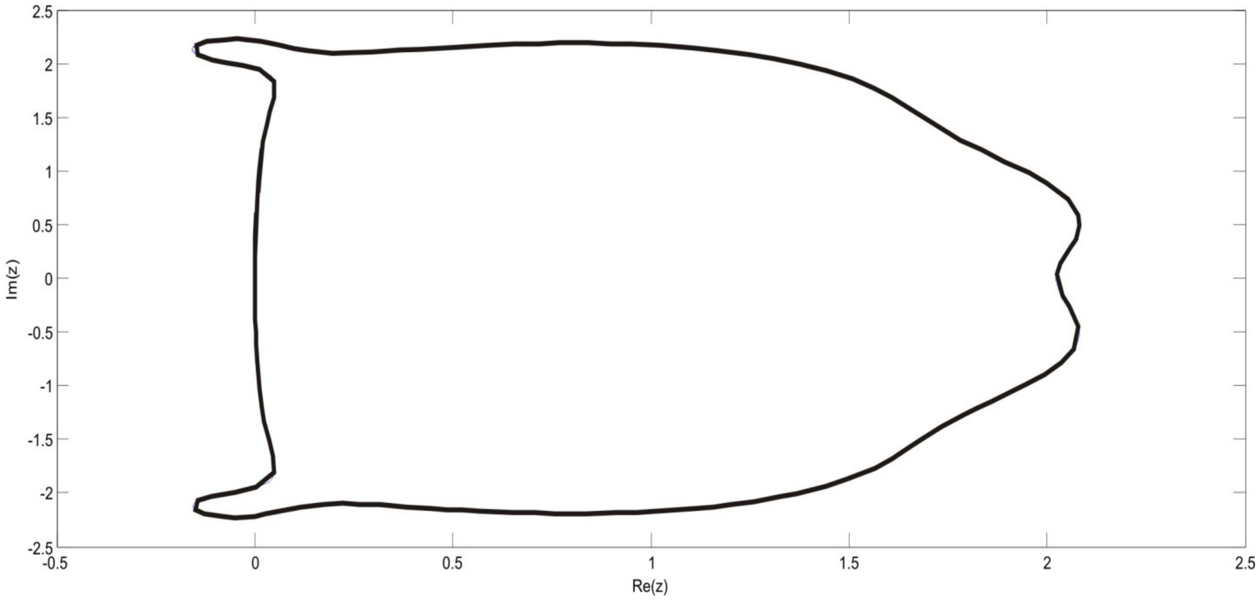

Fig.2 Absolute Stability Regions for the hybrid block methods (SDHBBDF Case$k = 5$)

Some numerical results to illustrate the performance of the methods executed in Matlab language

