

META-EPISTEMOLOGICAL DEFINITION OF MATHEMATICS AND ITS IMPLICATION TO THE MATHEMATICS TEACHING METHODOLOGY

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Abstract

Mathematics philosophers do not have a consensus understanding about the definition of Mathematics. An attempt to define Mathematics began in the days of Aristotle who stated that, "Mathematics is the science of quantity". Aristotle's definition prevailed until the 18th century when abstract topics in mathematics were discovered, the topics that had no direct relationship with quantities. In the 19th century, philosophers began to propose new definitions which however philosophically differ in many ways.

Since then up to date, the metaphysical and epistemological status is not well known, and hence the nature and scope of mathematics that could describe precisely mathematics is not known. Characterizing mathematics meta-epistemologically provides the philosophical foundation into which mathematics can be defined in a wide secure philosophical foundation and solve the everlasting problem in defining mathematics.

It is realized that precise philosophical definitions describe metaphysical nature of a thing together with its epistemological scope. In this basis, it is realized that the proposed definitions of mathematics do not satisfy the basis of definitions neither completely satisfy philosophical basis of definitions

Platonism's philosophy of mathematics and Aristotle's assertion that epistemology is the study of things that exist and happen provide the meta-epistemological basis with which any academic discipline, including mathematics, can be defined in terms of its nature and scope as the meaning of the term definition requires to develop its philosophical truth.

Platonism assertion that what exists lies beyond our experience, mathematical abstractions that reflect reality have themselves a reality and mathematics has its existence fundamentally describes the materialism, realism and idealistic nature of mathematics in the form of abstract materialistic-idealistic reality different to materialistic-idealistic reality nature of many disciplines.

Platonism's philosophy of mathematics and Aristotle's assertion that epistemology is the study of things that exist and happen provide the meta-epistemological basis with which any academic discipline, including mathematics, can be defined in terms of its nature and scope as the meaning of the term definition requires to develop its philosophical truth.

On the right of the meta-epistemological basis, mathematics can be viewed as the study of shape, qualities, form, structure, properties and applicability of mathematical objects derived from the object of our experience, forming the mathematical reality. Mathematics teachers' methodology of teaching mathematics should be based on studying the attributes of mathematical objects. And the choice of mathematics teaching methods should be guided by the methods that intend to develop the domains of learning on the six attributes of a mathematical object.

Keywords: Definition of Mathematics, Platonism, Nature of Mathematics, Scope of Mathematics, Metaphysical Strands; Materialism, Realism and Idealism of Mathematics, Meta-Epistemological Definition.

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1. Introduction

According to the Concise Oxford English Dictionary (2011), the term "definition" means the exact meaning of a word. It is the description of the nature and scope of something. That means, a definition of mathematics is a statement that describes the meta-epistemological nature and scope of mathematics.

Philosophically, the description of nature is the description of the metaphysical nature of mathematics in terms of its substance composition, whether composed of real substances non-real substances or both. And description of scope is the description of the scope of the epistemological attributes of the metaphysical thing that form the body of mathematical knowledge (Haule and Johnson, 2023a). Based on the meaning of the term "definition", the definition of Mathematics is perceived as a statement that describes precisely the Metaphysical nature and its epistemological scope.

The Concise Oxford English Dictionary (2011) also describes Metaphysics as the branch of Philosophy that deals with the first principle of things, real and ideal things. In other words, metaphysics is the study of real and non-real substance things in the ultimate reality. It includes studying things that are beyond the reach of our experience and cannot be studied by empirical means similar to the arguments of Nyirenda and Ishumi (2002). For instance, Theology which is a sub-sub branch of metaphysics, studies things that are beyond the reach of our experience, that cannot be studied by empirical means.

In discussing the definition of Mathematics, the researcher holds on both strands of Metaphysics that what exists lies beyond experience and the objects of experience do form the only reality of the existing ultimate reality. By holding this view, the researcher holds a view that the ultimate reality is formed by real and non-real substance metaphysical things. Therefore, the definition of mathematics can be stated based on this view and in the perspectives of the philosophy of mathematics.

It is thus conceived that Metaphysics is concerned with the nature of things that exist (real things) and things that are said to exist (non-real things). Real things are in nature composed of real substances and exist in real space. Non-real things are things composed of non-real substances and exist in an ideal space. Thus, any metaphysical thing must exist either in real substance form or in non-real substance form or both forms (Haule & Johnson, 2023a, 2023b, Nyirenda & Ishumi, 2002, Plato, 1952a, 1952b in Hutchins (Ed), Aristotle, (1952) In Hutchins (Ed,)). And therefore, mathematics must metaphysically be composed of either real substances, non-real substances or both.

According to Haule (2023a) in line with Plato (1995) and Plato (1952a, 1952b), mathematics is a metaphysical thing characterized by non-real mathematical objects derived from reality. according to Plato (1952a, 1952b), despite mathematics being derived from reality, mathematics itself is an idealistic reality existing in an ideal space in an abstract materialistic form due to mathematics reciprocating property, solving life real-life problems.

The same Dictionary defines epistemology as the theory of knowledge in terms of its methods, validity, and scope; and the distinction between justified belief and opinions. Epistemology is concerned with the theory of knowledge of metaphysical things, real substances, non-real substances or both. Knowledge of a metaphysical thing is described based on its inherent features, properties and characteristics (Haule & Johnson, 2023a, 2023b, Ernest, 1991, Plato, 1952). On the other hand, Aristotle (1952) stated that epistemology is the study of things that exist or happen in the world. The statement, "the study of" philosophically means the study of attributes of a thing. This directly implies that; mathematics epistemology is the study of mathematical metaphysical things. In other words, mathematical knowledge is the study of the attributes of a metaphysical thing.

Thus, Epistemology is concerned with the theory of knowledge of metaphysical things. Knowledge of a thing is described in terms of attributes of interest of that metaphysical thing in terms of inherent features, characteristics and properties. There does not exist universal knowledge in the absence of studying a metaphysical thing (Haule & Johnson, 2023a, Ernest, 1991, Plato, 1952a, 1952b in Hutchins (Ed).).

Then, the epistemological status of mathematics is described in terms of the scope of the features, characteristics and properties (attributes) of the metaphysical thing which build a body of an edifice of mathematical knowledge. According to Haule & Johnson (2023a) in line with Plato (1952a, 1952b), mathematics is composed of a set of mathematical objects which generally have shape, qualities, form, structure, properties and applicability. These attributes are considered as the general inherent features, properties and characteristics (attributes) of a mathematical object in each set of mathematical objects composing mathematics and describe the epistemological scope of mathematics.

