International Journal of Scientific and Technological Research
ISSN 2422-8702 (Online)
Vol 4, No.3, 2018

The Fundamental Groupoid as a Crossed Module in the Category of Local Groups

H. Fulya Akiz
Bozok University, Department of Mathematics,Yozgat
E-mail:fulya.gencel@bozok.edu.tr

Abstract
In this paper we define the notion of crossed module in the category of local groups and we prove that if L is a local topological group whose underlying topology has a simply connected cover, then the fundamental groupoid $\pi_1 L$ has a crossed module structure in the category of local groups.

Key Words: crossed module, local topological group, local group-groupoid

Introduction
Let X be a connected topological space, $\tilde{X}$ a connected and simply connected topological space, and let $p: \tilde{X} \to X$ be a covering map. We call such a covering simply connected. Further, if X is a topological space, then the fundamental groupoid $\pi_1 X$ becomes a groupoid. In [8] it was proved by Brown that if X is a semilocally simply connected topological space, i.e., each component has a simply connected covering, then the category TCov/X of topological coverings of X is equivalent to the category GpdCov/$\pi_1 X$ of groupoid coverings of the fundamental groupoid $\pi_1 X$.
Let X be a topological group, then the fundamental groupoid $\pi_1 X$ becomes a group-groupoid. Then we obtain a category GpGdCov/$\pi_1 X$ of group-groupoid coverings of $\pi_1 X$. In [23] it was proved that if X is a topological group whose underlying space is semilocally simply connected, then the category TGpCov/X of topological group coverings of X is equivalent to the category GpGdCov/$\pi_1 X$ of group-groupoid coverings of $\pi_1 X$.
If G is a group-groupoid, then the category GpGdCov/G of group-groupoid coverings and the category GpGdAct/G of group-groupoid actions of G are equivalent[9]. On the other hand the corresponding results for local topological groups and local group objects in the category of groupoids are given in [26].
The notion of local topological group-groupoid which is the group object in the category of local topological groups is given in Akız [1]. The category LTGpCov/L of covering morphisms $p: \tilde{L} \to L$ of local topological groups in which $\tilde{L}$ has also a universal cover and the category LTGpGdCov/$\pi_1 L$ of covering morphisms $q: \tilde{G} \to \pi_1 L$ of local topological group-groupoids based on $\pi_1 L$ are equivalent.[1].
A crossed module defined by Whitehead in [29, 30] can be viewed as a 2-dimensional group [24]. If X is a topological group, then the restriction $t: (\pi_1 X)^e \to X$ of the final point map to the star at the identity $e \in X$ is a crossed module of groups and the category GpGdCov/$\pi_1 X$ of covers of $\pi_1 X$ in the category of group-groupoids and the category of covers of $t: (\pi_1 X)^e \to X$ in the category of crossed modules within groups are equivalent [9]. In [2] the corresponding results for internal groupoids and crossed modules in the category of groups with operations are given.
In this paper we define the notion of crossed module in the category of local groups. We prove that if L is a local topological group whose underlying topology has a simply connected cover,then the fundamental groupoid $\pi_1 L$ has a crossed module structure in the category of local groups.

1 Preliminieries on groupoids and crossed modules
A *groupoid* is a (small) category in which each morphism is an isomorphism [8]. So a groupoid G has a set G of morphisms , which we call just elements of G, a set O_G of objects together with maps $s, t: G \to O_G$ and $\epsilon: O_G \to G$ such that $s\epsilon = t\epsilon = 1_{O_G}$. The maps *s, t* are called *initial and final point maps* respectively and the map is called object inclusion. If *a, b* ∈ G and *t(a) = s(b)*, then the composite *ab* exists such that *s(ab) = s(a)* and *t(ab) = t(b)*. So there exists a partial composition defined by $G_t \times_s G \to G$,*(a, b)* →*ab*, where $G_t \times_s G$ is the pullback of *t* and *s*. Further, this partial composition is associative, for x ∈ O_G the element $\epsilon(x)$ denoted by 1_x acts as the

identity, and each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $aa^{-1} = (s)(a)$, $a^{-1}a = t(a)$. The map $G \rightarrow G$, $a \rightarrow a^{-1}$ is called the *inversion*. In a groupoid G for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *transitive* if for $x, y \in O_G$, the set $G(x, y)$ is not empty. For $x \in O_G$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x.

