Meta-Epistemological Nature of Mathematics and Its Implication to the Empirical Learning Process in Mathematics: An Extension of Plato's Thoughts

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Abstract
Mathematics philosophers do not have a common agreed understanding about the nature of Mathematics. An attempt to describe the nature of Mathematics begun in the 4th C, BC, when Plato conceived that, Mathematics is constituted by abstract objects existing beyond the mind; and its mathematical knowledge is acquired empirically. Aristotle (Plato's student) extended the ideas of Plato by conceiving that mathematical knowledge can be acquired with or without senses.Since then up to 17th C, AD, debates continued, experimentalists standing on one side and rationalist on the other side, instead of focusing on the fundamental nature of Mathematics, they focused on whether mathematical knowledge is a an empirical knowledge or rational knowledge. However, from 17th C, AD, debates moved from experimentalist versus rationalist to an attempt in resolving paradoxes in real number and set theories. The focus in describing precisely nature of Mathematics was completely lost.In respect to the reviews of Plato's thoughts and related literature, it is concluded that, thoughts of Plato's thoughts are correct and consistent. Metaphysically, Mathematics is built upon a set of mathematical objects that exist beyond the mind. The mathematical objects possess six attributes: shape, qualities, form, structure, properties and applicability. Also, epistemologically, the attributes of mathematical objects build the body of mathematical empirical knowledge. The empirical mathematical knowledge is acquired in a cycle of five stages: perception, exposition, execution, assimilation and rationalization stages.
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1. Introduction

According to Concise Oxford English Dictionary (2007) nature is defined as the basic or inherent features, qualities, and characteristics of a thing. The dictionary also describes Metaphysics as the branch of Philosophy that deals with the first principle of things, real and ideal things. It further describes that Metaphysics has two main aspects: the first aspect considers that what exists lies beyond experience, advocated by Plato and the second aspect considers objects of experience form the only reality, advocated by Kant as argued and reflected in Plato (1952a, 1952b) in Hutchins (Ed), Kant (1984) Boyer (1968) and Kline (1972).

In extending Plato's thoughts about the nature of Mathematics, the researcher holds on the first strand of Metaphysical school of thought as argued by Plato. By holding this view, it is re-advocated that Plato's thoughts are correct and consistent; and that what exists lies beyond human experience, they are experiences of an ideal world of non-physical body of an individual, reflected by the individual's mental faculties of intelligence. And thus, what is experienced is only the outcome of an experiences of individual's non-physical body on the ideal world in the realm of existence of the mind.

Metaphysics concerns with the nature of things that exist (real things) and things that are said to exist (abstract things), real things are composed of real substances, and exist in real space. Abstract things are things composed by non-real substances, and exist in an ideal space. Thus, any metaphysical thing must exist either in real substance form or in non-real substance form (Plato, 1952a, 1952b in Hutchins (Ed), Aristotle, (1952) In Hutchins (Ed), Kant, 1984 Benacerraf and Putnam, 1966).

Epistemology concerns with the theory of knowledge of metaphysical things, be it real substance or non-real substance. Knowledge of a thing is described in terms of attributes of interest of that metaphysical thing. There does not exist knowledge out of existence of a metaphysical thing (Ernest, 1991, Plato, 1952a, 1952b in Hutchins (Ed), Aristotle, (1952) In Hutchins (Ed), Kant, 1984 Benacerraf and Putnam, 1966).

Therefore, meta-epistemological approach in philosophy of mathematics, intends to identify the metaphysical thing composing to mathematics, describe its epistemological attributes in the widely adopted approach, the attributes that build a body of mathematical knowledge.

Basing on the meaning of nature, nature of Mathematics is perceived as inherent features, qualities and characteristics of Mathematics. Accepting this meaning as a precise meaning of nature of Mathematics raises meta-epistemological problems. The questions arising from this metaphysical perspectives include; first, is Mathematics a single metaphysical object whose features, qualities, and characteristics can precisely be described? Second, which literature describe correctly the nature of Mathematics such that it's inherent features, qualities, and characteristics can be precisely described?

The questions arising from epistemological perspectives include; first, is it possible to have features, qualities, and characteristics of nothing? Second, is it possible to have features, qualities and characteristics of something which is neither metaphysically real substance nor metaphysically ideal substance? And thus, there must exist either real substance or ideal substance in the scope of metaphysics that has the inherent features, qualities, and characteristics.