From the meaning of the term definition and the philosophy of mathematics, the philosophical arising questions include: first: do the proposed definitions of mathematics satisfy the basis of definitions? Second: what is the meta-epistemological basis for defining mathematics? Third: do the proposed definitions of mathematics have a philosophy of mathematics foundation? Fourth: what is the precise definition of mathematics?

Therefore, this study intends to suggest the philosophical answers to these questions in the right of the philosophy of mathematics. And the meta-epistemological approach in the philosophy of mathematics intends to characterize mathematics meta-epistemologically. Characterizing mathematics meta-epistemologically provides a secure foundation to discuss the basis on which the definition of mathematics should philosophically rely, suggest the precise Meta-Epistemological definition of the term Mathematics and provide its implications to the basic mathematics teaching methodology.

Available literature about definitions of Mathematics includes the work of Haule (2023a, 2023b), Plato (1995), Plato (1952a, 1952b) in Hutchins (Ed), Aristotle (1952) in Hutchins (Ed), Courant and Robbins (1941), Korner (1960), Snapper (1979b), Kline (1972). Despite the available literature, Mathematicians and Philosophers do not come to a consensus about the definition of Mathematics nor do they provide a philosophical basis through which the term Mathematics could be defined. Lacking this philosophical foundation in the philosophy of mathematics breeds several proposed definitions, the definitions that philosophically differ in many ways, and most of them do not exactly reflect a sense of mathematics philosophy definitions.

Since all pedagogy of mathematics rests on the philosophy of mathematics, how do Mathematics teachers teach Mathematics subjects whose definition has not well been concluded or agreed upon among Mathematics philosophers? Which philosophical assumptions underpin Mathematics teachers' pedagogy? This indicates that a deliberate effort should be made among mathematicians and philosophers to determine a precision in the definition of Mathematics that has a great positive impact on Mathematics teaching methodology.

Therefore, it is imperative to review philosophically the meaning of the term Mathematics, and identify weak points in the available literature leading to diverse thoughts on the definition of Mathematics. Also, suggests the philosophical basis with which the term Mathematics should be defined. This work suggests the starting point definition of mathematics on meta-epistemological basis and provide its implication on the basic mathematics teachers' methodology in teaching mathematics.

2. Basic Review of Literature

Mathematics has had no accepted definition among mathematicians and philosophers since the days of Plato and Aristotle giving birth to several definitions which are however contradicting in many ways (Mura, 1993, Boyer, 1968, Richard & Herbert, 1996).

According to Platonism's view of the philosophy of mathematics, mathematical abstractions that reflect reality have themselves a reality that exists outside space and time. According to Aristotle, epistemology is the study of things that exist or happen in the world rising to the knowledge of the universe (Haule & Johnson, 2023a, Plato, 1952a,1952b & Aristotle, 1952). These assertions of Plato and Aristotle suggests that there exist mathematical objects whose attributes constitute mathematical knowledge. Moreover, these assertions on the philosophy of mathematics suggest that the definition of mathematics is centred on the meta-epistemological philosophy (Haule & Johnson, 2023a).

Apart from having no consensus about the definition of mathematics, nor its epistemological status, some mathematicians and philosophers place no interest in the definition of mathematics while others like Stewart (1996) view that mathematics is a vast of knowledge which can't be defined (Brown, 1998, Boyer, 2011).

The lack of consensus among mathematicians and philosophers about the definition of mathematics is rooted in a lack the consensus about the nature and scope of mathematics, leading to a lack of consensus on the meaning and definition of mathematics (Haule & Johnson, 2023a). This provided room for every mathematician or philosopher attempting to define mathematics without a strong philosophical basis that can widely be adopted. Due to a lack of a strong philosophical basis for defining mathematics, some modern mathematicians and philosophers attempted to define mathematics based on its object of study (Haule, 2023, Boyer, 1968, 2011).

Aristotle is considered the first philosopher and mathematician to define mathematics as "the science of quantity". This definition prevailed up to the 18th century when great mathematicians discovered abstract mathematics such as group theory and projective theory (Boyer, 2011, 1968, Brown, 1998, Hersh, 1977).

The discovery of group theory and projective geometry in the 19th century, the abstract mathematics that has no clear relationship with quantity and measurement drove the paradigm shift from considering mathematics as the science of quantity to the direction of proposing a modern definition of mathematics. Mathematicians and philosophers began to propose several new definitions that reflect different philosophies of mathematics. At the same time, the majority defined it basing on the object of study (Boyer, 1968, 2011, Hersh, 1977, Snapper, 1979).

While basing on the object of the study, the different definitions mainly focused on Logicism, intuitionism or formalism philosophy of mathematics. Focusing on the Logicism perspective of the philosophy of mathematics, Peirce (1870) defined mathematics as “the science that draws necessary conclusions” and Russell (1903) defined mathematics as a “symbolic logic”.

Intuitionist definitions are characterized by the construction of ideas in mind and suggest that mathematics can be sought in the construction obligation by watching what intuition allows and which doesn’t (Boyer, 1968, 2011, Snipper, 1979 & Russell, 1903).

Formalists proposed definitions of mathematics based on synthesizing logicism and intuitionism meaning, definitions and themes. One of the formalist definitions is that mathematics is a “science of formal systems” (Bowyer, 1968, 2011).

Other definitions are defined by the object of the study but emphasize pattern, order, structure, systems, relationships etc. Boyer (1968), for example, defines mathematics as “the classification and study of all possible patterns”. Some definitions seem as if mathematics is a politics. Such definitions include that of Russell (1901) who stated that mathematics is “the subject we never know what we are talking about whether what we are saying is true” and the definition of Darwin (1872) who stated that mathematics is “a blind man in a dark room looking for a black cat isn’t there”.

2.2 Review of Some Selected Definitions of Mathematics

As it was introduced in the previous section, the term definition is statement or description of the nature and scope of something. And the several proposed definitions of mathematics, like other concepts, are thought to satisfy the meaning of the term definition. Therefore, at this point, the review of the selected definitions is based on the meaning of the term definition and its philosophical implication to the philosophy of mathematics to identify the weak points from the selected definitions.

