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On Low Dimensional Leibniz Algebras

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Abstract
Leibniz algebras are generalization of Lie algebras. In literature, there are many studies on one dimensional and two dimensional Leibniz algebras. The structure of three dimensional Leibniz algebras are more complicated than the structure of one dimensional and two dimensional Leibniz algebras. In this study, our main aim is to investigate three dimensional non-Lie Leibniz algebras. Moreover, we prove that for any three dimensional non-Lie Leibniz algebra L, there exists at least one Leibniz algebra which is isomorphic to L.

Keywords: Leibniz algebra, Lie algebra, Leibniz kernel

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1. Introduction

Leibniz algebras have been a substantial topic of research since the 1960's. Leibniz algebras were first seen in the papers of A.M. Bloh [Bloh 1965, 1971] and they were called D-algebras. Then [Loday 1993] J.L. Loday rediscovered these algebras and he called them Leibniz algebras. There are many results on Leibniz algebras analogous to results on Lie algebras. For instance, the well-known Lie's Theorem, Engel's Theorem, Cartan's Criterion and Levi's Theorem on Lie algebras [Jacobson, 1979]. However, some of these results are proved for left Leibniz algebras and some of them are proved for right Leibniz algebras in literature. This paper is organized as follows. In Section 2 we define the basic definitions for Leibniz algebras and in Section 3 we investigate the structure of one dimensional and two dimensional Leibniz algebras. Then in Section 4 we focus on three dimensional non-Lie Leibniz algebras and we prove that if L is a three dimensional non-Lie Leibniz algebra, then there exists at least one Leibniz algebra which is isomorphic to L.

2. Preliminary

In this section we begin by setting up some definitions and notations for Leibniz algebras that will be needed in the sequal. A useful reference for more details is [Loday 1993]. Let L and R be algebras over a field F with binary operations + and [,]. L is called a left Leibniz algebra if it satisfies the Leibniz identity

$$[[x,y],z]=[x,[y,z]]-[y,[x,z]]$$

for all $x,y,z\in L$ and R is said to be a right Leibniz algebra if it satisfies the Leibniz identity

$$[x,[y,z]]=[[x,y],z]-[[x,z],y]$$

for all $x,y,x\in R$. Note that the classifications of left and right Leibniz algebras are different. Throughout of this paper, we prefer to work with left Leibniz algebra. It is possible to make a transfer from a left Leibniz algebra to a right Leibniz algebra or from a right Leibniz algebra to a left Leibniz algebra. An algebra L is said to be a symmetric Leibniz algebra if it is both a left and a right Leibniz algebra. Leibniz algebras are non-anticommutative generalization of Lie algebras. As an immediate consequence, every Lie algebras are Leibniz algebras. A Leibniz algebra L is a Lie algebra if and only if $[x, x] = 0$ for every element $x \in L$. A Leibniz algebra L is said to be abelian if $[x, y] = 0$ for all $x, y \in L$. Indeed, an abelian Leibniz algebra is a Lie algebra.

Since L is a vector space, for the subspaces X and Y of L, $[X, Y]$ will be a subspace generated by the elements $[x, y]$ where $x \in X$ and $y \in Y$. A subspace A is called a Leibniz subalgebra of L, if $[x, y] \in A$ for all $x, y \in A$. A subalgebra A is called a left (respectively right) ideal of L, if $[y, x] \in A$ (respectively $[x, y] \in A$) for all $x \in A$ and $y \in L$. If a subalgebra A is both a left and a right ideal, it is called an ideal of L, that is, $[x, y], [y, x] \in A$ for all $x \in A$ and $y \in L$. Then we can consider the cosets

$$x + A = \{x + a |\ a \in A\}$$

for $x \in L$ and a factor-algebra

$$L/A = \{x + A |\ x \in L\}$$

is also a Leibniz algebra.