Available literature describing the nature of Mathematics include the work of Plato (1952a, 1952b) in Hutchins (Ed), Aristotle (1952) in Hutchins (Ed), Courant and Robbins (1941), Korner (1960), Snapper (1979b), Kline (1972) and Kant (1984). Despite of the available literature, yet Mathematics philosophers do not come into agreement on the common thought about the nature of Mathematics, breeding to several schools of thoughts (Dorsey, 2007).

Then, how do Mathematics teachers teach Mathematics subject whose nature is not well known, neither agreed among Mathematics philosophers? Which pedagogy do Mathematics teachers apply in teaching Mathematics whose nature is not agreed among Mathematics philosophers? Which philosophical assumptions or Mathematics philosophical theories are applied in planning and implementation of Mathematics instructions? This indicates that precision on the nature of Mathematics has great positive impact in Mathematics education.

Therefore, it is found imperative to extend Plato's thoughts by identifying weak points on the available literature leading to diverse thoughts on the nature of Mathematics and provide its implication on the learning process in Mathematics education.

2. Basic Review of Literature

Ideal space, Mathematical space and Mathematical objects are used throughout in discussing and extending Plato's thoughts. Meaning of those concepts is guided by the Concise Oxford English Dictionary (2007) and Newbury House Dictionary of American English (1999). Space is regarded as dimension of area with freedom and scope for an organism to exist, live and move. An ideal space is perceived as a perfect space existing in imagination, representing an abstract space into which all things (abstract things) that are said to exist metaphysically exist.

Mathematical space is perceived as a set of all mathematical objects existing in an ideal space. And mathematical object is perceived as theoretical object with metaphysical nature and having epistemological inherent features, qualities and characteristics that represent the external recognized reality.

Great debate on the nature of Mathematics is traced back to the 4^{th} C, BC in the days of Plato and his student, Aristotle (Boyer, 1968, Kline, 1972, Ernest, 1991). The prominent Mathematics philosophers who attempted to describe the nature of Mathematics metaphysically are Plato and his student Aristotle. Each had perception of metaphysical nature of Mathematics, but having differences in the conception on means with which Mathematical knowledge is acquired (Robin and Courant, 1941, Boyer, 1968, Kline, 1972, Snipper, 1979b).

Plato conceived Mathematics as being in the position that the object of Mathematics has its own existence beyond the mind, in the external world; and that mathematical knowledge is acquired by senses (empirical knowledge) through experience with the real world (Plato, 1952a, 1952b in Hutchins (Ed), Korner, 1960, Kline, 1972, Snapper, 1979b). The first notion of Plato was consistent to view Mathematics as constituted by mathematical objects, but there is no clear description provided on the inherent features, qualities and characteristics of those mathematical objects, which in fact, are the epistemological attributes that constitute to the body of mathematical knowledge.

Second notion of Plato is based on the acquisition of knowledge through senses only; and that all mathematical knowledge is acquired through senses, implying that all mathematical knowledge is an empirical knowledge originating from observation and experimentation. The third notion of Plato describes the location into which the mathematical objects are stored for current and for future use is not within the mind itself, but beyond the mind, in the external world; the world which is experienced by mind: the ideal world.

Aristotle, a student of Plato, conceived Mathematics as one of the three genera into which knowledge can be divided into Physical, Mathematical and Theological (Aristotle, 1952 in Hutchins (Ed), Courant and Robbins, 1941, Boyer, 1968). Respective to the body of knowledge existed in the days of Plato and Aristotle, Aristotle generalizes that knowledge should be divided into the three mentioned branches, that gave birth to Mathematics as an independent discipline (Boyer, 1968).

Aristotle accepted the role of senses in abstracting mathematical ideas but extended to the perception that mathematical knowledge can also be acquired without senses. Aristotle's conclusion is quoted by Ptolemy (1952) in Boyer (1968), pointing out that, 'Mathematics is the one which shows quality, seeking figures, numbers and magnitudes, and similar things; and can be conceived through senses and without senses'. Aristotle's view of Mathematics was not based on the theory of external, independent and observable body of knowledge. But it was based on the experienced reality in which knowledge is acquired through experimentation, observation and abstraction: empirical and rational knowledge (Courant and Robbins, 1941, Boyer, 1968, Ernest, 1989, 1991).