Aristotle defined Mathematics as “a science of quantity”. Quantity is an epistemological attribute of something, a metaphysical thing. It implies that Aristotle views mathematics as the science of an epistemological attribute of something which is not clearly described. According to the meaning of a definition of an academic discipline, the scope of mathematics is described but the metaphysical nature of that thing whose science is studied is not described.

In its immediate meaning, the science of quantity may interchangeably mean the knowledge of the quantity of something since, meta-epistemologically, there is no knowledge in the absence of metaphysical things; material things, real-substance things. Aristotle’s view is based on the second strand of metaphysics implying that the object experience forms the only reality. Thus, he is implying that mathematics exists in materialistic-idealistic reality in the form of abstract materialistic-idealistic reality similar to the meaning of the object of experience.

However, Aristotle’s definition provides us with an insight into the consistence of Platonism's view that that there exist mathematical objects of which, according to Aristotle, the science of its quantity is one aspect of the mathematical knowledge that builds a body of mathematical knowledge. A major philosophical question from Aristotle’s definition is that, is all mathematics is built upon the science of quantities? Is a quantity a sufficient epistemological scope of mathematics?

Apart from this definition, Aristotle (1952) also states that epistemology is the study of things that exist or happen in the world. This statement provided Aristotle with an insight that there exist metaphysical things whose study of attributes rises to the mathematical knowledge. But then, according to Aristotle, what is the scope of metaphysics? What is the nature of metaphysical things? It implies that Aristotle stated the definition of mathematics without notion that there exist things in form of mathematics whose science of quantities is studied. And also implies that, Aristotle had no notion that metaphysics also studies things that are beyond our experience since there is no significant difference with Platonism.

Russell (1901) views mathematics as “the subject we never know what we are talking about whether what we are saying is true". It implies that mathematics is politics, it has no its own reality, has no materialistic-idealistic reality, has no abstract reality in idealistic-idealistic reality. The definition does not state the nature and scope of mathematics. It is like Russell who stated the definition of mathematics without notion of how definitions of academic disciplines are stated, neither notion of philosophical foundations of the philosophy of mathematics.

Peirce (1870) views mathematics as "the science of concluding". Philosophically, there exist no conclusions drawn of attribute of nothing. There must exist something, and upon its study of its attributes leads to concluding. This implies that mathematics is the science of drawing conclusion of unknown metaphysical things whose epistemological status is unknown. The statement implies that, according to the meaning of the term definition, nature and scope of mathematics is left unknown.

Curry (1951) views mathematics as "the science of formal systems". Philosophically, there exist no system of nothing; there must be a system of something. There must exist something, and upon its study leads to the formal systems. Therefore, according to the meaning of the term definition, the scope of mathematics has been described but the nature of mathematics is left unknown. That is, the metaphysical status of mathematics is left unknown.

Sawyer (1955) views mathematics as "the classification and the study of all possible patterns". This implies that mathematics is the study of patterns together with their classification. The definition philosophically implies that the scope of mathematics is patterns and classification. Again, philosophically, no possible patterns of nothing can be studied; there must exist something, and its study leads to obtaining the possible patterns; therefore, the metaphysical status of mathematics is left unknown. So, according to the meaning of the term definition, the scope of mathematics has been stated but its nature is left unknown.

Black (1933) views mathematics as "the study of all structures whose form can be expressed in symbols; it is a grammar of all symbolic systems". Structure in this sense means how parts are formed into a whole. It is to form ideas into a whole in which each part is related to other parts.

Black views that, mathematics is all about structures; and the symbols and relationships that exist between them are the stuff with which mathematics is made. It implies that structure is the epistemological attribute studied in mathematics, and expression of mathematics by symbols, is the mathematical object representation of mathematics.

Black attempts to define mathematics in terms of academic discipline definitions by attempting to provide the nature and scope of mathematics. He attempts to identify structure as an epistemological attribute studied in mathematics to build a body of mathematical knowledge and expression of symbols from mathematical objects. The philosophical questions here include, is a structure the only epistemological attribute studied in mathematics that builds a body of mathematical knowledge?

Courant and Robins (1941) view mathematics as "nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise can be created by the free will of the mathematicians". The definition implies that the System of conclusion is the scope of mathematics, but the nature of mathematics is left unknown. It is also philosophically interesting if mathematicians can create mathematics by free will without a sense of self-idealistic reality or self-materialistic reality or both, create mathematics, and then solve real-life problems.

Cline (1972) views mathematics as "a creative or inventive process, deriving ideas and suggestions from real problems, idealizing and formulation of the relevant concepts, posing questions, intuitively drawing a possible conclusion and only then, providing a bunch of intuitive argument deductively".

The definition implies that mathematics is a process of discovering mathematics from real-life problems based on intuition and construction. The metaphysical status of discovered mathematics in terms of nature and its scope is left unknown. According to the meaning of the term "definition", Cline does not describe the nature and scope of mathematics, as definitions of academic disciplines are stated.

As the comprehensive related literature shows, since the days of Plato and Aristotle to modern Mathematicians and Philosophers of nowadays, several definitions of mathematics have been proposed and yet no consensus has been reached. Some Mathematicians and philosophers view mathematics as politics that has no philosophical base in the philosophy of mathematics. Till now, Mathematicians and philosophers have not yet proposed the philosophical basis with which mathematics can be defined in more secure philosophical foundations and solve the problem of the definition of mathematics. These weaknesses in the philosophy of mathematics result in a persisting gap of all the time on the precise meaning and definition of Mathematics.

3. The Discussion on the Definition of Mathematics

Several philosophers attempted to define Mathematics despite not coming to a consensus about the stated definitions. We discuss the philosophical implication of these few selected definitions with the notion that the selected definitions are just a sample of many such definitions. We identify the weak points on the Meta-Epistemological basis that call for the need to review the definition of Mathematics, and then attempt to define precisely mathematics in Meta-Epistemological philosophy of mathematics.

3.1 Contradiction in Metaphysical Strands

According to the Concise Oxford English Dictionary (2011), the word experience has many meanings, but philosophically, the word has two main meanings. The first meaning is that experience is a practical contact and observation of facts. The second meaning is that experience is the event that leaves an impression on someone. In the same sense, the word object has two main meanings. The first meaning is that an object is a material thing that can be seen and touched. The second meaning is that an object is a thing external to the mind or subject.

