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Implicit Two Step Adam Moulton Hybrid Block Method with Two Off-Step Points for Solving Stiff Ordinary Differential Equations

Kumleng, G. M[1*] Adee S. O[2] Skwame Y[3]

1. Department of Mathematics, University of Jos, P.M.B. 2084, Plateau, State, Nigeria
2. Department of Mathematics, Modibbo Adama University of Technology, Yola, Adamawa state, Nigeria
3. Department of Mathematical Science, Adamawa State University, Mubi, Nigeria

* E-mail of the corresponding author: kumleng_g@yahoo.com

Abstract
A two step block hybrid Adam Moulton method of uniform order five is presented for the solution of stiff initial value problems. The individual schemes that made up the block method are obtained from the same continuous scheme which is applied to provide the solutions of stiff initial value problems on non overlapping intervals. The constructed block method is consistent, zero – stable and A – stable. Numerical results obtained using the new block method show that it is superior for stiff systems and competes well with existing ones.
Keywords: stiff ODEs, Block Method, Adam Moulton method, Stability

1. Introduction

Let us consider the stiff initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{1}$$

on the finite interval $I = [x_0, x_N]$ where $y:[x_0, x_N] \rightarrow \Re^m$ and $f:[x_0, x_N] \times \Re^m \rightarrow \Re^m$ is continuous and differentiable. It is a well known fact that (1) always occur in engineering and control systems. Fatunla (1998) noted that mathematical formulation of new models of physical situations in engineering and sciences often lead to systems of the form (1) and as such there was need to generate techniques to conveniently cope with these type of problems. The search for better methods for solving these stiff systems leads to the discovering of the Backward differentiation formulae (BDF) by Curtiss and Hirschfelder (1952). Since then, most of the improvements in the class of linear multistep methods have been based on the BDF, because of its special properties. Cash (1980) introduced the extended BDF. Ebadi et al. (2010) worked on the hybrid BDF with one additional off-grid point introduced in the first derivative of the solution to improve the absolute stability region of the method.

Akinfenwa et al.(2011) has proposed an A-stable block hybrid Adam Moulton method incorporating four off – step points to improve on the accuracy of the two step Adam Moulton method for the solution of stiff ordinary differential equations. However, this method has improved the accuracy of the two step Adam Moulton method, unfortunately it entails greater computational efforts due to the many off –step points involved.

In this paper, an attempt is made to reducing the computational efforts involved in Akinfenwa et al (2011) in solving stiff ordinary differential equations while retaining the accuracy of the method. We achieved this by incorporating just two off–step collocation points at $x_{n+\frac{5}{4}}$ and $x_{n+\frac{7}{4}}$ in the last subinterval $[x_{n+1}, x_{n+2}]$ of the two step Adam Moulton method as compared to the four ($x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$, $x_{n+\frac{4}{3}}$ and $x_{n+\frac{5}{3}}$) in Akinfenwa *et. al* (2011). The individual schemes that constitute the new two step block hybrid Adam Moulton method are obtained from the same continuous scheme which is applied to provide the solution for (1). Milne (1953) was the first to introduced block methods for use only as a means of obtaining starting values for predictor – corrector algorithms. Many researchers have also worked on the block methods (see Rosser, (1967), Shampine, *et al.* (1969)) for general use. The block method is self starting and so does not require starting values or special predictors to predict the off-grid values.

The paper is presented as follows: In section 2, we discuss the derivation of the block method and obtain the continuous scheme of the method which is used in generating the members of the block method. In section 3, the stability analysis of the block method is presented. In section 4, we briefly discuss the implementation of the method. In section 5, we give some concluding remarks.

2. Derivation of the Block Method

This section is concerned with the derivation of the two step hybrid block Adam Moulton method using the idea of interpolation and collocation proposed by Lambert (1973, 1991).The idea was later referred to as the multistep collocation method by Lie and Norsett (1989). Onumanyi *et al.* (1994, 1999) has also worked on the multistep collocation method.

The general k – step Linear multistep method is given as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2}$$

From (2), we obtain the general form of the k- step Adam Moulton as

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{3}$$

We obtain the continuous scheme for the two step Adam Moulton method incorporating the two off-step points $x_{n+\frac{5}{4}}$ and $x_{n+\frac{7}{4}}$ as

$$y(x) = \alpha_1(x)y_{n+1} + h\left(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_{\frac{5}{4}}(x)f_{n+\frac{5}{4}} + \beta_{\frac{7}{4}}(x)f_{n+\frac{7}{4}} + \beta_2(x)f_{n+2}\right) \tag{4}$$