Since Plato and his student Aristotle do not fundamentally differ on the metaphysical nature of mathematics, the discussion of this article bases on the thoughts of Plato by characterizing the nature of Mathematics meta-epistemologically in order to justify that thoughts of Plato were correct and consistent. Meta-epistemological characterization of Mathematics complies with Ernest (1991) and Dorsey (2007) in implying that all pedagogy of Mathematics rests on the meta-epistemology philosophy.

Boyer (1968) parallel to Kline (1972), Ernest (1991) and Snipper (1979a, 1979b) writes that since days of Plato and Aristotle, debates on the nature of Mathematics continued until at the early of 1,500AD when Francis Bacon commented on the distinction of the two schools when trying to separate Mathematics from pure and mixed Mathematics. Similar debates continued to the 17th century where Descartes worked to move Mathematics from the path of deductions of accepted axioms to experimentations, aiming at separating it from other sciences as it is justified by Brown (1998) who says that, 'Mathematics means exactly as a scientific study…. We see that almost everyone who has had the slightest schooling can easily distinguish what relates to Mathematics in any question from that which belongs to sciences'.

Boyer (1968) in line with Snipper (1979a, 1979b) points out that, experimentalists standing on the side of Platonic school of thought and rationalist standing on the side of Aristotelian school of thought, affected all branches of science in the 17th century and 18th century. It is at this period when Emmanuel Kant, a German philosopher held a view that all theorems and axioms of Mathematics were truths. The discovery of non-Euclidean geometry in the middle of 1,800AD excluded the conception that Mathematics is a shielded single set of axioms to the justification that man's mind is rationally powerful to construct new mathematical structures (of mathematical objects) that are free from bounds of the existing external world. According to Eves (1981), this conception brought in with a new notion of truth about the dynamic nature of Mathematics.

Boyer (1968) parallel with Snipper (1979a, 1979b) also points out that in the late of 19th century and 20th century, the investigation in Mathematics moved from debates on Platonic school (experimentalists) and Aristotelian schools of thoughts (rationalists) to dealing with paradoxes in the real number system and the set theory. Paradoxes refer to self-contradicting mathematical statements or propositions that may in fact be true. They are apparently sound statements or propositions that lead to a logically unaccepted conclusion.

As related literature shows, after the days of Plato and Aristotle, Mathematicians and philosophers debated on the basis of the two schools of thoughts, either Platonic or Aristotelian school of thoughts. The debates based on the means of acquisition of knowledge than debating on the fundamental nature of Mathematics. Thus, since then until the 20th century, where philosophers moved to finding solutions to paradoxes in Mathematics, no clear literature is available that verifies that the traditional school of thoughts were extended to determine a precise description of nature of Mathematics. Therefore, until 21st century, Mathematicians and philosophers have not come up to agree the nature of Mathematics, providing a persisting gap of all the time on the precise nature of Mathematics.

3. The Discussion of Plato's Thoughts

Dorsey (2007) points out that nature of Mathematics rests on the philosophy of Mathematics. And Korner (1960) identifies that one of the main purposes of philosophy of Mathematics is to reflect and account for the nature of Mathematics. Then, how do we reflect and account for it? According to Dorsey (2007) in line with Ernest (1991) and Ernest (1989), the philosophy of Mathematics attempt to provide a system into which Mathematics can systematically establish its truth depending on the widely adopted assumptions, be it implicitly or explicitly. He further points out that, the assumed role of philosophy of Mathematics is to provide a systematic and secure foundation for mathematical knowledge which is for Mathematical truth; and that the foundation of Mathematics is built upon justification of Mathematical knowledge.

These arguments of Dorsey (2007) in line with Korner (1960), Ernest (1989), Benacerraf and Putman (1964) and Ernest (1991) provide an insight on the basis with which the nature of Mathematics can be described meta-epistemologically from the fact that knowledge is an epistemological attribute of a metaphysical thing. And thus, the nature of any academic discipline can precisely be described meta-epistemologically. This basis grants a secure foundation and a secure approach to be adopted in extending Plato's thoughts on the nature of Mathematics.

In extending the Plato's thoughts; and attempt to reflect and account for the nature of Mathematics, meta-epistemological nature of Mathematics is adopted by assuming that all academic disciplines are reflected and accounted for its nature by identifying a metaphysical thing studied in the discipline whose attributes builds a body of knowledge of the academic discipline. It is considered as a standard measure on precise meaning and definition of any academic discipline. For instance, if Physics is defined as the study of matter in relation to energy, then matter is a metaphysical thing studied in Physics, and energy is an epistemological attribute of matter studied in Physics distinct to attributes of the same matter studied in Chemistry.

