The Modified Hadamard Products of Functions Belonging to the Class P (j, λ, α, n, z_0)

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Abstract

The subclass $P(j,\lambda,\alpha,n)$ of starlike functions with negative coefficients by using the differential D^n operator and functions of the form $f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k$ which are analytic in the open unit disk is considered. The subclass $P(j,\lambda,\alpha,n,z_0)$ for which $f(z_0) = z_0$ or $f'(z_0) = 1$, z_0 real is examined by Kiziltunc and Baba (Kiziltunc and Baba, 2012). The modified Hadamard products of functions belonging to the class $P(j,\lambda,\alpha,n,z_0)$ has been obtained.

Keywords: Fixed point, Starlike, Salagean operatör, Hadamard product.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \alpha, \quad (z \in \mathbb{U}). \tag{1.2}$$

Let $S^*(\alpha)$ define the class of all such functions. Adititionally, a function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{U}). \tag{1.3}$$

Let $K(\alpha)$ define the class of all those functions which are convex of order α in $\mathbb{U}$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the classes of starlike and convex functions in $\mathbb{U}$, respectively.

Let $\mathcal{A}(j)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N} = \{1,2,3,\ldots\}) \tag{1.4}$$

which are analytic in the open unit disk $\mathbb{U}$.

For a function $f(z)$ in $\mathcal{A}(j)$, the Sălăgean operator introduced by Sălăgean (Sălăgean, 1983) is as follows

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{k=j+1}^{\infty} k a_k z^k$$

$$D^2 f(z) = D(f(z)) = zf'(z) + z^2 f''(z) = z + \sum_{k=j+1}^{\infty} k^2 a_k z^k$$

and for $n = 1,2,3$ ... we can write

$$D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=j+1}^{\infty} k^2 a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.5}$$

With the help of the differential operatör D^n, M.K.Aouf and H.M.Srivastava (Aouf and Srivastava, 1996) said that a function $f(z)$ belonging to $\mathcal{A}(j)$ is in the class $Q(j, \lambda, \alpha, n)$ if and only if

$$Re\left\{\frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+1} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)}\right\} > \alpha \tag{1.6}$$

for some α $(0 \leq \alpha < 1)$ and λ $(0 \leq \lambda < 1)$, and for all $z \in \mathbb{U}$. Additionally, $T(j)$ denoted the subclass of $\mathcal{A}(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \qquad (a_k \geq 0; j \in \mathbb{N}) \tag{1.7}$$

Further, M.K.Aouf and H.M.Srivastava defined the class $P(j, \lambda, \alpha, n)$ by

$$P(j, \lambda, \alpha, n) = Q(j, \lambda, \alpha, n) \cap T(j). \tag{1.8}$$

For a function $f(z)$ in $P(j, \lambda, \alpha, n)$, M.K.A ouf and H.M.Srivastava define with

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) = z - \sum_{k=j+1}^{\infty} k a_k z^k$$

$$D^2 f(z) = D(f(z)) = zf'(z) + z^2 f''(z) = z - \sum_{k=j+1}^{\infty} k^2 a_k z^k$$

and for $n = 1,2,3$ ... we can write

$$D^n f(z) = D(D^{n-1} f(z)) = z - \sum_{k=j+1}^{\infty} k^2 a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.9}$$

In (Aouf and Srivastava, 1996), M.K.A ouf and H.M.Srivastava obtained coefficients inequalities, distortion theorems, closure thorems, and some properties involving the modified Hadamard products for functions belonging to the class $P(j, \lambda, \alpha, n)$. They also determined the radii of close-to-convexity, starlikeness, convexity for the class $P(j, \lambda, \alpha, n)$.

In order that prove our theorem, the following lemma is needed.

Lemma 1.1. (Aouf and Srivastava, 1996) Let the function $f(z)$ be defined by (1.7). Then $f(z) \in P(j, \lambda, \alpha, n)$ if and only if

$$\sum_{k=j+1}^{\infty} k^n (k - \alpha)\{1 + (k-1)\lambda\} a_k \leq 1 - \alpha \tag{1.10}$$

$(a_k \geq 0; n \in \mathbb{N}_0; 0 \leq \alpha < 1; z \in \mathcal{U}; 0 \leq \lambda < 1)$.