It further describes that Metaphysics has two main aspects: the first aspect considers that what exists lies beyond experience, advocated by Plato and the second aspect considers objects of experience form the only reality as reflected in Plato (1952a, 1952b) in Hutchins (Ed), Kant (1984) Boyer (1968, 2011) and Haule & Johnson (2023a).

Therefore, technically, the statement that "what exists lies beyond experience" would philosophically mean that reality is more than contact with something and collecting facts and more than encountering an event that leaves an impression on someone. And the statement that "the objects of experience form the only reality" has three philosophical meanings.

The first meaning is that it is a material thing that can be seen and touched, someone can contact with and make observations of facts. The second meaning is that it is a non-material thing external to the thinking mind causing events that can leave an impression on someone. The third meaning is the combination of both meanings. By asserting that "the objects of experience form the only reality" is a generalized philosophical statement, then the researcher holds on to this generalized statement with combinations of both meanings.

Standing on both meanings, it is implied that, the objects of experience form the only existing reality. The reality includes facts that are beyond the reach of our empirical means. It is further implied that the combined third meaning of the objects of experience form the existing ultimate reality does not significantly differ from the views of the first metaphysical strand that what exists lies beyond our experience (Plato, 1952a, 1952b, Aristotle, 1952, Haule and Johnson, 2023a, 2023b, Nyirenda & Ishumi, 2002).

This philosophical assertion means further that the ultimate reality is formed by real and non-real substance things facts as the meaning of metaphysics implies. Non-real substances are the substances that exist beyond our experience and cannot be studied nor justified by empirical means. Moreover, Platonism means that there exists reality in the form of images of objects of experience that lie beyond our experience (Haule & Johnson, 2023a, 2023b, Plato 1952a, 1952b). It means further that Such images, if they are mathematically reflecting reality, have their reality (Plato, 1995, 1952a, 1952b).

Plato claimed that Mathematics is in the position that the object of Mathematics has its existence beyond the mind, in the external world. According to Plato, a philosophical meaning of a word 'object' is a thing having representation in the mind, be it a real or ideal object, and existing externally to the mental faculties of intelligence, in the realms of mind (Plato, 1952a, 1952b, Aristotle, 1952, Haule & Johnson, 2023a).

Rejecting this philosophical assertion of Plato and Aristotle would mean that all that exists in the world is a result of dealing with the first principle of things, only in real terms and without the notion of the existence of ideal things that exist in non-real substance forms. The rejection of Plato's assertion and Aristotle's assertion on the reality of mathematics leads to difficulties in identifying the metaphysical things that constitute mathematics, and hence failure to describe precisely the nature of mathematics. But any definition describes the metaphysical nature and its epistemological attributional scope. And so mathematics, must be defined by stating the metaphysical nature and its epistemological attribution scope studied in mathematics.

Therefore, defining mathematics on a meta-epistemological basis, the definition of mathematics can never be obtained among mathematicians and philosophers without agreement on the metaphysical nature of mathematics and its epistemological attributes scope description.

Furthermore, rejection of the first strand of metaphysics and accepting only the second strand that objects of experience form the only reality would imply that the existence second aspect of non-real substance things in metaphysics is significantly odd and irrational which rises metaphysical contradiction among its branches.

In attempting to define Mathematics meta-epistemologically, the researcher holds on to the Platonism philosophy that mathematics has its reality in idealistic-idealistic form despite being derived from the materialistic-idealistic philosophy of mathematics. By holding this view, it is re-advocated that what exists lies beyond human experience, the objects of experience form the only reality in the scope of metaphysics, and mathematical objects that reflect reality have themselves a reality outside space and time.

Space in this Platonism philosophy of mathematics is conceived as a perfect ideal space existing in imagination, representing an abstract space into which all things that are said to exist metaphysically exist. Mathematical space is perceived as a set of all mathematical objects existing in an ideal space. And mathematical object is perceived as a theoretical object with a non-real substance metaphysical nature and having epistemological inherent features, qualities and characteristics that represent the externally recognized reality: Idealistic-Idealistic Realism (Haule & Johnson, 2023a).

Therefore, it is imperative to conceive that Metaphysics is concerned with the nature of things that exist (*real-substance things*) and things that are said to exist (*non-real substance things: abstract things*). Real-substance things are materialistic, can be studied by empirical means and exist in real space. Non-real substance things are idealistic, cannot be studied by empirical means, and exist in an ideal space. Thus, mathematics as metaphysical non-real substance thing must exist in non-real substance form in mathematical space in an ideal space (Haule & Johnson, 2023a, Plato, 1952a, 1952b in Hutchins (Ed), Aristotle, (1952) In Hutchins (Ed.)).

One the other hand, Aristotle claimed that Epistemology is the study of things in the world, rising to the knowledge of the universal. This assertion therefore metaphysically implies that knowledge is acquired by the process of studying metaphysical things in both strands of metaphysics. This implies further that there exists no knowledge in the absence of studying attributes of metaphysical things in both strands of metaphysics. That is, epistemology cannot exist if there doesn't exist metaphysical things to be studied.

Mathematicians and Philosophers' weakness of residing on one strand of metaphysics leads to defining mathematics based on the object of study. It is assumed to be a root cause of the failure to propose meta-epistemological definitions that could stand as the philosophical basis for defining mathematics for mathematics philosophical truth. As a consequence, lacking a precise meaning and definition of Mathematics has a great impact on Mathematics pedagogy which must be according to its metaphysical nature and epistemological scope.

3.2 Materialism, Realism and Idealism of Mathematics

According to Concise Oxford English Dictionary (2011), materialism is defined as the theory that advocates that nothing exists except matter and its movements and modifications. Idealism is the theory that advocates the representation of things in ideal form or idealized form. Realism is defined as the theory that advocates that the state of things should be accepted as they exist.

According to Nyirenda and Ishumi (2002), speculation activity in philosophy attempts to think in the most general and systematic way about *anything* in the universe. Metaphysical issues such as the nature of human beings are also speculated. They contend further that, some matters are beyond the realm of most disciplines, such as, "is a reality a material or spiritual?'. Since we cannot collect empirical data on spiritual cases and experiment by scientific means, then speculation activity attempts to fill this philosophical gap.