We use the notation *Leib(L)* to denote the subspace generated by the elements $[x, x]$ for $x \in L$. This subspace is called the Leibniz kernel of L. For all $[x, x] \in Leib(L)$ and $y \in L$, we obtain

(2.1) $$\big[[x, x], y\big] = \big[x, [x, y]\big] - \big[x, [x, y]\big] = 0.$$

Hence $\big[[x, x], y\big] = 0$. Moreover, a straightforward calculation shows that

$$\begin{aligned}[[x, x] + y, [x, x] + y] &= [[x, x], [x, x]] + [[x, x], y] + [y, [x, x] + \big[y, [x, x]\big] + [y, y] \\ &= \big[y, [x, x]\big] + [y, y].\end{aligned}$$

Then, we obtain $\big[y, [x, x]\big] = \big[[x, x] + y, [x, x] + y\big] - [y, y]$. Since $\big[[x, x] + y, [x, x] + y\big], [y, y] \in Leib(L)$, we have $[y, [x, x]] \in Leib(L)$. This result implies that *Leib(L)* is an ideal of *L*. Furthermore, from (2.1), we infer that *Leib(L)* is an abelian Leibniz algebra.

Say *K* = *Leib(L)*. Then in factor-algebra $L/K = \{x + K |\ x \in L\}$, we have $x + K, x + K] = [x, x] + K = K$

for each element $x \in L$. This means that L/K is a Lie algebra. Let L_1 and L_2 be two Leibniz algebras over a field *F*. A map $\varphi: L_1 \to L_2$ is called a homomorphism if φ is a linear map and $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$. If φ is also bijective, we say that φ is an isomorphism. Let *L* be a Leibniz algebra. Define the composition chain of ideals

$$L^1 = L, L^2 = [L, L], \dots, L^{k+1} = [L^k, L]$$

for $k \geq 1$. Then the Leibniz algebra L is called a nilpotent Leibniz algebra if there exists a positive integer $k \geq 1$ such that $L^k = 0$. Now, we define the composition chains of ideals

$$L^{(0)} = L, L^{(1)} = \big[L^{(0)}, L^{(0)}\big], \dots, L^{(n+1)} = [L^{(n)}, L^{(n)}]$$

for $n \geq 1$. If for some positive integer $n \geq 1$, we have $L^{(n)} = 0$, the Leibniz algebra L is said to be a solvable Leibniz algebra. Furthermore information about nilpotent Leibniz algebra and solvable Leibniz algebra can be found in [Demir, Misra & Stitzinger 2014, Kurdachenko & Chupordia 2017].

3. Low dimensional Leibniz algebras

In this section firstly we observe one and two dimensional Leibniz algebras and then we give some of results studied on three dimensional non-Lie Leibniz algebras. Recall that a Leibniz algebra L is a finite dimensional, if the dimension of L as a vector space over a field *F* is finite. In literature, there are many studies of the structure of one dimensional and two dimensional Leibniz algebras [Demir, Misra & Stitzinger 2014, Kurdachenko & Chupordia 2017]. Suppose that L is a one dimensional Leibniz algebra over a field F. Then we have $L = Fx$ for some $x \in L$ and so $[x,x] = \alpha x$ where $\alpha \in F$. Therefore we have

$$0 = \big[[x,x],x\big] = [\alpha x, x] = \alpha[x,x] = \alpha^2 x.$$

This shows that $\alpha^2 = 0$, that is, $\alpha = 0$. In the light of this, it is obvious that $[x,x] = 0$. Hence, L is an abelian Leibniz algebra.