Since Metaphysics concerns with the nature of things that exist (real things) and things that are said to exist (abstract things), the nature of Mathematics is approached metaphysically by accepting Plato's thoughts that, Mathematics is constituted by Mathematical objects. And therefore, the nature of Mathematics is approached metaphysically to describe the metaphysical objects which Plato claims to exist and which constitute to Mathematics. They are Mathematical objects whose attributes build a body of mathematical knowledge.

Since epistemology concerns with the theory of knowledge, the nature of mathematical knowledge is approached epistemologically to describe the attributes of a mathematical object that were claimed by Plato; which are associated to complete body of Mathematical knowledge.

This attempt, therefore, provides the basis into which the nature of mathematics can precisely be described systematically to establish its truth by relying on the widely adopted philosophical assumption. Thus, this article adopts meta-epistemological approach in reflecting and accounting for the nature of Mathematics in extension of Plato's thoughts.

Plato claimed that Mathematics is in the position that the object of Mathematics has its own existence beyond the mind, in the external world; and that Mathematical knowledge is acquired through senses in our experience with the real world. According to Plato, a philosophical meaning of a word 'object' is a thing having representation in the mind and existing externally to the mental faculties of intelligence, in the realms of mind.

Thom (1971) points out that, in fact, whether one wishes or not, all Mathematics pedagogy, even if scarcely coherent, rests on the philosophy of Mathematics. This thought of Plato partly matches with a precise meta-epistemological description of the nature of Mathematics, since, although the means of acquisition of empirical knowledge were well described but the attributes of the mathematical objects that are perceived empirically, which he claims to constitute to Mathematics and that which build the body of knowledge were not described. This weakness makes pedagogy experts lack the basis with which Mathematics pedagogy could rely on as Thom (1971) points out. Thus, lacking a precise meaning of the nature of Mathematics has a great impact on Mathematics pedagogy which must be according to its nature.

The thoughts of Plato have three philosophical implications: the Metaphysical characteristics in terms of mathematical objects having representation in the mind, the metaphysical characteristics of mathematical space into which those mathematical objects exist and the epistemological characteristics of acquisition of mathematical empirical knowledge by providing empirical means with which the mathematical knowledge is acquired.

The first implication of Plato signifies Meta-epistemologically that, there must exist metaphysical object, be it real or ideal, which fundamentally determines the nature of something. Trivially, any real object has five basic attributes: shape, qualities, form, structure and properties. If the Plato's thoughts are correct and consistent, that there exist mathematical objects beyond the mind in ideal space constituting to Mathematics, what then are the inherent features, qualities and characteristics of those mathematical objects? Intuitively, it is conceived that, in addition to real object attributes, a mathematical object must satisfy the purpose of studying Mathematics. And Ernest (1991) points out that Mathematics education intends to provide theory and science of teaching Mathematics for the purpose of achieving aims and objectives of studying Mathematics: to solve real the life problems.

It is therefore imperative to perceive that a mathematical object perceived by Plato has applicability property which satisfies the purpose of studying Mathematics. Thus, a mathematical object that Plato conceived to exist beyond the mind has six attributes: shape, qualities, form, structure, properties and applicability. And so, all axioms and propositions in Mathematics attempt to describe inherent features, qualities and characteristics of these six attributes of Mathematical objects.

The meaning of each attribute of a mathematical object is defined with reference to the Concise Oxford English Dictionary (2007) and Newbury House Dictionary of American English (1999). A shape of a mathematical object is a general appearance with which a mathematical object naturally appears to be. It is thus perceived that a shape of a mathematical object is a well mental proportionate visible way in which a mathematical object naturally appears to exist in a mathematical space in the ideal space.

Quality is the standard distinctive measure of a mathematical object as compared to other mathematical objects of the same species in the same mathematical space or with respect to other mathematical spaces in the ideal space. It is thus conceived that, a quality of a mathematical object is perceived as distinctive features and characteristics of a mathematical object with respect to other mathematical objects of similar species within a mathematical space of an ideal space.

Form is a mental visible configuration of a mathematical object as distinct from its components. It is thus perceived that, a form of a mathematical object is a way in which a mathematical object exists beyond the mind in a mathematical space of the ideal world.