According to this assertion, anything in the universe means anything that exists and that is said to exist. Things that exist are material things formed by real substances, and they can be experimented with by physical means. Things that are said to exist are any things formed by non-real substances that cannot be experimented with by physical means. Referring to Nyirenda and Ishumi (2002), anything that exists isn't only a materialistic thing in nature. Also, according to the second meaning of objects of experience, what exists is not only materialistic. Otherwise, the speculation activity in philosophy would be meaning less as it would mean that we are speculating on real things; no matters that are beyond most disciplines would be speculated. And the meaning of the term "metaphysics" as beyond physics would be distorted.

Moreover, speculation activity would be nonsense if whatever that exists would be of only real-substances material composition. And perhaps, the famous speculation questions such as "what is reality?" would be useless. Therefore, it is philosophically important to conceive that; first: reality is not only formed from the knowledge of materialistic things in materialistic-idealistic reality since metaphysically, anything that exists, doesn't exist only in real-substance forms. There also exists reality from knowledge of non-real substance things which is termed as non-materialistic-idealistic reality. Second: There also exists reality from the materialist nature of things and there exists reality from the idealistic nature of things; things that have their reality in abstract materialist-idealistic reality which is termed as idealist-idealistic reality similar to the philosophical meaning of both metaphysical stands implication.

Accepting this philosophical assertion directly implies that metaphysics is the study of things that exist and abstract things. Epistemology results from studying inherent features, properties and characteristics of anything that exists in the proposed meaning of what exists.

Otherwise, there would be contraction and irrationalities in philosophy and the philosophy of mathematics. For instance, theology and some aspects of anthropology wouldn't be part of metaphysics. Also, the proposed new assertion that both metaphysical strands metaphysically mean the same thing would contradict Nyirenda and Ishumi (2002) who define Metaphysics as the systematic study of the ultimate nature of reality; dealing with questions that are beyond our sensory impressions; studying matters (including things) that are beyond the reach of our empirical means.

Even mathematics is said to be metaphysical if it is part of the ultimate nature of reality, either materialistic in nature or idealistic in nature or both, and perhaps assumes the existence of facts that are outside the range of human observation in idealistic-idealistic reality form similar to Plato's assertion in line with Haule and Johnson (2023a) that mathematical abstractions that reflect reality have themselves a reality that exists outside space, in ideal space.

These metaphysical facts are assumed to be the root cause of difficulties in defining mathematics in the sense that, though mathematics is derived from real life, its abstractions aren't materialistic. Suggestions of Platonism philosophy of mathematics that, what exists lies beyond experience, mathematical abstractions that reflect reality have themselves a reality existing outside space and time; and Mathematics is in the position that the object of Mathematics has its existence beyond the mind philosophically makes sense. And the assertion that mathematics is composed of mathematical objects is even, rational and makes sense in broader suggested philosophical adoptions to meta-epistemological nature and the implication of the meaning of both metaphysical strands.

This Platonism assertion fundamentally describes the reality of mathematics and the philosophy of mathematics in the sense in which, the realism of mathematics is not of the form of materialistic- idealism in nature contrary to other common academic disciplines we know though it is derived from materialistic things. Therefore, this Platonism assertion together with Aristotle's assertion provides another philosophical foundation with which a definition of mathematics can be defined philosophically without philosophical facts distortion.

As discussed in the previous section, physics is defined as the study of nature and properties of matter and energy, the definition could be the study of nature, energy and properties of matter since energy isn't a metaphysical thing to be studied metaphysically. Energy is an attribute of matter.

Philosophically, the metaphysical thing studied in physics is matter. Matter is a real thing and it is a material thing, hence materialism and realism study of matter. On the other hand, nature, energy and properties are philosophically epistemological attributes of the metaphysical real and material thing, hence idealism of epistemological attributes of a metaphysical thing.

Therefore, many existing definitions of academic disciplines have been stated in materialistic-idealistic realism form. That is, matter is a real substance material, but properties and energy are idealistic realities derived from materialistic reality. The failure of mathematicians and philosophers to define mathematics is assumed to be caused by attempting to define mathematics in this philosophical perspective of materialist-idealistic realism without the notion that mathematics by itself is not a material real thing despite being derived from materialistic reality.

Accepting the Platonism philosophy of mathematics, mathematics isn't a real thing neither a material thing. It is an abstract thing constituted by mathematical objects, having representation in mind, and has its complex abstract reality which is related to the materialistic reality (applicability) from which it is derived (Haule & Johnson, 2023a). Therefore, the definition of mathematics can't be stated in materialistic-idealistic realism form as it is often stated in other academic disciplines such as physics and chemistry. It is to be stated in idealistic-idealistic realism. Mathematics is derived from the object of experience; it has its own complex abstract materialistic reality which is related to materialistic reality from which it is derived as Platonism philosophy and Aristotle assert.

3.3 Philosophical Basis for Defining Mathematics

According to Korner (1960), one of the main purposes of the philosophy of Mathematics is to reflect and account for the nature of Mathematics. According to Aristotle (1952) and the Concise Oxford English Dictionary (2011), epistemology is the study of things that exist or happen in the world, and rises to the knowledge of the universe. This notion implies that epistemology is the study of metaphysical things in the universe in terms of its nature and scope. In other words, epistemology itself is an idealistic reality acquired from the study of metaphysical things, be it real substances, non-real substances or both.

Then, how do we philosophically reflect and account for the definition of mathematics? According to Ernest (1991), Ernest (1989) and Haule & Johnson (2023a), the philosophy of Mathematics attempt to provide a system into which Mathematics can systematically establish its truth depending on the widely adopted assumptions. All together in line with Nyirenda and Ishumi (2002), they agree that the assumed role of philosophy of Mathematics is to provide a systematic and secure foundation for mathematical knowledge which is for Mathematical truth. And that the foundation of Mathematics is built upon justification of Mathematical knowledge acquired from inherent features, properties and characteristics of anything that exist and that is said to exist.

These views provide an insight on the basis with which the definition of Mathematics can be stated meta-epistemologically from the fact that knowledge is the study of epistemological attributes of a metaphysical thing or anything that exists. And thus, the definition of any academic discipline can precisely be stated meta-epistemologically. This foundation grants a secure foundation and a secure approach to be adopted in attempting to define the term Mathematics. And that, any mathematician or philosopher attempting to define mathematics should rely on the meta-epistemological philosophy of Mathematics.