2 Local groups and local group-groupoids

The following definition is given in [25].

Definition 2.1 A *local group* is a quintuple $L = (L, \mu, U, \imath, V)$, where

(1) a distinguish element $e \in L$, the identity element,

(2) a multiplication $\mu\colon U \rightarrow L, (x, y) \rightarrow x \circ y$ defined on a subset U of $L \times L$ such that $(\{e\} \times L) \cup (L \times \{e\}) \subseteq U$,

(3) an inversion map $\imath\colon V \rightarrow L$, $x \rightarrow \bar{x}$ defined on a subset $e \in V \subseteq L$ such that $V \times \imath(V) \subseteq U$ and $\imath(V) \times V \subseteq U$,

all satisfying the following properties:

(i) Identity: $e \circ x = x = x \circ e$ for all $x \in L$

(ii) Inverse: $\imath(x) \circ x = e = x \circ \imath(x)$, for all $x \in V$

(iii) Associativity: If $(x, y),(y, z),(x \circ y, z)$ and $(x, y \circ z)$ all belong to U, then $x \circ (y \circ z) = (x \circ y) \circ z$.

From now on we denote such a local group by $(L, \mu, U, \imath, V)$, or by L. Note that if $U = L \times L$ and $V = L$, then a local group becomes a group. It means that the notion of local group generalizes that of group.

Definition 2.2 Let $(L, \mu, U, \imath, V)$ and $(\tilde{L}, \tilde{\mu}, \tilde{U}, \tilde{\imath}, \tilde{V})$ be local topological groups. A map $f : L \rightarrow \tilde{L}$ is called a *local group morphism* if

(i) $(f \times f)(U) \subseteq \tilde{U}$, $f(V) \subseteq \tilde{V}$, $f(e) = \tilde{e}$,

(ii) $f(x \circ y) = f(x) \circ f(y)$ for $(x, y) \in U$,

(iii) $f(\imath(x)) = \tilde{\imath}(f(x))$ for $x \in V$.

We study on the topological version of Definition 2.1.

Definition 2.3 In Definition 2.1, if L is a topological space such that U is open in $L \times L$, V is open in L, the maps μ and $\imath$ are continuous, then $(L, \mu, U, \imath, V)$ is called a *local topological group*.
It is clear that if $U = L \times L$ and $V = L$, then a local topological group L becomes a topological group.

Example 2.4 [25]Let G be a topological group, L be an open neighbourhood of the identity element e. Then we obtain a local topological group taking $U = (L \times L) \cap \mu^{-1}(L)$ and $V = L \cap L^{-1}$. Here the group product μ and the inversion $\imath$ on G are restricted to define a local group product and inverse maps on L. Further if we choose U and V such that

$$(e \times L) \cup (L \times e) \subseteq U \subseteq (L \times L) \cap \mu^{-1}(L) \quad e \in V \subseteq L \cap \imath^{-1}(L)$$

and

$$V \times \imath(V)) \cup (\imath(V) \times V) \subseteq U$$

then we have a local topological group.

Definition 2.5 Let $(L, \mu, U, \imath, V)$ and $(\tilde{L}, \tilde{\mu}, \tilde{U}, \tilde{\imath}, \tilde{V})$ be local topological groups. A continuous map $f : L \rightarrow \tilde{L}$ is called a local topological group morphism if

(i) $(f \times f)(U) \subseteq \tilde{U}$, $f(V) \subseteq \tilde{V}$, $f(e) = \tilde{e}$,

(ii) $f(x \circ y) = f(x) \circ f(y)$ for $(x, y) \in U$

(iii) $f(\iota(x)) = \tilde{\iota}(f(x))$ for $x \in V$.

