

# Common Fixed Point Theorems in Non-Archimedean Normed Space

Vishal Gupta<sup>1\*</sup>, Ramandeep Kaur<sup>2</sup>

1. Department of Mathematics, Maharishi Markandeshwar University, Mullana, Ambala, Haryana, (India)
2. Department of Mathematics, Karnal Institutes of Technology and Management Karnal, Haryana, (India)

\* E-mail of the corresponding author: [vishal.gmn@gmail.com](mailto:vishal.gmn@gmail.com),  
[vk Gupta09@rediffmail.com](mailto:vk Gupta09@rediffmail.com)

## Abstract

The purpose of this paper is to prove some common fixed point theorem for single valued and multi-valued contractive mapping having a pair of maps on a spherically complete non-Archimedean normed space.

**Key-Words:** Fixed point, Contractive mapping, Non-Archimedean normed space, spherically complete metric space.

## 1. Introduction

C.Petals et al. (1993) proved a fixed point theorem on non-Archimedean normed space using a contractive condition. This result is extended by Kubiacyk(1996) from single valued to multi valued contractive mapping. Also for non expansive multi valued mapping, some fixed point theorems are proved. In 2008, K.P.R Rao(2008) proved some common fixed point theorems for a pair of maps on a spherically complete metric space.

## 2. Preliminaries

*Definition 2.1.* A non-Archimedean normed space  $(X, \|\cdot\|)$  is said to be spherically complete if every shrinking collection of balls in  $X$  has a non empty intersection.

*Definition 2.2.* Let  $(X, \|\cdot\|)$  be a normed space and  $T: X \rightarrow X$ , then  $T$  is said to be contractive iff whenever  $x$  &  $y$  are distinct points in  $X$ ,

$$\|Tx - Ty\| < \|x - y\|$$

*Definition 2.3.* Let  $(X, \|\cdot\|)$  be a normed space let  $T: X \rightarrow \text{Comp}(X)$  (The space of all compact subsets of  $X$  with Hausdroff distance  $H$ ), then  $T$  is said to be a multivalued contractive mapping if

$$H(Tx, Ty) < \|x - y\| \text{ for any distinct points in } X.$$

## 3. Main Results

*Theorem 3.1.* Let  $X$  be a non-Archimedean spherically complete normed space. If  $f$  and  $T$  are self maps on  $X$  satisfying  $T(X) \subseteq f(X)$

$$\|Tx - Ty\| < \|f(x) - f(y)\| \quad \forall x, y \in X, x \neq y$$

Then there exist  $z \in X$  such that  $fz = Tz$ .

Further if  $f$  and  $T$  are coincidentally commuting at  $z$  then  $z$  is unique common fixed point of  $f$  and  $T$ .

*Proof:* Let  $B_a = B[f(a), \|f(a) - Ta\|]$  denote the closed spheres centered at  $fa$  with radii  $\|f(a) - Ta\|$ , and let  $F$  be the collection of these spheres for all  $a \in X$ . The relation

$$B_a \leq B_b \text{ iff } B_b \subseteq B_a$$

is a partial order. Consider a totally ordered subfamily  $F_1$  of  $F$ . From the spherically completeness of  $X$ , we have

$$\bigcap_{B_a \in F_1} B_a = B \neq \emptyset$$

Let  $fb \in B$  and  $B_a \in F_1$  since  $fb \in B_a$  implies  $\|f(b) - f(a)\| \leq \|f(a) - T(a)\| \dots\dots\dots$   
 (3.1)

Let  $x \in B_b$ , then

$$\begin{aligned} \|x - fb\| &\leq \|fb - Tb\| \\ &\leq \max \{ \|fb - fa\|, \|fa - Ta\|, \|Ta - Tb\| \} \\ &= \{ \|fa - Ta\|, \|Ta - Tb\| \} \\ &< \|fa - Ta\| \quad [ \because \|Ta - Tb\| \leq \|fa - fb\| ] \end{aligned}$$

Now

$$\begin{aligned} \|x - fa\| &\leq \max \{ \|x - fb\|, \|fb - fa\| \} \\ &< \|fa - Ta\| \end{aligned}$$

Implies  $x \in B_a$ . Hence  $B_b \subseteq B_a$  for any  $B_a \in F_1$ .

Thus  $B_b$  is an upper bound in  $F$  for the family  $F_1$  and hence by Zorn's Lemma,  $F$  has a maximal element (say)  $B_z, z \in X$ .

Suppose  $fz \neq Tz$

Since  $Tz \in T(X) \subseteq f(X), \exists w \in X$  such that  $Tz = fw$ , clearly  $z \neq w$ . Now

$$\begin{aligned} \|fw - Tw\| &= \|Tz - Tw\| \\ &\leq \|fz - fw\| \end{aligned}$$

Thus  $fz \notin B_w$  and hence  $B_z \not\subseteq B_w$ , It is a contradiction to maximality of  $B_z$ .

Hence  $fz = Tz$

Further assume that  $f$  and  $T$  are coincidentally commuting at  $z$ .

$$\text{Then } f^2z = f(fz) = f(Tz) = Tf(z) = T(Tz) = T^2z$$

Suppose  $fz \neq z$

$$\text{Now } \|T(fz) - T(z)\| \leq \|f(fz) - f(z)\| = \|f^2z - fz\| = \|T(fz) - Tz\|$$

Hence  $fz = z$ . Thus  $z = fz = Tz$

Let  $v$  be a different fixed point, for  $v \neq z$ , we have

$$\begin{aligned} \|z - v\| &= \|Tz - Tv\| \leq \|fz - fv\| \\ &= \|z - v\| \end{aligned}$$

This is a contradiction. The proof is completed.

