Journal of Natural Sciences Research www.iiste.org
ISSN 2224-3186 (Paper) ISSN 2225-0921 (Online)
Vol.1, No.1, 2011

A Common Fixed Point Theorem for Compatible Mapping

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Abstract

The main aim of this paper is to prove a common fixed point theorem involving pairs of compatible mappings of type (A) using five maps and a contractive condition. This article represents a useful generalization of several results announced in the literature.

Key words: Complete metric space, Compatible mapping of type (A), Commuting mapping, Cauchy Sequence, Fixed points.

1. Introduction

By using a compatibility condition due to Jungck many researchers establish lot of common fixed point theorems for mappings on complete and compact metric spaces. The study of common fixed point of mappings satisfying contractive type conditions are also studied by many mathematicians. Jungck.G.(1986) prove a theorem on Compatible mappings and common fixed points. Also Sessa.S(1986) prove a weak commutativity condition in a fixed point consideration. After that Jungck, Muthy and Cho(1993) define the concept of compatible map of type (A) and prove a theorem on Compatible mapping of type (A) and common fixed points. Khan, M.S., H.K. Path and Reny George(2007) find a result of Compatible mapping of type (A-1) and type (A-2) and common fixed points in fuzzy metric spaces. Recently Vishal Gupta(2011) prove a common fixed point theorem for compatible mapping of type (A).

2.Preliminaries

Definition 2.1. Self maps S and T of metric space (X,d) are said to be weakly commuting pair

iff $d(STx,TSx)\leq d(Sx,Tx)$ for all x in X.

Definition 2.2. Self maps S and T of a metric space (X,d) are said to be compatible of type (A) if

$\lim d(TSx_n,SSx_n) =0$ and $\lim d(STx_n, TTx_n)=0$ as $n\to\infty$ whenever $\{x_n\}$ is a

Sequence in X such that $\lim Sx_n=\lim Tx_n =t$ as $n\to\infty$ for some t in X.

Definition 2.3. A function Φ: [0, ∞) → [0, ∞) is said to be a contractive modulus if Φ (0) =0 and

$$\Phi(t) < t \quad \text{for } t > 0.$$

3.Main Result

Theorem 3.1. Let S,R,T,U and I are five self maps of a complete metric space (X,d) into itself satisfying the following conditions:

(i) $SR \cup TU \subset I(x)$

(ii) $d(SRx, TUy) \le \alpha d(Ix, Iy) + \beta[d(SRx, Ix) + d(TUy, Iy)] + \gamma[d(Ix, TUy) + d(Iy, SRx)$

for all $x, y \in X$ and α, β and γ are non negative real's such that $\alpha + 2\beta + 2\gamma \prec 1$

(iii) One of S,R,T,U and I is continuous

(iv) (SR,I) and (TU,I) are compatible of type (A). then SR,TU and I have a unique common fixed point. Further if the pairs (S,R), (S,I), (R,I),(T,U), (T,I) ,(U,I) are commuting pairs then S,R,T,U,I have a unique common fixed point.

Proof: Let x_0 in X be arbitrary. Construct a sequence $\{Ix_n\}$ as follows:

$Ix_{2n+1} = SRx_{2n}$, $Ix_{2n+2} = TUx_{2n+1}$ n=0,1,2,3…

From condition (ii),we have

$$d(Ix_{2n+1}, Ix_{2n+2}) = d(SRx_{2n}, TUx_{2n+1}) \le \alpha d(Ix_{2n}, Ix_{2n+1}) + \beta[d(SRx_{2n}, Ix_{2n}) + d(TUx_{2n+1}, Ix_{2n+1})] + \gamma[d(Ix_{2n}, TUx_{2n+}$$
$$d(Ix_{2n+1}, SRx_{2n})]$$

$$= \alpha d(Ix_{2n}, Ix_{2n+1}) + \beta[d(Ix_{2n+1}, Ix_{2n}) + d(Ix_{2n+2}, Ix_{2n+1})] + \gamma[d(Ix_{2n}, I_{2n+2}) + d(Ix_{2n+1}, Ix_{2n+1})]$$

$$\le (\alpha + \beta + \gamma) d(Ix_{2n}, Ix_{2n+1}) + (\beta + \gamma) d(Ix_{2n+1}, Ix_{2n+2})$$

Therefore, we have

$$d(Ix_{2n+1}, Ix_{2n+2}) \le \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(Ix_{2n}, Ix_{2n+1})$$

i.e $d(Ix_{2n+1}, Ix_{2n+2}) \le h d(Ix_{2n}, Ix_{2n+1})$ where $h = \dfrac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \prec 1$

Similarly, we can show that

$$d(Ix_{2n+1}, Ix_{2n+2}) \le h^{2n+1} d(Ix_0, Ix_1)$$

For $k \succ n$, we have

$$d(Ix_n, Ix_{n+k}) \le \sum d(I_{n+i-1}, I_{n+i})$$

Journal of Natural Sciences Research www.iiste.org
ISSN 2224-3186 (Paper) ISSN 2225-0921 (Online)
Vol.1, No.1, 2011

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(I_0, I_1)$$

$$d(Ix_n, Ix_{n+k}) \leq \frac{h^n}{1-h} d(Ix_0, Ix_1) \to 0 \text{ as } n \to \infty$$

Hence $\{Ix_n\}$ is a Cauchy sequence. Since X is complete metric space $\exists z \in X$ such that $Ix_n \to z$. Then the subsequence of $\{Ix_n\}$,$\{SRx_{2n}\}$ and $\{TUx_{2n+1}\}$ also converges to z.

i.e $\{SRx_{2n}\} \to z$ & $\{TUx_{2n+1}\} \to z$.