Structure is a well-defined organization of arrangements of parts, together with the relationship between parts of a complex mathematical object. It is thus perceived that, a structure of a mathematical object is the existence of different parts of different forms that define a well organization of a mathematical object.

Properties are inherent features and characteristics of a mathematical object described in terms of axioms and propositions. It is thus perceived that, properties are sets of accepted axioms and propositions with an adequate grounds for asserting and justifying them.

Applicability is the appropriateness of the mathematical object to be capable to be of use to solve real life problems. It is thus perceived that, applicability of a mathematical object is the fitness of a mathematical object to be used to solve real life problems from which mathematical ideas were perceived: reciprocation property of a mathematical object.

Reciprocation property refers to the process of mind abstracting reality to construct mental ideas and then using the constructed mental ideas to solve real life problems. And therefore, reciprocation property of a mathematical object is the process of perceiving reality to construct mathematical abstract objects and then using the constructed mathematical abstract objects to solve real life problems.

Second implication of Plato's thoughts attempt to describe the existence of storage facility of processed information exist beyond the mind itself. That is, the processed information due to perception from reality by empirical means: through senses, exist beyond the mind itself. In other words, Plato believes that the constructed mathematical objects exist beyond the mind, existing externally of the mind itself, in the ideal space. This thought of Plato basically suggested the existence of mathematical space in the ideal space of mind in which all processed information is stored for current and for future use.

To understand clearly what actually Plato meant, we must have a brief review on the mind in terms of the philosophy of mind, which is claimed to be modern branch of Metaphysics. Tracing back to the days of Plato, in the 4th century, BC, Plato attempted to describe the nature of mind in terms of mind-body relationship in a school of dualism. Plato held a view that mind is a non-physical substance of an individual that can not be described physically neither be measured by empirical scientific means (Plato, 1995 in Deke, *et al* (Eds) and Hart (1996).

Descartes (1984) in Cottingham, *et al* (Eds) points out that, metaphysically, mind is a non-physical substance of an individual. Descartes (1984) in Cottingham, *et al* (Eds) held a view that mind is an independent ideal substance in a physical body that is capable of distinguishing itself from the brain; implying that mind is an independent ideal substance of an individual that has its own space of existence, in the realms of existence of the mind.

Epistemologically, Descartes (1984) in Cottingham, *et al* (Eds) conceives that the mind is a non-substance material compacted in brain which can not be distinguished physically in the scope of the real world. Descartes (1984) in Cottingham, *et al* (Eds) also conceived to mean that brain and mind are two ontologically distinct entities that are located and compacted in the same, one substance, in the head of an individual.

In fact, basing on the views of Plato (1995) in Duke, *et al* (Eds) and Descartes (1984) in Cottingham, *et al* (Eds), the brain and the mind are inseparable ontologically distinct entities of an individual. The brain as a part of physical body and the mind as a part of non-physical body. Thus, separation is not possible since the two entities exist in different states: one exists in real substance state and the other exists in ideal substance state; and existing in different spaces though both are compacted in one substance, compacted in real substance.

The mind plays two major complex roles: first, enabling consciousness of an individual with the world and perform all human functions; and second distinguishing itself from the brain and performing independently all the mental functions.

Referring to the Concise Oxford English Dictionary (2007), the mind is a component of a non-physical body of a person. The entire non-physical body of a person is a complete independent entity which is a seat of emotions and character, which is capable of distinguishing itself from the brain. It is regarded as a true self of an individual and as capable of surviving the physical death and manifested as an apparition after death. And therefore, non-physical body is a complete ideal substance body of a person in which mind is just one component of it.

According to the Concise Oxford English Dictionary (2007), Plato (1995) in Duke, *et al* (Eds) and Descartes (1984) in Cottingham, *et al* (Eds), mind is perceived as the brain of the non-physical body of an individual with a complete own intelligence and ability to distinguish itself from the biological brain. And that, the mind is the centre of intelligence of an individual, consisting of five intelligence faculties: consciousness, thinking, judgement, language and memory.

Since the mind is a part of non-physical body and has the ability to distinguish itself from the brain, then the mind has its own world of existence beyond the real world, in which mathematical objects exist as Plato conceived. And since the memory is one of the faculties of mind that performs a role of storing temporarily or permanently all the mental perceived or processed information, then the memory itself cannot be a storage facility. The storage facility must exist beyond the memory itself. And therefore, as Plato suggested, the storage facility (location) of processed information (mathematical objects) cannot exist within the memory itself, neither exist within the mind itself; it must exist beyond the mind but in the realm of the mind in the ideal space. And therefore, a Mathematics teacher just attempts to transform real mathematical phenomena to ideal mathematical phenomena, and then applying the ideal mathematical phenomena to solve problems of real mathematical phenomena: Reciprocate property of a mathematical object.

