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On the Periodic Solutions of Certain Fifth Order Nonlinear Vector Differential Equations

Melike Karta
Department of Mathematics, Faculty of Science and Arts,
Agri Ibrahim Cecen University, Agri, E-mail: melike_karta2010@hotmail.com

Abstract
The purpose of this paper is to show that under some sufficient conditions of equation (1.1) have no periodic solution other than the trivial solution X=0. Namely, we improve some well-known periodic results for certain fifth order nonlinear vector differential equations.

Keywords: Nonlinear vector differential equation of fifth, Lyapunov function, Periodic solutions.

1. Introduction

Periodic solutions of high order scalar and vector differential equations have emerged in the applied sciences as some practical mechanical problems, physics, chemistry, biology, ecomomics, and control theory. According to observations in the literature, the problems related to the periodic behavior of solutions of a higher order non-linear scalar and vector differential equation have been investigated by many authors. For some related contributors to the subject, we refer to the papers of Li & Duan [1], Bereketoğlu [2-3], Ezeilo [4-6], Tejumola [8], Tiryaki [9] and Tunç [10]. In all of the papers mentioned above, authors used Lyapunov's second (or direct) method [11].

In this article, based on study of form

$$\dot{x}_i = x_{i+1} \qquad (i = 1,2,3,4)$$

$$\dot{x}_i = -a_5 x_5 - f_4(x_1,x_2,x_3,x_4,x_5)x_4 - f_3(x_2)x_3 - f_2(x_1,x_2,x_3,x_4,x_5)x_2 - f_1(x_1)$$
$$(f_2(0) = f_1(0) = 0)$$

produced by Li & Duan [5], we investigate that have no periodic solution other than the trivial solution $X = 0$ of the fifth- order non-linear vector differential equations of the form

$$X^{(5)} + AX^{(4)} + \theta(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dddot{X} + F(\dot{X})\ddot{X} + G(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dot{X} + H(X) = 0 \qquad (1.1)$$

in the real Euclidean space $\mathbb{R}^n$ (with the usual norm denoted in what follows by $\|.\|$),

in which $X \in \mathbb{R}^n$; A is constant $n \times n$-symmetric matrix; Θ, F and G are $n \times n$-symmetric continuous matrix functions depending, in each case, on the arguments shown; $H: \mathbb{R}^n \to \mathbb{R}^n$ is continuous n-vector function. It is supposed that the function H is continuous and $H(0) = 0, H(X) \neq 0$ for $X \neq 0$. Let $J_H(X)$ and $J_F(Y)$ display the Jacobian matrices corresponding to the functions $H(X)$ and $F(Y)$, respectively, that is,

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right),\ J_F(Y) = \left(\frac{\partial f_i}{\partial y_j}\right) \quad , \quad (i,j=1,2,\ldots,n)$$

in which $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$, $(f_1, f_2, , \dots, f_n)$ and $(h_1, h_2, \dots, h_n)$, are the components of X, Y, F and H, respectively. The symbol $\langle X, Y\rangle$ corresponding to any pair X,Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the traspose of X.

Throughout this paper, we consider the following differential systems which are equivalent to the equation (1.2) which was attained as usual by setting $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W, X^{(4)} = U$ from (1.2):

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U$$

$$\dot{U} = -AU - \theta(X, Y, Z, W, U)W - F(Y)Z - G(X, Y, Z, W, U)Y - H(X) \tag{1.2}$$

2. Main Result

Our main result is the following theorem. Here we show that equation (1.1) have no periodic solution other than the trivial solution $X = 0$.

Theorem 2.1. In addition to the basic conditions given above for coefficients A, Θ, F, G and H of (1.2) equation, it is assumed that A and $J_H(X)$ are symmetric matrices and there is constant a_1 , a positive constant as the following conditions hold:

(i) $\lambda_i(A) > a_1$, $(i = 1,2,\dots,n)$

(ii) $\lambda_i(G(X, Y, Z, W, U)) - \frac{1}{4}[\lambda_i \Theta(X, Y, Z, W, U)]^2 > 0$, $(i = 1,2, \dots, n)$

Then trivial solution $X = 0$ of (1.2) is instability.

In prove main result, we use the following lemma.

Lemma2.2. Let A be a real symmetric $n \times n$-matrix and

$$a^\iota \geq \lambda_i(A) \geq a > 0 \qquad (i = 1,2,\dots,n)\,, \tag{2.1}$$

in which a^ι, a are constants.

Then

$$\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle\,, \quad a^{\iota^2}\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle. \tag{2.2}$$

See [12] for proof .

Proof. Let

$$(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4, \boldsymbol{\varepsilon}_5) = (\boldsymbol{\varepsilon}_1(\boldsymbol{t}), \boldsymbol{\varepsilon}_2(\boldsymbol{t}), \boldsymbol{\varepsilon}_3(\boldsymbol{t}), \boldsymbol{\varepsilon}_4(\boldsymbol{t}), \boldsymbol{\varepsilon}_5(\boldsymbol{t}))$$

be an arbitrary α-periodic solution of (1.2), that is

$$(\boldsymbol{\varepsilon}_1(\boldsymbol{t}), \boldsymbol{\varepsilon}_2(\boldsymbol{t}), \boldsymbol{\varepsilon}_3(\boldsymbol{t}), \boldsymbol{\varepsilon}_4(\boldsymbol{t}), \boldsymbol{\varepsilon}_5(\boldsymbol{t})) = (\boldsymbol{\varepsilon}_1(\boldsymbol{t}+\boldsymbol{\alpha}), \boldsymbol{\varepsilon}_2(\boldsymbol{t})+\boldsymbol{\alpha}, \boldsymbol{\varepsilon}_3(\boldsymbol{t}+\boldsymbol{\alpha}), \boldsymbol{\varepsilon}_4(\boldsymbol{t}+\boldsymbol{\alpha}), \boldsymbol{\varepsilon}_5(\boldsymbol{t}+\boldsymbol{\alpha})) \tag{2.3}$$

for some $\alpha > 0$.It will be shown that, under the conditions in Theorem (2.1),