In (Schild, 1974), Schild investigated the class of univalent polynomials of the form $f(z) = z -$

$\sum_{n=2}^{\mathbb{N}} a_n z^n$, where $a_n \geq 0$ and in the disk $|z| < 1$. In (Silverman, 1975), Silverman dealt with functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\mathbb{N}} a_n z^n \tag{1.11}$$

where either

$$a_n \geq 0, f(z_0), \quad (-1 < z_0 < 1;\ z_0 \neq 0) \tag{1.12}$$

or

$$a_n \geq 0, f'(z_0) = 1, \quad (-1 < z_0 < 1). \tag{1.13}$$

$S_0^*(\alpha, z_0)$ studied the subclass of functions starlike of order α that satisfy (1.12), and $S_1^*(\alpha, z_0)$ examined the subclass of functions starlike of order α that satisfy (1.13). Also defined by $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ the subclasses of functions convex of order α that satisfy, respectively, (1.12) and (1.13).

We denote by $P(j, \lambda, \alpha, n, z_0)$ the subclass of $P(j, \lambda, \alpha, n)$ involving any fixed point.

In order that prove our theorem, the following lemma is needed.

Lemma 1.2. (Kiziltunc and Baba, 2012) Let the function $f(z)$ be defined by (1.7). Then $f(z)$ is $P(j, \lambda, \alpha, n, z_0)$ if and only if

$$\sum_{k=j+1}^{\infty} \left[k^n \left(\frac{k-\alpha}{1-\alpha}\right)\{1 + (k-1)\lambda\} - z_0^{k-1}\right] a_k \leq 1 \tag{1.14}$$

($a_k \geq 0;\ j \in \mathbb{N};\ n \in \mathbb{N}_0;\ 0 \leq \alpha < 1;\ z \in \mathcal{U};\ 0 \leq \lambda < 1;\ z_0 \in \mathbb{R}$ fixed point).

In (Kiziltunc and Baba, 2012), H.Kızıltunc and H.Baba, studied the class $P(j, \lambda, \alpha, n, z_0)$ involving any fixed point and we determined coefficient inequalities for functions belonging to the class $P(j, \lambda, \alpha, n, z_0)$. As special cases, the results of this paper reduced to Silverman (Silverman, 1975).

In the present paper, I shall prove that the class $P(j, \lambda, \alpha, n, z_0)$ is closed under convex linear combinations. Then the modified Hadamard products of functions belonging to the class $P(j, \lambda, \alpha, n, z_0)$ has been obtained.

2. Convex Linear Combinations

In this section, I shall prove that the class $P(j, \lambda, \alpha, n, z_0)$ is closed under convex linear combinations.

Theorem 2.1. $P(j, \lambda, \alpha, n, z_0)$ is a convex set.

Proof. Let functions

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{v,k} z^k \quad (a_{v,k} \geq 0; v = 1,2) \tag{2.1}$$

be in the class $P(j, \lambda, \alpha, n, z_0)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leq \mu \leq 1) \tag{2.2}$$

is also in the class $P(j, \lambda, \alpha, n, z_0)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} \{\mu a_{1,k} + (1-\mu) a_{2,k}\} z^k, \tag{2.3}$$

with the aid of Lemma 1.1, I have

$$\sum_{k=j+1}^{\infty}[k^n(k-\alpha)\{1+(k-1)\lambda\}]\{\mu a_{1,k}+(1-\mu)a_{2,k}\}\leq 1-\alpha \tag{2.4}$$

which implies that $h(z)\in P(j,\lambda,\alpha,n)$. Further, the function $h(z)\in P(j,\lambda,\alpha,n)$ given by $h(z_0)=z_0$ is class of $P(j,\lambda,\alpha,n,z_0)$. Hence $P(j,\lambda,\alpha,n,z_0)$ is a convex set.