In attempting to reflect and account for the definition of Mathematics, a meta-epistemological definition of Mathematics is adopted by assuming that all academic disciplines are reflected and accounted for its definition by identifying a metaphysical thing studied in the discipline whose attributes build a body of knowledge of the academic discipline (Haule & Johnson, 2023a). And also, it is adopted by assuming that both metaphysical strands do not significantly differ in the sense in which the object of experience forms the reality of the universe. Moreover, it is adopted by assuming that epistemology rises from the study of objects of experience in the context of both meanings of the objects of experience as Aristotle (1952) asserts. It is considered as a philosophical foundation for stating the definition of any academic discipline.

For instance, according to the Concise Oxford English Dictionary (2011), physics is defined as the study of nature and properties of matter and energy. Although energy is an attribute of matter, and matter is a metaphysical thing, the definition satisfies the meta-epistemological basis for defining an academic discipline. This means, matter is a metaphysical real thing, and nature, properties and energy are the aspects of matter studied in physics. As such, this definition is considered a meta-epistemological precise definition of physics.

Similarly, the same dictionary defines chemistry as the study of chemical composition and properties of a substance (matter). The definition satisfies the meta-epistemological basis for defining an academic discipline. This means, that matter is a metaphysical real thing, and chemical composition and properties are the aspects of matter studied in chemistry. Therefore, this definition is considered a meta-epistemological precise definition of chemistry. So, matter has many aspects to be studied, and some of them specifically are studied in distinguished terms of the attributes studied in physics and chemistry.

In other words, both Physics and Chemistry studies matter, which is a metaphysical real thing. but in Physics; nature, energy and properties are studied while in Chemistry; chemical composition and properties are studied. Similarly, if the nature of mathematics together with its epistemological status is not concluded among mathematicians and philosophers, mathematics can never philosophically be defined.

On the other hand, Concise Oxford English Dictionary (2011), defines history as the study of past events in human affairs. Grammatically, the word event is a synonymity of proceedings, actions, measures, traits, procedures etc. which are the attributes of a metaphysical thing called a human being. The definition satisfies the meta-epistemological basis for defining an academic discipline because human being is a metaphysical real thing, and an event is an attribute in human affairs that are encountered by a human being. Therefore, this definition is considered a meta-epistemological precise definition of history.

Since Metaphysics is concerned with the nature of things that exist and things that are said to exist, the definition of Mathematics is approached metaphysically by accepting that Mathematics is a metaphysical thing, be it real thing or non-real things or both. Epistemology is built upon the knowledge of attributes of these metaphysical things. In the realism philosophy of mathematics, it is conceived that reality exists in two forms: materialistic reality and idealistic reality formed from objects of experience; idealistic reality like mathematics has its reality which is related to the materialistic reality. Therefore, it is accepted that all mathematics abstractions reflecting reality have their own reality beyond space and time as Plato (1952a, 1952b) asserts.

Accepting Plato's assertion implies that mathematics, contrary to other academic disciplines, it is the study of non-real metaphysical things that are derived from experience, and has its reality in idealistic-idealistic reality. This attempt, therefore, provides the basis into which the definition of mathematics can precisely be stated systematically to establish its truth by relying on the widely adopted philosophical assumption. Thus, the researcher adopts a meta-epistemological approach in reflecting and accounting for the definition of then term Mathematics.

2.4 Shortcomings of the Selected Definitions

As it was introduced in the previous section, on a philosophical basis the term definition means a statement that describes the meta-epistemological nature and scope of something, and so does mathematics. Therefore, the analysis of the selected definitions based on the meta-epistemological philosophy of mathematics to identify the weak points from the selected definitions.

Aristotle define Mathematics as "a science of quantity". A quantity is an epistemological attribute of a metaphysical thing. According to this definition, mathematics is a science of an epistemological attribute which contradicts the precise meanings of academic discipline since the nature and scope of that thing are not clearly described.

In addition, a quantity of a thing is not the only attribute of metaphysical things that compose mathematics (Haule & Johnson, 2023a). Moreover, this statement provides us with an insight into the consistency of Platonism's view that that there exist mathematical objects of which, according to Aristotle, the science of its quantity is one aspect of the mathematical knowledge that builds a body of mathematical knowledge. Furthermore, the definition lacks some meta-epistemological basis as it is about the study of an attribute of a metaphysical thing instead of being the study of a metaphysical thing in terms of its attributes.

On the other hand, apart from this definition, Aristotle (1952) also acknowledges that epistemology is the study of things that exist or happen in the world. This statement provided Aristotle with an insight that there exist metaphysical things whose study of attributes rises to mathematical knowledge. It was therefore expected that Aristotle could have stated the definition of mathematics with the notion in mind that there exist metaphysical things in nature in the form of mathematics whose science of quantities is studied.

Russell (1901) views mathematics as "the subject we never know what we are talking about or whether what we are saying is true". The definition has not been stated on meta-epistemological basis. It is like Russell stated the definition of mathematics without the notion of how definitions of academic disciplines are stated, neither notion of the basis of the philosophy of mathematics. No metaphysical status nor epistemological status has been described.

Peirce (1870) views mathematics as the science of concluding. Defining something by beginning with the "science of" will always lead to stating a science of an epistemological attribute. Defining something by beginning with "the study of" will obviously lead to the study of a metaphysical thing in terms of its epistemological attribute, and thus it is possible to state the nature and scope of something as a philosophical definition requires.

Philosophically, there exist no conclusions drawn from nothing. There must exist something, and the study of its epistemological attributes leads to concluding. Therefore, this definition doesn't reflect the existence of a philosophical foundation in meta-epistemological basis since the metaphysical status of mathematics is left unknown, and also meta-epistemologically, no conclusion of nothing can be drawn.

Curry (1951) views mathematics as "the science of formal systems". A system isn't a metaphysical thing, it is an epistemological attribute of something. Philosophically, there exists no system of nothing; there must be a system of something. There must exist something, and the study of its epistemological attributes leads to the formal systems. The metaphysical status of mathematics is left unknown. Furthermore, it doesn't reflect the existence of a philosophical foundation meta-epistemologically since there exist no systems of nothing.

Sawyer (1955) views mathematics as the classification and the study of all possible patterns. Classification is an epistemological attribute of something, and so does patterns. The statement "the study of possible patterns" alone philosophically means the study of an epistemological attribute, and the metaphysical status of mathematics is left unknown. However, no possible patterns of nothing can be studied; there must exist something, and its study leads to obtaining the possible patterns. Therefore, it doesn't reflect the existence of a philosophical foundation since meta-epistemologically, no classification of an attribute of nothing can be done and possible patterns can be studied.