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0.$$

As basic tool the proof of Theorem (2.1), we will use Lyapunov function $V_1 = V_1(X, Y, Z, W, U)$ given as

$$V = -\langle Y, U\rangle + \langle Z, W\rangle - \langle AY, W\rangle + \frac{1}{2}\langle AZ, Z\rangle - \int_0^1 \langle F(\sigma Y)Y, Y\rangle\, d\sigma - \int_0^1 \langle H(\sigma X), X\rangle d\sigma \quad (2.4)$$

Consider the function

$$\theta(t) = \boldsymbol{V_1}\big(\boldsymbol{\varepsilon_1(t)}, \boldsymbol{\varepsilon_2(t)}, \boldsymbol{\varepsilon_3(t)}, \boldsymbol{\varepsilon_4(t)}, \boldsymbol{\varepsilon_5(t)}\big).$$

Since $\theta(t)$ is continuous and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ are periodic in t, $\theta(t)$ is clearly bounded. An elementary differentiation will show that

$$\dot{V} = \frac{d}{dt} V(X, Y, Z, W, U)$$

$$= \langle W, W\rangle + \langle Y, \theta(X, Y, Z, W, U)W\rangle + \langle Y, F(Y)Z\rangle$$

$$= Y, G(X, Y, Z, W, U)Y + \langle Y, H(X)Y\rangle - \frac{d}{dt}\int_0^1 \langle \sigma F(\sigma Y, Y)\rangle d\sigma - \frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma \quad (2.5)$$

Now, we recall that

$$\frac{d}{dt}\int_0^1 \langle \sigma F(\sigma Y, Y)\rangle d\sigma = \sigma^2 \langle \sigma F(\sigma Y)Z, Y\rangle \Big|_0^1 = \langle F(Y)Z, Y\rangle \quad (2.6)$$

and

$$. \quad \frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \sigma \langle H(\sigma X), Y\rangle \Big|_0^1 = \langle H(X), Y\rangle. \quad (2.7)$$

Substituting (2.5) and (2.6) into (2.4) and taking into account the conditions of the Theorem 2.1, we obtain

$$\dot{V} = \langle W, W\rangle + \langle Y, \theta(X, Y, Z, W, U)W\rangle + \langle Y, G(X, Y, Z, W, U)Y\rangle$$

$$= \left\| W + \frac{1}{2}\theta(X, Y, Z, W, U)Y \right\|^2 - \frac{1}{4}\langle \theta(X, Y, Z, W, U)Y, \theta(X, Y, Z, W, U)Y\rangle$$
$$+ \langle Y, G(X, Y, Z, W, U)Y\rangle$$

$$\geq \langle Y, G(X,Y,Z,W,U)Y\rangle - \tfrac{1}{4}\langle \theta(X,Y,Z,W,U)Y, \theta(X,Y,Z,W,U)Y\rangle \geq 0. \quad (2.8)$$

Hence $\dot{\theta}(t) \geq 0$;so that $\theta(t)$ is monotone in *t*, and therefore, being bounded, tends to a limit, θ_0

say, as $t \to \infty$.That is $\lim_{t\to+\infty}\theta(t) = \theta_0$ .it is readily showed that

$$\theta(t) = \theta_0 \quad \text{for all to t.} \tag{2.9}$$

from by (2.1),

$$\theta(t) = \theta(t+m\alpha) \tag{2.10}$$

for any arbitrary fixed *t* an for arbitrary integer *m*, and then letting $m \to \infty$ in the right-hand side of (2.10) leads to (2.9).

the result (2.7) itself implies that

$$\dot{\theta}(t) = 0 \quad \text{for all } t.$$

Furthermore $\dot{\theta}(t) = 0$ necessarily implies that ε_2=0 for all t.Thus, if ε_2=ε_3=ε_4=ε_5=0 are written in (1.2) system, we can clearly see that it is $H(\varepsilon_1) = H(X_0) = 0$, because of $H(0) = 0$ and $H(X) \neq 0$ for $X \neq 0$ under conditions of Theorem (2.1) and ε_1=X_0.So that ε_1=$X_0 = 0$.It is $\varepsilon_1 = \varepsilon_2$=$\varepsilon_3$=$\varepsilon_4$=$\varepsilon_5$=0 for all t.Since $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ is a solution of (1.2), it is evident that

$$\big(\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), \varepsilon_4(t), \varepsilon_5(t)\big) = \big(X(t), Y(t), Z(t), W(t), U(t)\big) = (0,0,0,0,0)$$

This completes the proof of Theorem (2.1).

3.Conclusion

The proof of the Theorem2.1 showed that the equation (1.1) have no periodic solution other than the trivial solution $X = 0$. So, this has helped us to find a new result.

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