**Theorem 3.2** Let  $X$  be a non-Archimedean spherically complete normed space. Let  $f : X \rightarrow X$  and  $T : X \rightarrow C(X)$  (the space of all compact subsets of  $X$  with the Hausdroff distance  $H$ ) be satisfying

$$T(X) \subseteq f(X) \quad (3.2)$$

$$H(Tx, Ty) \leq \|fx - fy\| \quad (3.3)$$

For any distinct points  $x$  and  $y$  in  $X$ . then there exist  $z \in X$  such that  $fz = Tz$ .

Further assume that

$$\|fx - fw\| \leq H(T(fx), Tw) \quad \text{for all } x, y, w \in X \text{ with } fx \in Ty \quad (3.4)$$

And  $f$  and  $T$  are coincidentally commuting at  $z$ . (3.5)

Then  $fz$  is the unique common fixed point of  $f$  and  $T$ .

*Proof:* Let  $B_a = (fa, d(fa, Ta))$  denote the closed sphere centered at  $fa$  with radius  $d(fa, Ta)$  and  $F$  be the collection of these spheres for all  $a \in X$ . Then the relation

$$B_a \leq B_b \text{ iff } B_b \subseteq B_a$$

is a partial order on  $F$ . Let  $F_1$  be a totally ordered subfamily of  $F$ . From the spherically completeness of  $X$ , we have

$$\bigcap_{B_a \in F_1} B_a = B \neq \emptyset$$

Let  $fb \in B$  and  $B_a \in F_1$ , then  $fb \in B_a$

$$\text{Hence } \|fb - fa\| \leq d(fa, Ta) \quad (3.6)$$

If  $a = b$ , then  $B_a = B_b$ . Assume that  $a \neq b$

$$\text{Since } Ta \text{ is complete, } \exists w \in Ta \text{ such that } \|fa - w\| = d(fa, Ta) \quad (3.7)$$

Consider  $x \in B_b$ , then

$$\begin{aligned} \|x - fb\| &\leq d(fb, Tb) = \inf_{c \in Tb} \|fb - c\| \\ &\leq \max \left\{ \|fb - fa\|, \|fa - w\|, \inf_{c \in Tb} \|w - c\| \right\} \\ &\leq \max \left\{ d(fa, Ta), H(Ta, Tb) \right\} \\ &\leq d(fa, Ta) \end{aligned} \quad (3.8)$$

Also

$$\begin{aligned} \|x - fa\| &\leq \max \{ \|x - fb\|, \|fb - fa\| \} \\ &\leq d(fa, Ta) \end{aligned}$$

Thus  $x \in B_a$  and  $B_b \subseteq B_a$  for any  $B_a \in F_1$ . Thus  $B_b$  is an upper bound in  $F$  for the family  $F_1$  and hence by Zorn's Lemma  $F$  has maximal element say  $B_z$ , for some  $z \in X$

Suppose  $fz \notin Tz$ , since  $Tz$  is compact, there exist  $k \in Tz$  such that

$$d(fz, Tz) = \|fz - k\|$$

Since

$$T(X) \subseteq f(X), \exists u \in X \text{ Such that } k = fu$$

$$\text{Thus } d(fz, Tu) = \|fz - fu\| \quad (3.9)$$

Clearly  $z \neq u$ .

Now

$$\begin{aligned} d(fu, Tu) &\leq H(Tz, Tu) \\ &= \|fz - fu\| \end{aligned}$$

Hence  $fz \notin B_u$ , thus  $B_z \not\subseteq B_u$

It is a contradiction to the maximality of  $B_z$ , hence  $fz \in Tz$ .

Further assume (3.4) and (3.5)

Write  $fz = p$ , then  $p \in Tz$ , from (3.6)

$$\begin{aligned} \|p - fp\| &= \|fz - fp\| \leq H(Tfz, Tp) \\ &= H(Tp, Tp) \\ &= 0 \end{aligned}$$

Implies  $fp = p$ . From (3.7)  $p = fp \in fTz \subseteq Tfz = Tp$

Thus  $fz = p$  is a common fixed point of  $f$  and  $T$ .

Suppose  $q \in X, q \neq p$  is such that  $q = fq \in Tq$

From (3.6) and (3.7)

$$\|p - q\| = \|fp - fq\| \leq H(Tfp, Tq) = H(Tp, Tq) \leq \|fp - fq\| = \|p - q\|$$

Implies  $p = q$ . Thus  $p = fz$  is the unique common fixed point of  $f$  and  $T$ .

### References

Kubiacyk, N. Mostafa Ali (1996), A multivalued fixed point theorems in Non-Archimedean vector spaces, Nov Sad J. Math, 26(1996).

K.P.R Rao, G.N.V Kishore (2008), Common fixed point theorems in Ultra Metric Spaces, Punjab Uni. J.M, 40, 31-35 (2008).

Petals.C., Vidalis.T. (1993), A fixed point theorem in Non-Archimedean vector spaces, Proc. Amer. Math. Soc. 118(1993).

This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE's homepage:

<http://www.iiste.org>

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. **Prospective authors of IISTE journals can find the submission instruction on the following page:**

<http://www.iiste.org/Journals/>

The IISTE editorial team promises to review and publish all the qualified submissions in a fast manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

### **IISTE Knowledge Sharing Partners**

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