Black (1933) views mathematics as "the study of all structures whose form can be expressed in symbols; it is a grammar of all symbolic systems". Structure in this sense means parts that are formed into a whole. It is to form ideas into a whole in which each part is related to other parts (Haule & Johnson, 2023a).

Symbols and relationships that exist between them is the stuff of which mathematical objects are made, in idealistic-idealist reality construction using symbolic systems. And that mathematics is a self-contained body of abstract reality formed of symbols. But mathematics is not all about structures that describe the epistemological scope of mathematics; there exist other epistemological attributes of which mathematical knowledge is made (Haule & Johnson, 2023a).

The structure is an epistemological attribute of something, and metaphysically, the expression of mathematics by symbols, is the metaphysical mathematical object representation in idealistic-idealistic realism. However, Black attempts to define mathematics on the meta-epistemological basis by attempting to state the nature and scope of mathematics. He attempts to identify structure as an epistemological attribute studied in mathematics to build a body of mathematical knowledge and expression of mathematics by symbols as the stuff that metaphysically form mathematical objects of mathematics.

The philosophical questions here include, is a structure the only epistemological attribute of mathematics that builds a body of mathematical knowledge? Mathematics is for whom and for what? However, despite trying to describe a metaphysical thing composing mathematics it doesn't completely describe the epistemological attributes of mathematics which build a complete body of mathematical knowledge.

Courant and Robins (1941) view mathematics as "nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise can be created by the free will of the mathematicians". A system is an epistemological attribute of something, and so does postulates and definitions in this sense. As discussed before, and based on the philosophy of mathematics, there exists no definition or postulate of nothing. It therefore implies that mathematics is the study of an epistemological attribute while the metaphysical status of mathematics is left unknown.

That is to say, it doesn't reflect the existence of a philosophical basis since meta-epistemologically, no epistemological attribute of nothing can be studied. It is also philosophically interesting if mathematicians can create mathematics by free will without existence of objects of experience: materialistic or idealistic realities from which mathematical ideas are drawn, then create mathematics in certain forms of reality, and then use the obtained certain form of reality to solve real life problems. Real-life problems cannot be solved by certain forms of reality if it isn't reality, otherwise, epistemology would be useless.

Cline (1972) views mathematics as a creative or inventive process, deriving ideas and suggestions from real problems, idealizing and formulation of relevant concepts, posing questions, intuitively drawing a possible conclusion and only then, providing a bunch of intuitive argument deductively.

Cline (1990) sees mathematics as a process of discovering mathematics from real-life problems based on intuition and construction. The definition is based on materialistic-idealism realism as if mathematics on its own is materialistic-idealism reality. However, the meta-epistemological status of discovered mathematics is left unknown. Cline doesn't describe the metaphysical status of mathematics nor describe the epistemological attributes whose study rise to mathematical knowledge as Aristotle (1952) contends.

Therefore, the definition doesn't reflect the presence of the meta-epistemologically basis since no metaphysical status of mathematics has been described and no epistemological attributes of nothing can be studied to build a body of mathematical knowledge.

3.5 Meta-Epistemological Definition of Mathematics

The harmonization of the metaphysical strands is considered as the first philosophical adoption in the attempt to define mathematics. It is conceived that the object of experience has two philosophical meanings. The first meaning is that it is a material thing that can be seen and touched, someone can contact with and make observations of facts. The reality associated with the study of these metaphysical things is of the materialistic-idealistic form. The epistemology rises from the study of the attributes of real-substance metaphysical things. The second meaning is that it is a non-material thing external to the thinking mind causing events that can leave the impression to someone. The reality associated with the study of these metaphysical things is of the abstract materialistic-idealistic form which, it has been termed as the idealistic-idealistic reality. The epistemology rises from the study of the attributes of non-real-substance metaphysical things.

Since real-life problems can't be solved by certain forms of reality if it doesn't have reality, then the Plato's assertion is considered consistent in the realism perspective of the philosophy of mathematics. The two forms of reality: materialistic-idealist and idealistic-idealistic realities must be related since one of the roles of the philosophy of mathematics is to solve real-life problems from which mathematics is derived (Haule & Johnson, 2023, Plato (1952a, 1952b, Aristotle, 1952). By asserting that "the objects of experience form the only reality" in either case or both cases of its philosophical meaning, then the researcher holds that the definition of mathematics can be stated on the right of this philosophical foundation.

The Platonism philosophy of mathematics and the Metaphysical nature of mathematics in terms of existence of mathematical objects having representation in the mind is considered as the second philosophical adoption in the attempt to define mathematics (Haule & Johnson, 2023, Plato, Plato (1952a, 1952b, Aristotle, 1952).

If the Plato's thoughts are correct and consistent, that there exist mathematical objects beyond the mind in ideal space constituting to Mathematics, respective to Aristotle's views, then what are the inherent features, qualities and characteristics of those mathematical objects? Accepting that a mathematical object that Plato conceived to exist beyond the mind has six attributes: shape, qualities, form, structure, properties and applicability provides the epistemological basis for defining precisely mathematics (Haule, 2023, Plato, 1952a, 1952b, Aristotle, 1952, Hart, 1996). This implies that all definitions, postulates, axioms and propositions in Mathematics attempt to describe the epistemological attributes of mathematics in terms of inherent features, qualities and characteristics. And therefore, it is imperative to conceive that mathematics is the study of these epistemological attributes of a mathematical object in its mathematical ideal space. The mathematical objects have its own reality in abstract materialistic-idealistic form derived from objects of experience. The discovery of mathematics would simply mean discovering a mathematical object in a mathematical space together with its attributes, mathematical objects that reflect reality.

Having the notion in mind that objects of experience in terms of their multiple meanings form mathematical reality, meta-epistemologically, mathematics can be defined as the study of shape, qualities, form, structure, properties and applicability (epistemological attributes) of mathematical objects (abstract metaphysical object) derived from objects of experience. A body of mathematical knowledge is assumed to be developed and acquired through studying these attributes of mathematical objects for complete mathematical knowledge acquisition.

It is concluded that any mathematician or philosopher interested in attempting to define mathematics, must first describe precisely the metaphysical nature of mathematics and then define mathematics philosophically on a meta-epistemological basis. This adoption will provide a widely secure foundation in the philosophy of mathematics.