Now suppose that L is a two dimensional non-Lie Leibniz algebra. Then $K = Leib(L) \neq 0$ and so it is clear to see that there exists an element x in L such that $y = [x,x] \neq 0$. Thus, $L = Fx \oplus Fy$ and we have $[y,x] = 0$. The fact that K is an ideal of L implies that $[x,y] = \beta y$ for some $\beta \in F$. Assume that $\beta \neq 0$ and say $z = \beta^{-1}x$. Then $[z,y] = [\beta^{-1}x, y] = \beta^{-1}[x,y] = \beta^{-1}\beta y = y$. Therefore, we have

$$[z,z] = [\beta^{-1}x, \beta^{-1}x] = \beta^{-2}[x,x] = \beta^{-2}y = t$$

and

$$[z,t] = [z, \beta^{-2}y] = \beta^{-2}[z,y] = \beta^{-2}y = t.$$

By these choice, $\{z,t\}$ is a basis of L. Consequently, we obtain the following two non-isomorphic algebras $L_1 = Fx \oplus Fy$ with the products given by

$$[x,x] = y, [x,y] = \beta y, [y,y] = [y,x] = 0$$

and $L_2 = Fz \oplus Ft$ with the products given by

$$[z,z] = t, [z,t] = t, [t,t] = [t,z] = 0.$$

Suppose that $\beta = 0$, then we have $[x,y] = 0$ and so the Leibniz algebra L is nilpotent.

4. Main Result

The structure of three dimensional Leibniz algebras are more complicated than the structure of one dimensional and two dimensional Leibniz algebras. In [Demir, Misra & Stitzinger 2014], I. Demir, K.C. Misra and E. Stitzinger showed the existence of Leibniz algebras isomorphic to three dimensional nilpotent non-Lie Leibniz algebras. In this paper, this result is generated to any three dimensional non-Lie Leibniz algebra. Now by the following theorem we give our main result on three dimensional non-Lie Leibniz algebras.

Theorem 4.1. Let L be a non-Lie Leibniz algebra and dim$L$ = 3. Then there exists at least one Leibniz algebra which is isomorphic to L.

Proof. Let L be a three dimensional non-Lie Leibniz algebra. Afterwards, K = *Leib(L)* is non-zero. Since K is abelian, $K \neq L$. It follows that there exists an element a in L such that $b = [a,a] \neq 0$ for $a \notin K$. If $a \in K$, then since K is abelian, $[a,a] = 0$. Then we have $[b,b] = 0, [b,a] = 0$ and since K is an ideal of L, $[a,b] \in K$. Namely, this implies that $[a,b] = \beta b$ for some $\beta \in F$. Now we take an arbitrary element c in L. We have two cases :

Case 1: If $c \in K$, then we have $[c,c] = 0$ and $c = [d,d] \neq 0$ for $d \notin K$. Here, there are two possibilities: if $d = a$, then $c = [a,a] = b$. Therefore, we have

$$[a,b] = [a,c] = \beta b, [b,a] = [c,a] = 0, [c,c] = 0.$$

It follows that L is a two dimensional Leibniz algebra, this is a contradiction. If $d \neq a$, namely, $c \neq b$, then $L = Fa \oplus Fb \oplus Fc \oplus Fd$, a contradiction.

Case 2: If $c \notin K$, then $L = Fa \oplus Fb \oplus Fc$ and we have $[b,c] = \big[[a,a],c\big] = 0$. Since K is an ideal of L, $[c,c] \in K$, $[c,c] = \alpha b$ and $[c,b] \in K, [c,b] \in K, [c,b] = \gamma b$ for some $\alpha, \gamma \in F$.

$$[a,c] = \alpha_1 a + \alpha_2 b + \alpha_3 c, \qquad [c,a] = \beta_1 a + \beta_2 b + \beta_3 c.$$

Now we choose an element in L, say $d = \beta^{-1}a$. Suppose that $\beta \neq 0$. We compute