A local topological group morphism is called a *homeomorphism* if it is one-to-one and onto , with continuous inverse. Since the composition of local topological group homomorphisms is again a local group homomorphism, we obtain a category of local topological groups denoted by LTGp.
A notion of local group-groupoid is given in [26].

Definition 2.6 A *local group-groupoid* G is a groupoid in which O_G and G both have local group structures such that the following maps are the local morphisms of groupoids:

(i) $\mu\colon U \rightarrow G, (a, b) \rightarrow a \circ b$

(ii) $\iota\colon V \rightarrow G,\ a \rightarrow \overline{a}$

(iii) e: $\{*\} \rightarrow G$, where * is singleton.

We obtain that in a local group-groupoid G, $(ac) \circ (bd) = (a \circ b)(c \circ d))$ for $a, b, c, d \in G$ such that the necessary composition and multiplications are defined.

Example 2.7 If L is a local topological group, then the fundamental groupoid $\pi_1 L$ is a local group-groupoid.
Let L be a local topological group such that $U \subseteq L \times L$ ve $e \in V \subseteq L$ are open. Then $\pi_1 L$ is a groupoid. Since the maps on local topological group structure

$$\mu\colon U \rightarrow L, (x, y) \rightarrow x \circ y$$

and

$$\iota\colon V \rightarrow L,\ x \rightarrow \overline{x}$$

are continuous, then the induced maps

$$\pi_1(\mu)\colon \pi_1 U \rightarrow \pi_1 L,\ [(a, b)] \rightarrow [a] \circ [b]$$

and

$$\pi_1(\iota)\colon \pi_1 V \rightarrow \pi_1 L,\ [a] \rightarrow \overline{[a]}$$

are well defined. Note that since (a, b) is defined in U, then $a \circ b$ is defined. So $\pi_1 L$ is a local group-groupoid[26].

3 Crossed modules in the category of local groups

The notion of crossed module in the category of groups was given in [29]. Further a crossed module in groups with operations is formulated in [28]. Now we define this notion in the category of local groups.

Definition 3.1 Let L and K be local groups. If the function

$$L \times K \rightarrow L, (l, k) 7\rightarrow\ l^k$$

is satisfying the following contitions, then we say the local group K *acts(right)* on the local group L:

(i) $l^{ks} = (l^k)^s$, for $l \in L$, $k,s \in K$,

(ii) $l^e = l$, for $l \in L$, $e \in K$.

Definition 3.2 Let L and K be local groups and $\theta : K \rightarrow L$ be a local group morphism. Let

$$L \times K \rightarrow L, (l, k) \rightarrow l^k$$

be a right action of L on K. Then the triple (θ, K, L) is called a *crossed module* of local groups if the followings are satisfied:

(i) $\theta(l^k) = k^{-1}\theta(l)k$, for $l \in L$, $k \in K$,

(ii) $k^{\theta(s)} = s^{-1}ks$ for $k, s \in K$.

Example 3.3 Let G be a local group and N be a normal subgroup of G. The inclusion $\theta : N \to G$ indicates a crossed module.
Since G is a local group then also N becomes a local group. The action of G on N is defined by

$$N \times G \to G, (n,g) \to g^{-1}ng.$$

We know from [29] that $\theta : N \to G$ is a crossed module in the category of groups. So it is enough to show that $\theta : (L, \mu, U, \iota, V) \to (\tilde{L}, \tilde{\mu}, \tilde{U}, \tilde{\iota}, \tilde{V})$ is a morphism of local groups.