Teorem 2.2. Let functions $f_j(z)=z$ and

$$f_k(z)=z-\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}z^k \qquad (k\geq j+1; n\in\mathbb{N}_0) \tag{2.5}$$

for $0\leq\alpha<1$ and $0\leq\lambda<1$. Then $f(z)$ in the class $P(j,\lambda,\alpha,n,z_0)$ if and only if it can be expressed in the form:

$$f(z)=\sum_{k=j}^{\infty}\mu_k f_k(z), \tag{2.6}$$

where

$$\mu_k\geq 0 \quad (k\geq j) \quad \text{ve} \quad \sum_{k=j}^{\infty}\mu_k=1. \tag{2.7}$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=j}^{\infty}\mu_k f_k(z) \\ &= \sum_{k=j}^{\infty}\mu_k z-\sum_{k=j+1}^{\infty}\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}\mu_k z^k \\ &= z-\sum_{k=j+1}^{\infty}\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}\mu_k z^k \end{aligned} \tag{2.8}$$

and by Lemma 1.2

$$\sum_{k=j+1}^{\infty}a_k z_0^{k-1}=0.$$

Then it follows that

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}-z_0^{k-1}\right].\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}}\mu_k=\sum_{k=j}^{\infty}\mu_k=1-\mu_j\leq 1.$$

So, by Lemma 1.2, $f(z)\in P(j,\lambda,\alpha,n,z_0)$.

Conversely, assume that the function $f(z)$ defined by (1.7) belongs to the class $P(j,\lambda,\alpha,n,z_0)$. Then

$$a_k\leq\frac{1-\alpha}{k^n(k-\alpha)(1+(k-1)\lambda)-(1-\alpha)z_0^{k-1}} \qquad (k\geq j+1; n\in\mathbb{N}_0). \tag{2.9}$$

Setting

$$\mu_k=\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}-z_0^{k-1}\right]a_k \qquad (k\geq j+1; n\in\mathbb{N}_0) \tag{2.10}$$

and

$$\mu_j=1-\sum_{k=j}^{\infty}\mu_k, \tag{2.11}$$

I can see that $f(z)$ can be expressed in the form (2.6). This completes the proof of Theorem 2.2.

3. Modified Hadamard Products

Let functions $f_v(z)$ $(v=1,2)$ be defined by (2.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=j+1}^{\infty} a_{1,k}a_{2,k}z^k. \tag{3.1}$$

Theorem 3.1. Let each of the functions $f_v(z)$ $(v = 1,2)$ defined by (2.1) be in the class $P(j, \lambda, \alpha, n, z_0)$. Then $(f_1 * f_2)(z)$ is in the class $P(j, \lambda, \varphi(j, \lambda, \alpha, n), n, z\_0)$ where

$$\varphi(j, \lambda, \alpha, n) = \frac{(j+1)^n(1+j\lambda)-(j+1)\{(1-\alpha)/(j+1-\alpha)\}^2}{(j+1)^n(1+j\lambda)-\{(1-\alpha)/(j+1-\alpha)\}^2}. \tag{3.2}$$

Proof. Employing the technique used earlier by Schild and Silverman (Shild and Silverman, 1975), I need to find the largest $\varphi = \varphi(j, \lambda, \alpha, n)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\varphi)\{1+(k-1)\lambda\}}{1-\varphi} a_{1,k}a_{2,k} \leq 1. \tag{3.3}$$

From

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} a_{1,k} \leq 1 \tag{3.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} a_{2,k} \leq 1 \tag{3.5}$$

using the Cauchy-Schwarz inequality, I leads to

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha} \sqrt{a_{1,k}a_{2,k}} \leq 1. \tag{3.6}$$

Thus it is sufficient to show that

$$\frac{k^n(k-\varphi)}{1-\varphi} a_{1,k}a_{2,k} \leq \frac{k^n(k-\alpha)}{1-\alpha} \sqrt{a_{1,k}a_{2,k}} \qquad (k \geq j+1), \tag{3.7}$$

or equivalently

$$\sqrt{a_{1,k}a_{2,k}} \leq \frac{(k-\alpha)(1-\varphi)}{(1-\alpha)(k-\varphi)} \qquad (k \geq j+1). \tag{3.8}$$

Note that I have from (3.6)

$$\sqrt{a_{1,k}a_{2,k}} \leq \frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \qquad (k \geq j+1). \tag{3.9}$$