From Onumanyi *et al.* (1994), we obtain the continuous coefficients $\alpha_1(x),\ \beta_0(x),\ \beta_1(x),\ \beta_{\frac{5}{4}}(x),\ \beta_{\frac{7}{4}}(x)$ and $\beta_2(x)$ and (4) becomes

$$\left.\begin{aligned} y(x) = y_{n+1} &+ \left(-\tfrac{571}{2100}h + \lambda - \tfrac{201\lambda^2}{140h} + \tfrac{211\lambda^3}{210h^2} - \tfrac{12\lambda^4}{35h^3} + \tfrac{8\lambda^5}{175h^4}\right) f_n \\ &+ \left(-\tfrac{122}{45}h + \tfrac{35\lambda^2}{3h} - \tfrac{131\lambda^3}{9h^2} + \tfrac{20\lambda^4}{3h^3} - \tfrac{16\lambda^5}{15h^4}\right) f_{n+1} \\ &+ \left(\tfrac{616}{225}h - \tfrac{224\lambda^2}{15h} + \tfrac{928\lambda^3}{45h^2} - \tfrac{152\lambda^4}{15h^3} + \tfrac{128\lambda^5}{75h^4}\right) f_{n+\frac{5}{4}} \\ &+ \left(-\tfrac{376}{315}h + \tfrac{160\lambda^2}{21h} - \tfrac{736\lambda^3}{63h^2} + \tfrac{136\lambda^4}{21h^3} - \tfrac{128\lambda^5}{105h^4}\right) f_{n+\frac{7}{4}} \\ &+ \left(\tfrac{79}{180}h - \tfrac{35\lambda^2}{12h} + \tfrac{836\lambda^3}{18h^2} - \tfrac{8\lambda^4}{3h^3} + \tfrac{8\lambda^5}{15h^4}\right) f_{n+2} \end{aligned}\right\} \tag{5}$$

Evaluating (5) at the following points $\lambda = 0,\ \frac{5}{4}h,\ \frac{7}{4}h\ ,2h$ yields the following discrete schemes which constitute the two step hybrid block Adam Moulton method.

$$\left.\begin{aligned} y_{n+1} - y_n &= \tfrac{h}{6300}\left(1713 f_n + 17080 f_{n+1} - 17248 f_{n+\frac{5}{4}} + 7520 f_{n+\frac{7}{4}} - 2765 f_{n+2}\right) \\ y_{n+\frac{5}{4}} - y_{n+1} &= \tfrac{h}{100800}\left(-33 f_n + 11095 f_{n+1} + 15218 f_{n+\frac{5}{4}} - 1570 f_{n+\frac{7}{4}} + 490 f_{n+2}\right) \\ y_{n+\frac{7}{4}} - y_{n+1} &= \tfrac{h}{11200}\left(9 f_n + 315 f_{n+1} + 5586 f_{n+\frac{5}{4}} + 2910 f_{n+\frac{7}{4}} - 420 f_{n+2}\right) \\ y_{n+2} - y_{n+1} &= \tfrac{h}{6300}\left(3 f_n + 280 f_{n+1} + 2912 f_{n+\frac{5}{4}} + 2720 f_{n+\frac{7}{4}} + 385 f_{n+2}\right) \end{aligned}\right\} \tag{6}$$

The new block method is of order $(5,5,5,5)^T$ and error constant of $\left(-\frac{253}{115200}, -\frac{131}{9830400}, \frac{159}{3276800}, \frac{1}{38400}\right)^T$.

3. Stability Analysis

From (6), we arrange the block method as a matrix finite difference equation of the form

$$A^{(1)}y_{u+1} - A^{(0)}y_u = hB^{(1)}F_u \tag{7}$$

where

$$Y_{u+1} = \left(y_{n+1}, y_{n+\frac{5}{4}}, y_{n+\frac{7}{4}}, y_{n+2}\right)^T$$

$$Y_u = \left(y_{n-1}, y_{n+\frac{1}{4}}, y_{n+\frac{3}{4}}, y_n\right)^T$$

$$F_u = \left(f_{n+1}, f_{n+\frac{5}{4}}, f_{n+\frac{7}{4}}, f_{n+2}\right)^T$$

for $u = 0,...$ and $n = 0,...$ and the matrices $A^{(1)}, A^{(0)}, B^{(1)}$ are 4 by 4 matrices whose entries are given by the coefficients of (6).

3.1 Zero stability block method

According to Fatunla (1994), a block method is said to be zero stable if $\lambda_{ij} = 1, 2, ..., k$ specified as

$$\rho(\lambda) = \left| \sum_{i=0}^{k} A^{(i)} \lambda^{k-i} \right| = 0 \tag{8}$$

satisfies $\rho(\lambda) = |\lambda_j| \le 1$, the multiplicity must not exceed two.