(i) $\theta(\tilde{e}) = e$, $(\theta \times \theta)(\tilde{U}) \subseteq U$, $\theta(\tilde{V}) \subseteq V$,

(ii) $\theta(\tilde{n}\tilde{m}) = \tilde{n}\tilde{m} = \theta(\tilde{n})\theta(\tilde{m})$, for $\tilde{n}\tilde{m} \in \tilde{U}$,

(iii) $\theta(\tilde{\iota}(\tilde{n})) = \theta((\tilde{n}^{-1})) = (\tilde{n}^{-1}) = (\theta(\tilde{n}))^{-1} = \tilde{\iota}(\theta(\tilde{n}))$ for $\tilde{n} \in \tilde{V}$.

We know from Example 2.7 that if L is a local topological group, then the fundamental groupoid $\pi_1 L$ becomes a local group-groupoid. Let $(\pi_1 L)^e$ be the set of morphisms with initial point e. So it has a local group structure. Then we can give the following proposition.

Proposition 3.4 Local group $(\pi_1 L)^e$ acts on local group L via the function

$$L \times (\pi_1 L)^e \to L, (l, [\alpha]) \to l^{[\alpha]} = \overline{1_{[\alpha]}}\, l 1_{[\alpha]}.$$

Let us show that the conditions are satisfied:

(i) for $l \in L$, $[\alpha] \in (\pi_1 L)^e$,

$[l^{[\alpha][\beta]}] = \overline{1_{[\alpha]}}\,\overline{1_{[\beta]}}\, l 1_{[\alpha]} 1_{[\beta]}$

$(l^{[\alpha]})^{[\beta]} = (\overline{1_{[\alpha]}}\, l 1_{[\alpha]})^{\beta}$

$= \overline{1_{[\alpha]}}\,\overline{1_{[\beta]}}\, l 1_{[\alpha]} 1_{[\beta]}$.

(ii) for $l \in L$, $[e] \in (\pi_1 L)^e$, $l^{[e]} = \overline{1_{[e]}}\, l 1_{[e]} = l$.

Proposition 3.5 Let L be a local topological group whose underlying topology has a simply connected cover. Then we have a crossed module with the final point function t: $(\pi_1 L)^e \to$ L, $[\alpha] \to t[\alpha] = t(\alpha)$.

We first prove that the conditions of crossed module in the category of local groups are satisfied:

(i) for $l \in L$, $[\alpha] \in (\pi_1 L)^e$,

$t(l^{[\alpha]}) = t(\overline{1_{[\alpha]}}\, l 1_{[\alpha]}) = t((\overline{1_{[\alpha]}})t(l)t(1_{[\alpha]})$

$= \overline{1_{[\alpha]}}\, t(l) 1_{[\alpha]}$.

(ii) for $[\alpha], [\beta] \in (\pi_1 L)^e$,

$[\beta]^{t([\alpha])} = \overline{[\alpha]}[\beta][\alpha]$.

It is easy to see that the final point function t is a morphism of local groups. So we have a crossed module with the function t: $(\pi_1 L)^e \to$ L, $[\alpha] \to t[\alpha] = t(\alpha)$.

References

[1] Akız, H.F., Covering Morphisms of Local Topological Group-Groupoids, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, (2017)pp.1- 4.

[2] Akız, H.F., Alemdar, N., Mucuk O., Şahan, T. , Coverings Of Internal Groupoids And Crossed Modules In The Category Of Groups With Operations, Georgian Mathematical Journal, vol.20 (2013) pp.223-238.

[3] Mucuk, O., Akız, H.F., Monodromy Groupoid of an Internal Groupoid in Topological Groups with Operations, FILOMAT, vol.29 (2015) pp.2355-2366.

[4] Pontrjagin, L., Topological Groups, Princeton University Press (1946).

[5] Taylor, R. L., Covering Groups of Non-connected Topological Groups , Proc. Amer. Math. Soc., 5, 753-768 (1954).