4. Implications to the Mathematics Teaching Methods

With the philosophical assertion that mathematics is the study of shape, qualities, form, structure, properties and applicability of mathematical objects derived from objects of experience, then all pedagogy and the methods of teaching must rest on this meta-epistemological basis. The objects of experience range from simple symbols and digits to complex relationships of the symbols with different form of attributes specified by a space into which they belong.

4.1 Shape

The shape of a mathematical object is a well mental proportionate visible way in which a mathematical object naturally appears to exist in abstract materialistic realism in a mathematical space of the ideal space. Real objects, where possible, are correctly perceived in the mind to construct mental images in the form of mathematical objects that have their reality (Haule & Johnson, 2023a, Kant, 1984, Cangelosi, 1996).

For instance, an algebraic quadratic function has a general shape represented by its general form $f(x) = ax^2 + bx + c$. In classroom instructions, a mathematics teacher must be able to enable learners to perceive the correct shape of this algebraic quadratic equation distinct from other such shapes of the same species. The choice of methodology should be methods that serve this purpose.

4.2 Qualities

A quality of a mathematical object is a distinctive feature and characteristic of a mathematical object concerning other mathematical objects of similar species within a mathematical space of an ideal space (Haule & Johnson, 2023a). The qualities of a mathematical object are identified, described and analyzed for basic understanding of the mathematical object. Correct mental representations enable learners to distinguish a mathematical object from other objects of the same species of the sub-mathematical space.

In classroom instructions, a mathematics teacher must enable learners identify and describe of the algebraic quadratic equation of the same species. Choice of methodology should be methods that serve this purpose. A mathematics teacher must make sure that correct qualities have been analyzed by the learners, and they have corrected understanding.

Qualities are described from the perceived learning experiences and constructing conceptual content in abstract substance forms to generate a basic understanding of the facts comprehended from algebraic quadratic functions learning phenomena. The qualities of an algebraic quadratic equation are identified, analyzed, described and explained through correct thinking and judgement to generate knowledge (Haule, 2023, Kant, 1984, Cangelosi, 1996).

That is, distinctive qualities are identified, described and analyzed to the other functions of a similar kind. Perhaps, a mathematics teacher will end up leading learners to the conclusion that an algebraic quadratic function is a polynomial of degree two. In other words, a mathematics teacher will probably aim at enabling learners to conceive that an algebraic quadratic function has all the qualities of functions polynomial function except that it is of degree two.

In classroom instructions, a mathematics teacher must enable learners to identify, describe and analyze the distinctive qualities of this algebraic quadratic equation distinct from the qualities of other algebraic quadratic equations of the same species. The choice of methodology should be methods that serve this purpose. A mathematics teacher must make sure that the correct qualities have been analyzed by the learners, and that they have corrected understanding.

4.3 Form

A form of a mathematical object is a way in which a mathematical object exists in different ways in abstract materialistic realism in a mathematical space of the ideal mathematical space. It is a mental visible formation of a mathematical object as distinct from its components. Algebraic quadratic functions, for example, exist in different forms of the same species; $f(x) = ax^2 + bx + c$, $f(x) = ax^2$, $f(x) = ax^2 + bx$ and $f(x) = ax^2 + c$.

In classroom instructions, a mathematics teacher must enable learners to identify and describe different forms of the algebraic quadratic equation of the same species. The choice of methodology should be methods that serve this purpose. A mathematics teacher must make sure that the correct qualities have been analyzed by the learners, and that they have corrected understanding.

4.4 Structure
The structure of a mathematical object is the existence of different parts of different forms that define a good organization of a mathematical object in abstract materialistic realism. It is a well-defined organization of the arrangements of parts of a complex mathematical object in abstract realism form. Algebraic quadratic functions, for example, are composed of linear factors of the general form $f(x) = ax + b$; and exist in different shapes, forms and structures.

In classroom instructions, a mathematics teacher must enable learners to identify and describe different structures of the algebraic quadratic equation of the same species. The choice of methodology should be methods that serve this purpose. A mathematics teacher must make sure that the correct qualities have been analyzed by the learners, and that they have corrected understanding.

4.5 Properties
Properties of a mathematical object are sets of accepted definitions, axioms, propositions and postulates with an adequate ground for asserting and justifying them. For instance, one of the properties of an algebraic quadratic function is that there exist zeros of at $f(x) = 0$. And the zeros may be real roots or complex roots or both.

In classroom instructions, a mathematics teacher must enable learners to identify and describe properties of the algebraic quadratic equation of the same species. The choice of methodology should be methods that serve this purpose. A mathematics teacher must make sure that the correct qualities have been analyzed by the learners, and that they have corrected understanding.

4.5 Applicability
Applicability of a mathematical object is the appropriateness and fitness of a mathematical object to be used to solve real-life problems from which mathematical ideas were perceived. It is about matching the abstract materialistic reality to the materialistic reality, from which the mathematical objects were derived, to solve real-life problems. It is the reciprocate property of mathematics which is the relationship between idealistic-idealistic reality and objects of experience in the form of materialistic-idealistic reality.

In simpler meaning, learners are subjected to a state of using the mentally constructed mathematical abstract objects to solve real-life problems from which the mathematical objects were derived (Haule & Johnson, 2023a). Learners are subjected to a state of imagination, think logically and rationally, judge the worthiness and appropriateness of mental constructs in reciprocate property and gain self-knowledge and skills according to their own understanding. Learning in classrooms is expected to involve using mathematics to solve related real-life problems. In classroom instructions, a mathematics teacher must be able to enable learners to apply mathematics to solve real-life problems. The choice of methodology should be methods that serve this purpose.

5. Conclusion and Recommendations
Based on meta-epistemological characterization of the nature of mathematics and related literature, it is concluded that mathematics can be thought of as the study of the attributes of a mathematical object in terms of its shape, qualities, form, structure, properties and applicability to real life. All learning process and pedagogy of mathematics rests on this meta-epistemological assertion in the philosophy of mathematics.

In the right of this study, it is recommended that;

(a) Mathematics philosophers should review the definition of Mathematics on a meta-epistemological basis to solve the everlasting dilemma of the precise definition of Mathematics, which has an impact and implication for Mathematics pedagogy.
(b) Mathematics pedagogy experts should review the pedagogy of Mathematics from the philosophical perspective of studying the attributes of mathematics objects.
(c) Higher learning Institutions should design and implement degree programmes in Mathematics education that will produce mathematics philosophers who will play a role in in reviewing the pedagogy of mathematics education in meta-epistemological basis.

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