Consequently, if

$$\frac{1-\alpha}{k^n(k-\alpha)\{1+(k-1)\lambda\}} \leq \frac{(k-\alpha)(1-\varphi)}{(1-\alpha)(k-\varphi)} \qquad (k \geq j+1). \tag{3.10}$$

or, if

$$\varphi \leq \frac{k^n\{1+(k-1)\lambda\}-k\{(1-\alpha)/(k-\alpha)\}^2}{k^n\{1+(k-1)\lambda\}-\{(1-\alpha)/(k-\alpha)\}^2} \qquad (k \geq j+1). \tag{3.11}$$

then (3.6) is satisfied. Since

$$\Gamma(k) = \frac{k^n\{1+(k-1)\lambda\}-k\{(1-\alpha)/(k-\alpha)\}^2}{k^n\{1+(k-1)\lambda\}-\{(1-\alpha)/(k-\alpha)\}^2} \tag{3.12}$$

is non-decreasing for $k \geq j+1$, letting $k = j+1$ in (3.12). I obtain

$$\varphi \leq \Gamma(j+1) = \frac{(j+1)^n(1+j\lambda)-(j+1)\{(1-\alpha)/(j+1-\alpha)\}^2}{(j+1)^n(1+j\lambda)-\{(1-\alpha)/(j+1-\alpha)\}^2} \tag{3.13}$$

which proves assertion of Theorem 3.1.

Finally, by taking the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1} \qquad (v = 1,2), \tag{3.14}$$

we can see that the result is sharp.

Theorem 3.2. Let

$f_1(z) \in P(j,\lambda,\alpha,n,z_0)$ and $f_2(z) \in P(j,\lambda,\gamma,n,z_0)$.

Then $(f_1 * f_2)(z)$ is in the class $P(j,\lambda,\tau(j,\lambda,\alpha,\gamma,n),n,z_0)$ where

$$\tau(j,\lambda,\alpha,\gamma,n) = \frac{(j+1)^n(1+j\lambda)-(j+1)\left\{\frac{(1-\alpha)}{(j+1-\alpha)}\right\}.\{(1-\gamma)/(j+1-\gamma)\}}{(j+1)^n(1+j\lambda)-\left\{\frac{(1-\alpha)}{(j+1-\alpha)}\right\}.\{(1-\gamma)/(j+1-\gamma)\}} \tag{3.15}$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1}, \tag{3.16}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n(j+1-\gamma)(1+j\lambda)} z^{j+1}. \tag{3.17}$$

Proof. Proceeding as in the proof of Theorem 3.1, I get

$$\tau \le \frac{k^n\{1+(k-1)\lambda\}-k\{(1-\alpha)/(k-\alpha)\}.\left\{\frac{(1-\gamma)}{(k-\gamma)}\right\}}{k^n\{1+(k-1)\lambda\}-\{(1-\alpha)/(k-\alpha)\}.\left\{\frac{(1-\gamma)}{(k-\gamma)}\right\}} \qquad (k \ge j+1). \tag{3.18}$$

Since the right-hand side of (3.18) is non-decreasing for $k \ge j+1$, letting $k = j+1$ in (3.18), I obtain (6.15).

This completes the proof of Theorem 3.2.

Corollary 3.1. Let the functions $f_v(z)$ defined by

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{v,k} z^k \qquad \left(a_{v,k} \ge 0; v = 1,2,3\right) \tag{3.19}$$

be in the class $P(j,\lambda,\alpha,n,z_0)$. Then

$(f_1 * f_2 * f_3)(z) \in P(j,\lambda,\mu(j,\lambda,\alpha,n),n,z_0)$,

where

$$\mu(j,\lambda,\alpha,n) = \frac{(j+1)^{2n}(1+j\lambda)^2-(j+1)\{(1-\alpha)/(j+1-\alpha)\}^3}{(j+1)^{2n}(1+j\lambda)^2-\{(1-\alpha)/(j+1-\alpha)\}^3}. \tag{3.20}$$

For the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)(1+j\lambda)} z^{j+1}, \; (v = 1,2,3), \tag{3.21}$$

the result is the best possible.