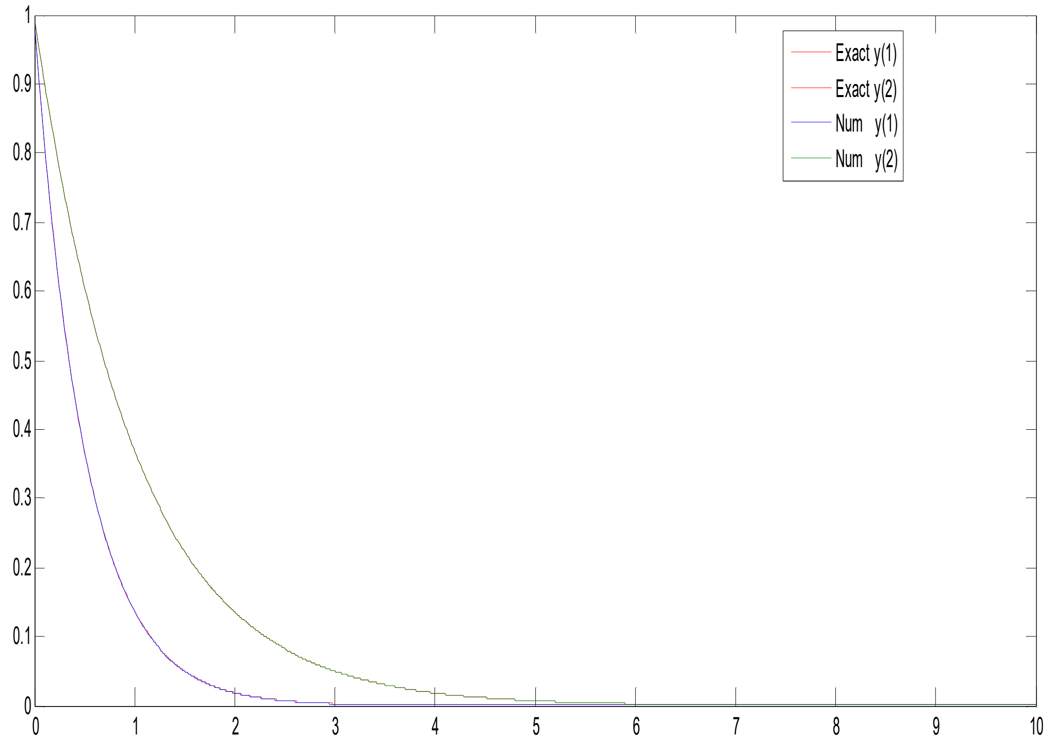

Fig.3 Problem 1: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 4$)

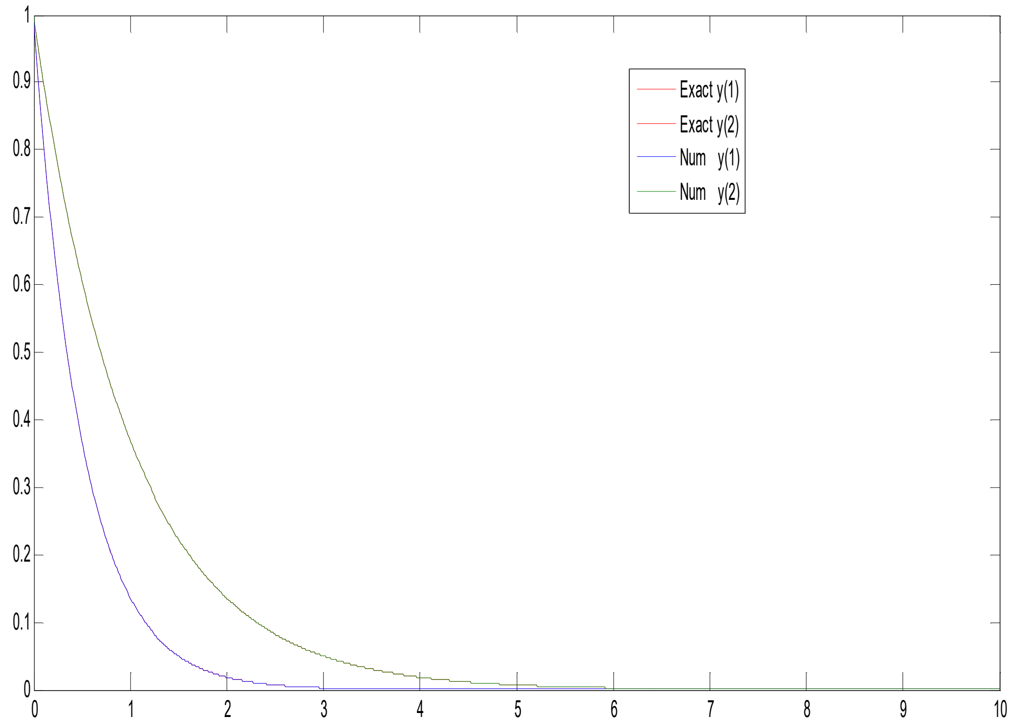

Fig.4 Problem 1: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 5$)