Suppose that I is continuous and the pair {SR,I} is compatible of type (A),

then from condition (ii), we have

$$d(SR(Ix_{2n}), TUx_{2n+1}) \leq \alpha d(I^2x_{2n}, Ix_{2n+1}) + \beta[d(SR(Ix_{2n}), I^2x_{2n}) + d(TUx_{2n+1}, Ix_{2n+1})] + \gamma[d(I^2x_{2n}, TUx_{2n+1}) + d(Ix_{2n+1}, SR(Ix_{2n}))]$$

Since I is continuous, $I^2x_{2n} \to Iz$ as $n \to \infty$.

The pair (SR,I) is compatible of type (A), then $SRIx_{2n} \to Iz$ as $n \to \infty$

Letting $n \to \infty$, we have

$$d(Iz, z) \leq \alpha d(Iz, z) + \beta[d(Iz, Iz) + d(z, z)] + \gamma[d(Iz, z) + d(z, Iz)]$$

$$= (\alpha + 2\gamma) d(Iz, z)$$

Hence
$d(Iz, z) = 0$ and $Iz = z$ since $\alpha + 2\gamma < 1$

Again, we have

$$d(SRz, TUx_{2n+1}) \leq \alpha d(Iz, Ix_{2n+1}) + \beta[d(SRz, Iz) + d(TUx_{2n+1}, Ix_{2n+1})] + \gamma[d(Iz, TUx_{2n+1}) + d(Ix_{2n+1}, SRz)]$$

Letting $n \to \infty$ and using $Iz = z$, we have

$$d(SRz, z) \leq \alpha d(z, z) + \beta[d(SRz, Iz) + d(z, z)] + \gamma[d(z, z) + d(z, SRz)]$$

$$= (\beta + \gamma) d(SRz, z)$$

Hence $d(SRz, z) = 0$ and $(SR)z = z$ since $\beta + \gamma < 1$

Since $SR \subset I$, $\exists z' \in X$ such that

$z = SRz = Iz' = Iz$

Now

$$d(z, TUz') = d(SRz, TUz')$$

$$\leq \alpha d(Iz, Iz') + \beta[d(SRz, Iz) + d(TUz', Iz')] + \gamma[d(Iz, TUz') + d(Iz', SRz)]$$

$$= \alpha d(z,z) + \beta[d(z,z) + d(TUz',z)] + \gamma[d(z,TUz') + d(SRz,SRz)]$$

Implies $d(TUz',z) = 0$ as $\alpha + \gamma < 1$

Hence $TUz' = z = Iz'$

Take $y_n = z'$ for $n \geq 1$, we have $TUy_n \to Tz'$

Now, again

$$d(z,TUz) = d(SRz,TUz)$$
$$\leq \alpha d(Iz,Iz) + \beta[d(SRz,Iz) + d(TUz,Iz)] + \gamma[d(Iz,TUz) + d(Iz,SRz)]$$
$$= \alpha d(z,z) + \beta[d(Iz,Iz) + d(TUz,z)] + \gamma[d(z,TUz) + d(z,z)]$$
$$= (\beta + \gamma)d(TUz,z)$$

Hence $TUz = z$

For uniqueness, Let z and w be the common fixed point of SR,TU and I, then by condition (ii)

$$d(z,w) = d(SRz,TUw)$$
$$\leq \alpha d(Iz,Iw) + \beta[d(SRz,Iz) + d(TUw,Iw)] + \gamma[d(Iz,TUw) + d(Iw,SRz)]$$
$$= \alpha d(z,w) + \beta[d(z,z) + d(w,w)] + \gamma[d(z,w) + d(w,z)]$$
$$= (\alpha + 2\gamma)d(z,w)$$

Since $\alpha + 2\gamma < 1$ we have $z = w$.

Further assume that (S, R), (S, I), (T, U) and (T, I) are commuting pairs and z be the unique common fixed point of pairs SR, TU and I then

$Sz = S(SRz) = S(RSz) = SR(Sz)$, $Sz = S(Iz) = I(Sz)$

$Rz = R(SRz) = (RS)(Rz) = SR(Rz)$, $Rz = R(Iz) = I(Rz)$

Which shows that Sz and Rz is the common fixed point of SR and I, yielding thereby

$Sz = z = Rz = Iz = SRz$

In view of uniqueness of the common fixed point of SR and I. therefore z is the unique common fixed point of S, R and I. Similarly using commutatively of (T, U) and (T, I) it can be shown that

$Tz = z = Uz = Iz = TUz$

Thus z is the unique common fixed point of S, R, T, U and I.

Hence the proof.

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