Proof. From Theorem 3.1, I have

$(f_1 * f_2)(z) \in P(j,\lambda,\varphi(j,\lambda,\alpha,n),n,z_0)$,

where φ is given by (3.2). Now, using Teorem 3.2, I get

$(f_1 * f_2 * f_3)(z) \in P(j,\lambda,\mu(j,\lambda,\alpha,n),n,z_0)$,

where

$$\mu(j,\lambda,\alpha,n) = \frac{(j+1)^n(1+j\lambda)-(j+1)\left\{\frac{(1-\alpha)}{(j+1-\alpha)}\right\}.\{(1-\varphi)/(j+1-\varphi)\}}{(j+1)^n(1+j\lambda)-\left\{\frac{(1-\alpha)}{(j+1-\alpha)}\right\}.\{(1-\varphi)/(j+1-\varphi)\}} = \frac{(j+1)^{2n}(1+j\lambda)^2-(j+1)\{(1-\alpha)/(j+1-\alpha)\}^3}{(j+1)^{2n}(1+j\lambda)^2-\{(1-\alpha)/(j+1-\alpha)\}^3}.$$

This completes the proof of corollary 3.1.

Theorem 3.3. Let the functions $f_v(z)$ $(v = 1,2)$ defined by (2.1) be in the class $P(j,\lambda,\alpha,n,z_0)$. Then the function

$$h(z) = z - \sum_{k=j+1}^{\infty}\left(a_{1,k}^2+a_{2,k}^2\right)z^k \tag{3.22}$$

belongs to the class $P(j,\lambda,\sigma(j,\lambda,\alpha,n),n,z_0)$, where

$$\sigma(j,\lambda,\alpha,n) = \frac{(j+1)^n(1+j\lambda)-2(j+1)\{((1-\alpha))/((j+1-\alpha)\,)\}^2}{(j+1)^n(1+j\lambda)-2\{((1-\alpha))/((j+1-\alpha)\,)\}^2}. \tag{3.23}$$

For the functions $f_v(z)$ $(v = 1,2)$ defined by (3.14), the result is sharp.

Proof. By virtue of Lemma 1.1, I obtain

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2 a_{1,k}^2 \le \left[\sum_{k=j+1}^{\infty}\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}a_{1,k}\right]^2 \le 1 \tag{3.24}$$

and

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2 a_{2,k}^2 \le \left[\sum_{k=j+1}^{\infty}\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}a_{2,k}\right]^2 \le 1. \tag{3.25}$$

From (3.24) and (3.25), I have

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2\left(a_{1,k}^2+a_{2,k}^2\right) \le 1. \tag{3.26}$$

Thus, I need to find the largest $\sigma = \sigma(j,\lambda,\alpha,n)$ such that

$$\frac{k^n(k-\sigma)\{1+(k-1)\lambda\}}{1-\sigma} \le \frac{1}{2}\left[\frac{k^n(k-\alpha)\{1+(k-1)\lambda\}}{1-\alpha}\right]^2 \quad (k \ge j+1)\,, \tag{3.27}$$

or equivalently

$$\sigma \le \frac{k^n(1+(k-1)\lambda)-2k\{(1-\alpha))/((k-\alpha)\}^2}{k^n(1+(k-1)\lambda)-2\{(1-\alpha))/((k-\alpha)\}^2} \quad (k \ge j+1). \tag{3.28}$$

Since

$$\nabla(k) = \frac{k^n(1+(k-1)\lambda)-2k\{(1-\alpha))/((k-\alpha)\}^2}{k^n(1+(k-1)\lambda)-2\{(1-\alpha))/((k-\alpha)\}^2} \tag{3.29}$$

is non-decreasing for $k \ge j+1$, I have

$$\sigma = \nabla(j+1) = \frac{(j+1)^n(1+j\lambda)-2(j+1)\{((1-\alpha))/((j+1-\alpha))\}^2}{(j+1)^n(1+j\lambda)-2\{((1-\alpha))/((j+1-\alpha))\}^2}, \tag{3.30}$$

which proves the assertion of Theorem 3.3.

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