4. Implications to the Learning Process in Mathematics

Although Plato described the means of acquisition of empirical knowledge, he did not describe the learning process of acquisition of the empirical knowledge he attempted to describe. Guided by Kant (1984) and the philosophy of mind, domains of learning and theories of learning, empirical learning process is a cycle of five stages: perception, exposition, execution, assimilation and rationalization.

4.1 Perception Stage

Perception refers to the mental process of becoming aware of mathematics learning phenomena leading to a natural understanding. Learning process begins when an individual is exposed to an interesting mathematics learning phenomena that raises curiosity to explore the learning phenomena. Thus, learners are exposed to a state of being aware of learning activities that enable them interpret the learning phenomena. Real objects are correctly perceived in mind through senses to construct mental images, ready for identification, description and analysis of the attributes for basic understanding of the mathematical object together with its attributes.

This is implied from Kant (1984), that there is relationship between conscious sensory experiences of an objective mathematics learning phenomena. And that, real world is transformed to mental representations using appropriate senses. And self-conscious promote attitude to explore learning phenomena.

4.2 Exposition Stage

Exposition refers to the comprehensive description of inherent features and characteristics of the mathematical object and explains the guiding theory drawn from the learning phenomena. After interpretation of the learning phenomena, the mind constructs correct understanding of a mathematical object together with its attributes by means of personal judgement. The attributes of a mathematical object are identified, analyzed, described and explained by means of correct thinking and judgement to generate knowledge.

This is implied from Kant (1984) that real features are described from epistemological attributes of the perceived experiences and constructing conceptual content in abstract substance forms to generate basic understanding of the facts comprehended from mathematics learning phenomena.

4.3 Execution Stage

Execution refers to the physical performance of the commands form the mind; brain and mind communicating to execute knowledge to generate skills. The brain accepts commands from the mind (knowledge) for physical execution (psychomotor) to generate skills. A learner practice knowledge to generate skills. Skills are more solid than knowledge.

This is implied from Kant (1984) that facts are discriminated and differentiated; and decisions are made of which facts are relevant to be used to execute the mathematical problem of the mathematical phenomena. Correct execution attains content correction conditions for precision in execution: basic mathematical skills.

4.4 Assimilation Stage

Assimilation refers to the process of absorbing in mind the gained skills in doing Mathematics for current and for future use, as a result of practicing knowledge. This stage involves fitting the acquired knowledge into skills on using the mathematical object in real situations.

This is implied from Kant (1984), that learners understand and reason about mathematical content acquired and making judgement about the precisely executed mathematical problems, by intuition with mathematical objects or without mathematical objects. The acquired skills are grasped in mind; knowledge acquired is fitted into skills.

4.5 Rationalization Stage

Rationalization refers to the process of thinking systematically, logically and rationally; and making personal judgement about the fitness and worthiness of the constructed mathematical object to real world from which the ideas were perceived. Mathematical objects are tested for their appropriateness in real world. Personal judgement is made about the worthiness of constructed mathematical objects into solving real life problems. Learners justify the compatibility of the mathematical object in real life

This is implied from Kant (1984), that learners imagine, think rationally and judge the worthiness and appropriateness of mental constructs in reciprocate property and gain self-knowledge according to own understanding in meta-epistemological substance forms.

5. Conclusion and Recommendations

On the basis of Plato's thoughts and related literature, it is concluded that, thoughts of Plato are consistent about of the nature of Mathematics. Mathematics consists of mathematical objects having six attributes: shape, qualities, form, structure, properties and applicability. These attributes constitute to the mathematical knowledge in the meta-epistemological basis. Mathematical objects exist beyond the mind, in a mathematical space of ideal space, in the realm of existence of the mind. Empirical knowledge is fundamentally acquired through senses in a process of five stages cycle: perception, exposition, execution, assimilation and rationalization stages.

