Common Fixed Point Theorems in Non-Archimedean Normed Space

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Abstract

The purpose of this paper is to prove some common fixed point theorem for single valued and multi-valued contractive mapping having a pair of maps on a spherically complete non-Archimedean normed space.

Key-Words: Fixed point, Contractive mapping, Non-Archimedean normed space, spherically complete metric space.

1. Introduction

C.Petals et al. (1993) proved a fixed point theorem on non-Archimedean normed space using a contractive condition. This result is extended by Kubiaczyk(1996) from single valued to multi valued contractive mapping. Also for non expansive multi valued mapping, some fixed point theorems are proved. In 2008, K.P.R Rao(2008) proved some common fixed point theorems for a pair of maps on a spherically complete metric space.

2. Preliminaries

Definition 2.1. A non-Archimedean normed space $(X,\| \|)$ is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.

Definition 2.2. Let $(X,\| \|)$ be a normed space and $T:X\to X$, then T is said to be contractive iff whenever x & y are distinct points in X,

$$\|Tx-Ty\|<\|x-y\|$$

Definition 2.3. Let $(X,\| \|)$ be a normed space let $T:X\to Comp(X)$ (The space of all compact subsets of X with Hausdroff distance H), then T is said to be a multivalued contractive mapping if

$H(Tx,Ty)<\|x-y\|$ for any distinct points in X.

3. Main Results

Theorem 3.1. Let X be a non-Archimedean spherically complete normed space. If f and T are self maps on X satisfying $T(X)\subseteq f(X)$

$$\|Tx-Ty\|<\|f(x)-f(y)\| \qquad \forall x,y\in X, x\neq y$$

Then there exist $z\in X$ such that $fz=Tz$.

Further if f and T are coincidentally commuting at z then z is unique common fixed point of f and T.

Proof: Let $B_a = B\left[f(a), \| f(a) - Ta\| \right]$ denote the closed spheres centered at fa with radii $\| f(a) - Ta\|$, and let $\mathcal{F}$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \le B_b \text{ iff } B_b \subseteq B_a$$

is a partial order. Consider a totally ordered subfamily $\mathcal{F}_1$ of $\mathcal{F}$. From the spherically completeness of X, we have

$$\bigcap_{B_a \in \mathcal{F}_1} B_a = B \ne \phi$$

Let $fb \in B$ and $B_a \in \mathcal{F}_1$ since $fb \in B_a$ implies $\| f(b) - f(a)\| \le \| f(a) - T(a)\|$ ………….. (3.1)

Let $x \in B_b$, then

$$\begin{aligned} \|x - fb\| &\le \| fb - Tb\| \\ &\le \max\left\{ \| fb - fa\|, \| fa - Ta\|, \|Ta - Tb\| \right\} \\ &= \left\{ \| fa - Ta\|, \|Ta - Tb\| \right\} \\ &< \| fa - Ta\| \qquad \left[\because \|Ta - Tb\| \le \| fa - fb\| \right] \end{aligned}$$

Now

$$\begin{aligned} \|x - fa\| &\le \max\left\{ \|x - fb\|, \| fb - fa\| \right\} \\ &< \| fa - Ta\| \end{aligned}$$

Implies $x \in B_a$. Hence $B_b \subseteq B_a$ for any $B_a \in \mathcal{F}_1$.

Thus B_b is an upper bound in $\mathcal{F}$ for the family $\mathcal{F}_1$ and hence by Zorn's Lemma, $\mathcal{F}$ has a maximal element (say) $B_z, z \in X$.

Suppose $fz \ne Tz$

Since $Tz \in T(X) \subseteq f(X), \exists w \in X$ such that $Tz = fw$, clearly $z \ne w$. Now

$$\begin{aligned} \| fw - Tw\| &= \|Tz - Tw\| \\ &\le \| fz - fw\| \end{aligned}$$

Thus $fz \notin B_w$ and hence $B_z \not\subset B_w$, It is a contradiction to maximality of B_z.

Hence $fz = Tz$

Further assume that f and T are coincidently commuting at z.

Then $f^2 z = f(fz) = f(Tz) = Tf(z) = T(Tz) = T^2 z$

Suppose $fz \ne z$

Now $\|T(fz) - T(z)\| \le \| f(fz) - f(z)\| = \left\| f^2 z - fz\right\| = \|T(fz) - Tz\|$

Hence $fz = z$. Thus $z = fz = Tz$

Let v be a different fixed point, for $v \ne z$, we have

$$\begin{aligned} \|z - v\| = \|Tz - Tv\| &\le \| fz - fv\| \\ &= \|z - v\| \end{aligned}$$

This is a contradiction. The proof is completed.