[6] Montgomery, D. and Zippin, L., Topological Transformation Groups, Interscience, New York (1955).

[7] Bourbaki, N., Elements of Mathematics, General Topolgy, 1, Addison-Wesley , London (1966).

[8] Brown, R., Topology and groupoids, Booksurge PLC (2006).

[9] Brown, R. and Mucuk, O., Covering groups of non-connected topological groups revisited, Math. Proc. Camb. Phill. Soc. 115 (1994) 97-110.

[10] R. Brown and O. Mucuk, The monodromy groupoid of a Lie groupoid, Cah. Top. G´eom. Diff. Cat. 36 (1995) 345-370.

[11] Brown, R. and Spencer C. B., G-groupoids, crossed modules and the fundamental groupoid of a topological group, Proc. Konn. Ned. Akad. v. Wet. 79 (1976) 296-302.

[12] Brown R. and Hardy J. P. L., Topological Groupoids I: Universal Constructions, Math. Nachr. 71 (1976) 273-286.

[13] Brown R., Danesh-Naruie, G. and Hardy J. P. L., Topological groupoids II: Covering morphisms and G-spaces, Math. Nachr. 74 (1976) 143-156.

[14] Brown R., Danesh-Naruie, G. , The Fundamental Groupoid as a Topological Groupoid, Proc. Edinb. Math. Soc., Vol. 19, (series 2), Part 3, 237-244. 1975.

[15] Chevalley, C., Theory of Lie groups, Princeton University Press, 1946.

[16] Hardy J. P., Topological Groupoids:Coverings and Universal Constractions, University Colloge of North. (1974).

[17] K.C.H. Mackenzie, Lie groupoids and Lie algebroids in differential geometry, London Math. Soc. Lecture Note Series 124, Cambridge University Press, 1987.

[18] Danilov , V.I., Monodromy, in Hazewinkel, M., Encyclopaedia of Mathematics, Springer, (2001). [19] Mucuk, O., Kılıçarslan, B., Şahan, T. and Alemdar, N., Group-groupoid and monodromy groupoid, Topology and its Applications, 2011.

[20] Mucuk, O. and Şahan, T., Group-groupoid actions and liftings of crossed modules, Georgian Mathematical Journal, DOI 10.1515/gmj-2018.0001.

[21] Brown, R., İçen, İ., and Mucuk, O., Holonomy and monodromy groupoids, in Lie Algebroids, Banach Center Publications Institute of Mathematics, Polish Academy of Sciences, Warsaw, 54 (2001) 9-20.

[22] İçen, İ., Özcan, A. F. and Gürsoy, M.H. , Topological group-groupoids and their coverings, Indian Journal of Pure and Applied Mathematics, 36 (2005) 493-502. 7

[23] Mucuk, O., Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid, PhD Thesis, University of Wales, 1993.

[24] Brown, R., em Higher dimensional group theory, in Low Dimensional Topology, London Math. Soc. Lect. Notes, 48, Cambridge Univ. Press, 1982, 215-238.

[25] Olver, P.J., Non-associatibe local Lie groups, J. Lie Theory 6 (1996), 23-51.

[26] Mucuk, O., Ay, H.Y. and Kılıçarslan, B., Local group-groupoids, ˙Istanbul University Science Faculty the Journal of Mathematics, 97 (2008).

[27] Mucuk O., Star Topological Groupoids, Turkish Journal of Mathematics,, vol.20, pp.227-232, 1996.

[28] Porter, T., Extensions, crossed modules and internal categories in categories of groups with operations, Proc. Edinb. Math. Soc. 30 (1987) 373-381.

[29] Whitehead, J. H. C., Note on a previous paper entitled “On adding relations to homotopy group”, Ann. of Math., t. 47, (1946) 806-810.

[30] Whitehead, J. H. C., Combinatorial homotopy II, Bull. Amer. Math. Soc. t. 55 (1949) 453-496 8.