In the right of this study, it is recommended that;

(a) Mathematics philosophers should review the definition of Mathematics in meta-epistemological basis in order to solve the everlasting dilemma on the precise definition of Mathematics, which has an impact and implication to Mathematics pedagogy.
(b) Mathematics pedagogy experts, if they are available, should review the pedagogy of Mathematics in order to describe relevant pedagogy for mathematics which is according to its nature.
(c) Higher learning Institutions should design and implement degree programmes in Mathematics education that will produce mathematics philosophers who will play a role in in reviewing the current philosophy of Mathematics, philosophy of Mathematics education and pedagogy of mathematics education. The recommended programmes are Bachelor of Education in Mathematics, Master of Mathematics Education and PhD in Mathematics Education.

References

Aristotle (1952). Metaphysics. In Hutchins (Ed), *Great Books of the Western World: Vol. 8. Aristotle 1.* Chicago: Encyclopedia Britannica, Inc.

Hart (1996). *Dualism*. In Guttenplan (Org). A Companion to the Philosophy of Mind. Oxford: Blackwell, pp 265-267.

Amazon (2011). *Concise Oxford English Dictionary*. Oxford: Oxford University Press

Benacerraf, P. and Putnam, H. (1964), *Philosophy of Mathematics.* Englewood Cliffs, NJ; Prentice Hall.

Boyer, C. (1968). *A History of Mathematics*. New Yolk: Wiley.

Courant, R. and Robbins, H. (1941). *What is Mathematics?* New York: Oxford University Press.

Descartes, R. (1952). Rules for the Direction of the Mind. In Hutchins (Ed). *Great Books of the Western World: Vol. 31. Descartes & Spinoza 1.* Chicago: Encyclopedia Britannica, Inc.

Descartes, R. (1994). In Cottingham, J; Stoothoff, R; Murdoch, D (Eds). The Philosophical Writings of Descartes. Cambridge: *Cambridge University Press, Vol. 20.*

Ernest, P. (1989). Philosophy, Mathematics and Education. *International Journal of Mathematics Education in Science and Technology, Vol. 20.*

Ernest, P. (1991). *The Philosophy of Mathematics Education.* New York: Tayler and Francis Co.

Hart (1996). *Dualism*. In Guttenplan (Org). A Companion to the Philosophy of Mind. Oxford: Blackwell, pp 265-267.

Ishumi (2004). *Philosophy of Education*: An introduction to Concepts, Principles and Practice. Dar Es Salaam. Dar Es Salaam University Press.

Kant, I. (1984). The Critique on Pure Reason. In Hutchins, R (Ed). *Great Books of the Western World: Vol. 42. Kant (pp. 1 – 250).* Chicago: Encyclopedia Britannica, Inc

Kline, M. (1972). *Mathematical Thoughts from Ancient to Modern Times.* New Yolk: Oxford University Press.

Korner, S. (1960). *The Philosophy of Mathematics: An Introduction.* New York: Harper & Row.

Mc Lear, C. *Internet Encyclopidia of Philosophy*. ISSN 2161-0002; Retrieved on 22/04/2022, www.iep.utm.edu.wp-content/media/cant2.

Plato. (1952a). The Republic. In Hutchins, M. (Ed). *Great Books of the Western World. Vol. 7. Plato (pp. 295-441)*. Chicago: Encyclopedia Britannica, Inc.

Plato. (1952b). Timaeus. In Hutchins, M. (Ed). *Great Books of the Western World. Vol. 7. Plato (pp. 442-477).* Chicago: Encyclopedia Britannica, Inc.

Plato. (1995). In Duke, E; Hicken, W; Nicoll, W; Robinson, D; Strachan, J. (Eds). Phaedo: Clarendon Press.

Snapper, E. (1979a). The Three Crises in Mathematics: Logicism, Intuitionism and Formalism. *Mathematics Magazine. Vol. 52 (pp. 207- 2016).*

Snapper, E. (1979b). What is Mathematics? *American Mathematical Monthly. Vol. 86 (pp. 551- 557).*

Rideout, P (2004). *Newbury House Dictionary of American English*. Boston, M.A: Heinle & Heinle Publishers.

Stewart, I. (1987). *The Problems of Mathematics.* New Yolk: Oxford University Press.

Timoczko, T. (1986). *New Direction in the Philosophy of Mathematics.* Boston: Birkhauser.

Von Glaserfeld, E. (1987). *Learning as a Constructive Activity*. In Janvier, C (Ed). Problems of Representation in the Teaching and Learning of Mathematics (pp. 3 – 17). Hillsdale, NJ: Lawrence Erbaum.