$[d,a] = [\beta^{-1}a, a] = \beta^{-1}$b=e,

$[a,d] = [\beta^{-1}a, b] = b,$

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$[a,d] = [a, \beta^{-1}a] = e,$
$[d,b] = [\beta^{-1}a, b] = b,$
$[b,d] = [b, \beta^{-1}a] = 0,$
$[d,c] = [\beta^{-1}a, c] = \beta^{-1}[a,c] = \beta^{-1}(\alpha_1 a + \alpha_2 b + \alpha_3 c) = \alpha_1 d + \beta^{-1}\alpha_2 b + \beta^{-1}\alpha_3 c,$
$[c,d] = [c, \beta^{-1}a] = \beta^{-1}[c,a] = \beta^{-1}(\beta_1 a + \beta_2 b + \beta_3 c) = \beta_1 d + \beta^{-1}\beta_2 b + \beta^{-1}\beta_3 c,$
$[d,d] = [\beta^{-1}a, \beta^{-1}a] = \beta^{-2}b = \beta^{-1}e.$

Say $e = \beta^{-1}b$, so we have

$$[a,e] = [a, \beta^{-1}[a,b] = \beta e,$$
$$[e,a] = [\beta^{-1}b, b] = 0,$$
$$[e,b] = [\beta^{-1}b, b] = 0,$$
$$[b,e] = [b, \beta^{-1}b] = 0,$$
$$[e,c] = [\beta^{-1}b, c] = 0,$$
$$[c,e] = [c, \beta^{-1}b] = \gamma e,$$
$$[d,e] = \beta^{-2}[a,b] = 0,$$
$$[e,d] = \beta^{-1}[b,d] = 0,$$
$$[e,e] = \beta^{-2}[b,b].$$

Now we say $f = \beta^{-1}c$, we have

$[a,f] = [a, \beta^{-1}c] = \alpha_1 d + \alpha_2 e + \alpha_3 f,$
$[a,f] = [a, \beta^{-1}c] = \beta^{-1}(\alpha_1 d + \alpha_2 e + \alpha_3 f),$
$[f,a] = \beta^{-1}[c,a] = \beta_1 d + \beta_2 e + \beta_3 f,$
$[b,f] = \beta^{-1}[b,c] = 0,$
$[f,b] = \beta^{-1}[c,b] = \alpha e,$
$[f,c] = \beta^{-1}[c,c] = \alpha e,$
$[d,f] = \beta^{-1}[d,c] = \beta^{-1}(\alpha_1 d + \alpha_2 e + \alpha_3 f),$
$[f,d] = \beta^{-1}[c,d] = \beta^{-1}(\beta_1 d + \beta_2 e + \beta_3 f),$
$[e,f] = \beta^{-2}[b,c] = 0,$
$[f,e] = \beta^{-2}[c,b] = \beta^{-1}\gamma e,$
$[f,f] = \beta^{-2}[c,c] = \beta^{-1}\alpha e.$

By this choice, $\{d,e,f\}$ is a basis of L. Therefore, we obtain the following two isomorphic algebras $L_1 = Fa \oplus Fb \oplus Fc$, with the products given by

$$[a,a] = b, [b,b] = 0, [c,c] = \alpha b,$$
$$[a,b] = \beta b, [c,b] = \gamma b, [a,c] = \alpha_1 a + \alpha_2 b + \alpha_3 c,$$
$$[b,a] = 0, [b,c] = 0, [c,a] = \beta_1 a + \beta_2 b + \beta_3 c,$$

and $L_2 = Fd \oplus Fe \oplus Ff$, $(d = \beta^{-1}a,\ e = \beta^{-1}b,\ f = \beta^{-1}c)$ with the products given by

$$[d,d] = \beta^{-1}e, [e,e] = 0, [f,f] = \beta^{-1}\alpha e,$$
$$[d,e] = e, [f,e] = \gamma\beta^{-1}e, [d,f] = \beta^{-1}(\alpha_1 d + \alpha_2 e + \alpha_3 f),$$
$$[d,e] = 0, [e,f] = 0, [f,d] = \beta^{-1}(\beta_1 d + \beta_2 e + \beta_3 f)$$

which completes the proof of the theorem.

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