The block method (6) is expressed in matrix equation form to obtain

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first characteristic polynomial of the new block method (6) is given by

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)}) \tag{9}$$

substituting $A^{(0)}$ *and* $A^{(1)}$ into the characteristic polynomial (9) gives

$$\rho(\lambda) = \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = \begin{vmatrix} \lambda & 0 & 0 & -1 \\ -\lambda & \lambda & 0 & 0 \\ -\lambda & 0 & \lambda & 0 \\ -\lambda & 0 & 0 & \lambda \end{vmatrix}$$

$$= \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$. Following Henrici (1962), the new block method (6) is zero stable and consistent since its order $p = 5 > 1$. The method is therefore convergent.

3.2 Region of Absolute Stability of the Block method

The absolute stability region of the block method is obtained using Chollom et al. (2007) and is as shown below,

4. Numerical Experiment

In this section, some numerical experiments are presented in order to ascertain the performance of our new block method (6). The problems considered are the ones solved by Akinfenwa *et. al*. (2011) and the results obtained shall be compared with this method .

Example 1

Consider the stiffly linear problem

$$y_1' = -29998 y_1 - 59994 y_2 \qquad y_1(0) = 1$$

$$y_2' = 9999 y_1 + 19997 y_2 \qquad y_2(0) = 0$$

$0 \le t \le 10$

The eigenvalues of the system are $\lambda_1 = -10000$ and $\lambda_2 = -1$ with exact solution

$$y_1(x) = \tfrac{1}{9999}(29997e^{-10000x} - 19998e^{-x})$$
$$y_2(x) = e^{-x} - e^{-10000x}$$

Example 2
Consider the Stiffly nonlinear problem

$$\begin{aligned} y'_1 &= -\left(\varepsilon^{-1}+2\right)y_1 + \varepsilon^{-1}y_2^2 \qquad && y_1(0)=1 \\ y'_2 &= y_1 - y_2 - y_2^2 && y_2(0)=1 \\ 0 &\le t \le 10 \end{aligned}$$

where $\varepsilon = 10^{-6}$, the smaller ε is, the more serious the stiffness of the system. The exact solution is $y_1(x) = y_2^2(x)$, $y_2(x) = e^{-x}$

5. Conclusion
The numerical results obtained in Table1 and 2 for the problems solved suggest that the new proposed block hybrid method (6) is suitable for solving stiff problems and perform competitively well with less computational effort compared with the method of Akinfenwa et al (2011).

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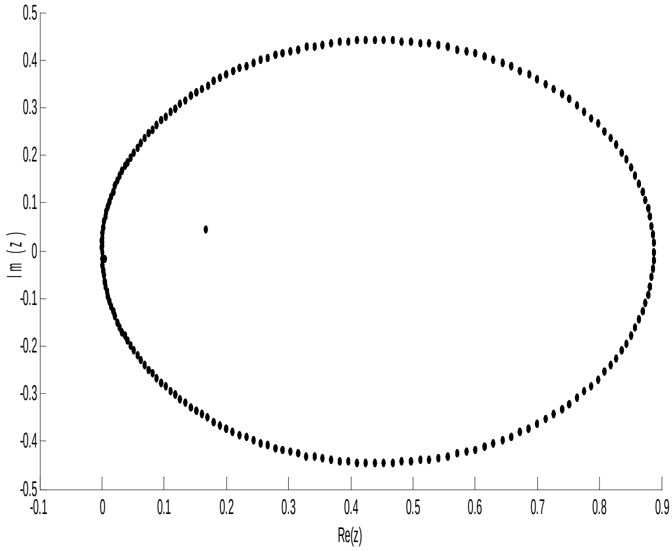

Fig.1 Absolute Stability Region of the New Block Method

The absolute stability region consists of the set of points in the complex plane outside the enclosed figure. Therefore, the block method is A - stable since the left –half complex plane is contained in S. where $S = \{ z \in \mathbb{C} : R(z) \le 1 \}$.

Table 1. Example 1 Absolute Error For New Block Hybrid Method With Two Off-Steps Compared With Akinfenwa Et Al.(2011) With Four Off-Steps At The End Point T = 10

error = $|y_i - y(X_i)|$

h	error 1 New block method	Akinfenwa et al (2011) block method	New block method	error 2 Akinfenwa et al (2011) block method
0.01	1×10^{-15}	8.26×10^{-15}	0	4.13×10^{-15}
0.001	0	4.66×10^{-15}	0	2.33×10^{-15}

Table 2: Example 2 Absolute Error For New Block Hybrid Method With Two Off-Steps Compared With Akinfewa Et Al.(2011) With Four Off-Steps At The End Point T = 10

error = $|y_i - y(X_i)|$

h	error 1 New block method	Akinfenwa et al (2011) block method	New block method	error 2 Akinfenwa et al (2011) block method
0.1	2.16×10^{-11}	4.5×10^{-15}	9.2×10^{-12}	4.8×10^{-15}
0.01	2.0×10^{-13}	1.4×10^{-16}	1.42×10^{-14}	2.6×10^{-15}