Theorem 3.2 Let X be a non-Archimedean spherically complete normed space. Let $f : X \to X$ and $T : X \to C(X)$ (the space of all compact subsets of X with the Hausdroff distance H) be satisfying

$$T(X) \subseteq f(X) \qquad (3.2)$$

$$H(Tx, Ty) \le \| fx - fy\| \qquad (3.3)$$

For any distinct points x and y in X. then there exist $z \in X$ such that $fz = Tz$.

Further assume that

$$\| fx - fw\| \le H\left(T(fx), Tw\right) \quad \text{for all } x, y, w \in X \text{ with } fx \in Ty \qquad (3.4)$$

And f and T are coincidentally commuting at z. (3.5)

Then fz is the unique common fixed point of f and T.

Proof: Let $B_a = \left(fa, d(fa, Ta)\right)$ denote the closed sphere centered at fa with radius $d(fa, Ta)$ and $\mathcal{F}$ be the collection of these spheres for all $a \in X$. Then the relation

$$B_a \le B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{F}$. Let $\mathcal{F}_1$ be a totally ordered subfamily of $\mathcal{F}$. From the spherically completeness of X, we have

$$\bigcap_{B_a \in \mathcal{F}_1} B_a = B \ne \varnothing$$

Let $fb \in B$ and $B_a \in \mathcal{F}_1$, then $fb \in B_a$

Hence $\| fb - fa\| \le d(fa, Ta)$ (3.6)

If $a = b$, then $B_a = B_b$. Assume that $a \ne b$

Since Ta is complete, $\exists w \in Ta$ such that $\| fa - w\| = d(fa, Ta)$ (3.7)

Consider $x \in B_b$, then

$$\begin{aligned} \|x - fb\| \le d\left(fb, Tb\right) &= \inf_{c \in Tb} \| fb - c\| \\ &\le \max\left\{\| fb - fa\|, \| fa - w\|, \inf_{c \in Tb} \|w - c\|\right\} \\ &\le \max\left\{d(fa, Ta), H(Ta, Tb)\right\} \\ &\le d(fa, Ta) \qquad (3.8) \end{aligned}$$

Also

$$\begin{aligned} \|x - fa\| &\le \max\left\{\|x - fb\|, \| fb - fa\|\right\} \\ &\le d(fa, Ta) \end{aligned}$$

Thus $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in \mathcal{F}_1$. Thus B_b is an upper bound in $\mathcal{F}$ for the family $\mathcal{F}_1$ and hence by Zorn' Lemma $\mathcal{F}$ has maximal element say B_z, for some $z \in X$

Suppose $f_z \notin Tz$, since Tz is compact, there exist $k \in Tz$ such that

$$d(fz, Tz) = \| fz - k\|$$

Since

$$T(X) \subseteq f(X), \exists u \in X \text{ Such that } k = fu$$

Thus $d(fz, Tu) = \| fz - fu\|$ (3.9)

Clearly $z \ne u$.

Now

$$\begin{aligned} d(fu, Tu) &\le H(Tz, Tu) \\ &= \| fz - fu\| \end{aligned}$$

Hence $fz \notin B_u$, thus $B_z \not\subset B_u$

It is a contradiction to the maximality of B_z, hence $fz \in Tz$.

Further assume (3.4) and (3.5)

Write $fz = p$, then $p \in Tz$, from (3.6)

$$\begin{aligned} \|p - fp\| = \| fz - fp\| &\le H(Tfz, Tp) \\ &= H(Tp, Tp) \\ &= 0 \end{aligned}$$

Implies $fp = p$. From (3.7) $p = fp \in fTz \subseteq Tfz = Tp$

Thus $fz = p$ is a common fixed point of f and T.

Suppose $q \in X, q \ne p$ is such that $q = fq \in Tq$

From (3.6) and (3.7)

$$\|p-q\| = \|fp - fq\| \le H(Tfp, Tq) = H(Tp, Tq) \le \|fp - fq\| = \|p-q\|$$

Implies $p = q$ .Thus $p = fz$ is the unique common fixed point of f and T.

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