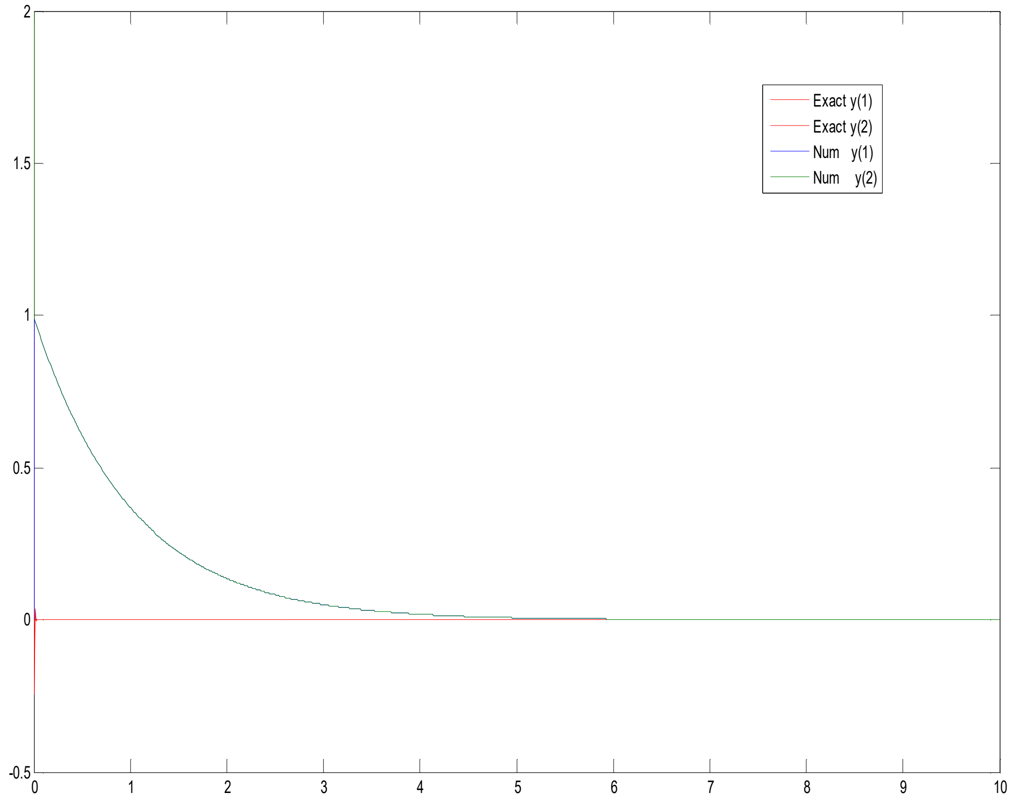

Fig.5 Problem 2: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 4$)

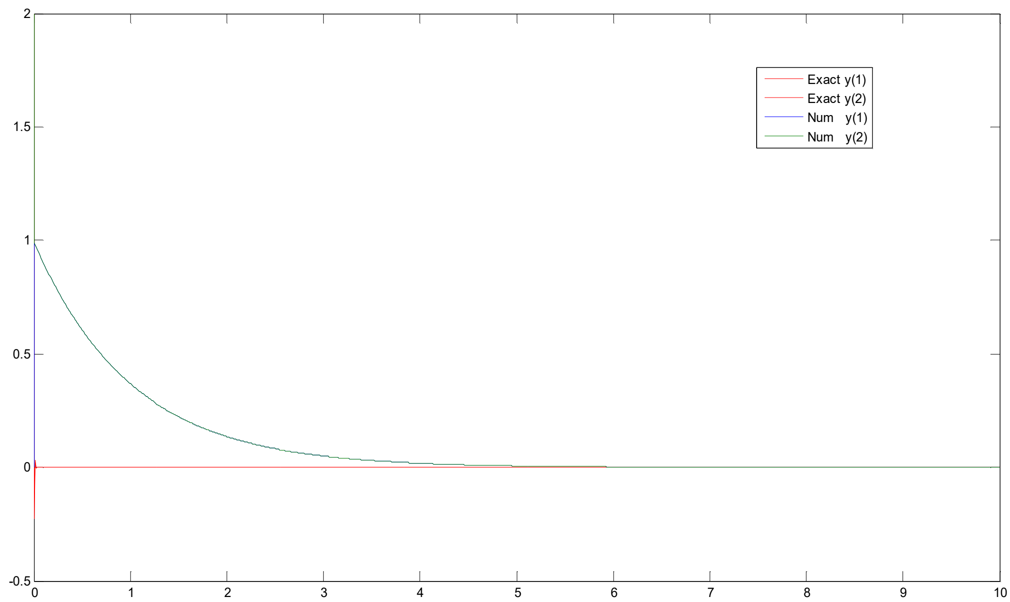

Fig.6 Problem 2: Computed Solution for the hybrid block methods (SDHBBDF Case $k = 5$)