 

# New Type Ultra-Banach Spaces

Ibrahim Sanlibaba (Corresponding author)
Institute of Sciences, Nevsehir Haci Bektas Veli University
50100, Nevşehir, Turkey
E-mail: ibrahimsanlibaba@gmail.com

**Abstract**
In this paper, we have given some definitions and theorems about ultra-metric, ultra-normed spaces which are shown in mathematical literature. Then, we have defined some new type ultra-Banach spaces and we have shown that these spaces are ultra-isomorphic and studied some interesting properties. Furthermore, some inclusion theorems are proved about these new type ultra-Banach spaces.

**Keywords:** Ultra-metric, ultra-norm, non-Archimedean space, ultra-convergent.

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## 1. Introduction and Background

One of the important branch as of the normed space theory is ultra-normed spaces. In general, we know that ordinary ultra-metric spaces are special kind of the metric spaces and one can obtain an ultra-normed space from the ultra-metric space. It can be seen that, in literature, the concept of ultra-metric is used in different areas of science, for instance, Flagg-Kopperman [1], Lemin [2], Priess-Crampe-Ribenboim [3] and Rammal [4]. Recently, Diagana has obtained some new type ultra-normed spaces and given some interesting properties of them [8-10].

In this paper, in the light of developments mentioned above, we have investigated a new type ultra-Banach space. Now we will give some preliminary and definitions.
Let $X$ be a linear space, real valued function $\tilde{d}: X\text{x}X \to \mathbb{R}^+$ and $\tilde{d}$ satisfies the following conditions:

$(d_1)$ $\tilde{d}(x,y) = 0$ if and only if $x = y$,

$(d_2)$ $\tilde{d}(x,y) = \tilde{d}(y,x)$,

$(d_3)$ $\tilde{d}(x,y) \leq \max\{ \tilde{d}(x,z), \tilde{d}(z,y)\}$

Then the function $\tilde{d}$ is called ultra-metric or super-metric on the set $X$. It is clear that, if the expression $\tilde{d}(x,y) \leq \max\{\tilde{d}(x,z), \tilde{d}(z,y)\}$ is held then $\tilde{d}(x,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y)$ also held but vice versa is not true. The conditions $d_1$-$d_3$ are called ultra-metric conditions and the couple $(X,\tilde{d})$ is called ultra-metric or super-metric space. From here, we can say that every ultra-metric space is a metric space but vice versa is not true. Let $F$ be a field. Let us consider function $|.|: F \to \mathbb{R}$ is satisfying following conditions:

$(a_1)$ $|x| \geq 0$ ; $|x| = 0$ if and only if $x = 0$

$(a_2)$ $|xy| = |x||y|$

$(a_3)$ $|x + y| \leq |x| + |y|$

then the function $|.|$ is called absolute value function and $|x|$ is an absolute value of $x \in F$. In this case, the couple $(F, |.|)$ is called Archimedean field [8].

If we take

$(a_3)'$ $|x+y| \leq \max\{|x|, |y|\}$

instead of $(a_3)$ then the function $|.|$ is called non-Archimedean absolute value. The condition $(a_3)'$ is called strong triangle inequality or ultra-metric inequality**.** In this case, the field $F$ is also called as a non-Archimedean field. From here, the non-Archimedean absolute value we will be denoted with $\langle|.|\rangle$. A simple examples of Archimedean fields are $(\mathbb{R}, |.|)$ and $(\mathbb{C}, |.|)$, where the function $|.|$ does not provide the condition $(a_3)'$.

**Example 1.1.** Let $F$ be a field and consider the function

$$\langle|x|\rangle = \begin{cases} 1 & , x \neq 0 \\ 0 & , x = 0 \end{cases} \tag{1.1}$$

defined from $F \to \mathbb{R}^+$.

In this case the function $\langle|.|\rangle$ satisfies the conditions $(a_1)$, $(a_2)$ and $(a_3)'$ so, the pair $(F, \langle| \quad |\rangle)$ is called trivial non-Archimedean field.

**Example 1.2.** Let us consider function $d$: $\mathbb{R}$x$\mathbb{R} \to \mathbb{R}^+$, defined as $d(x,y) = |x-y|$, where the notation $|.|$ is absolute value function. Clearly, $d$ satisfies axioms of ordinary metric space. But the function $d$ does not satisfy the axiom $(a_3)'$, therefore the couple $(\mathbb{R}, d)$ is not an ultra-metric space.

Let $X$ be a non empty set and consider function $\tilde{d}$: $X$x$X \to \mathbb{R}^+$ defined by

$$\tilde{d}(x,y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases} \tag{1.2}$$

Then the function $\tilde{d}$ satisfies the conditions of ultra-metric, so $X$ is an ultra-metric space [9], [10]. Thus we can suggest that " Every discrete metric space is an ultra-metric space" . Let $X$ be a linear space $F$ be a field and $x, y \in X$, $\alpha \in F$. If a single argument non negative real valued function $u$ satisfies the following conditions then $u$ is a called ultra-norm.

$(n_1')$ $u(x) = 0 \Leftrightarrow x = \theta$,

$(n_2')$ $u(\alpha x) = |\alpha|\, u(x)$,

$(n_3')$ $u(x+y) \leq \max\{u(x), u(y)\}$

It is clear that, if the inequality $u(x+y) \leq \max\{u(x), u(y)\}$ is held then the $u(x+y) \leq u(x) + u(y)$ is held but vice versa is not true. Let $u$ be an ultra-norm on $X$ and for all $x, y \in X$ we define function $\tilde{d}(x,y) = u(x-y)$ from $X$ x $X$ to $\mathbb{R}$. Then the function $\tilde{d}$ satisfies ultra-metric conditions [16]. This shows that every ultra-normed space are ultra-metric space.

**Definition 1.1** Let us suppose that $(X, u)$ be an ultra-normed space and $(x_k)$ be a sequence $X$.

(i) If for all $\varepsilon > 0$ there exists a $k_0$ positive integer such that $u(x_k - x_0) < \varepsilon$ for all $k \geq k_0$, then $(x_k)$ is called ultra-convergent (or super-convergent) to $x_0$ and denoted by $x_k \xrightarrow{u} x_0,$ $k \to \infty$. If $x_0$ equal to zero then $(x_k)$ is called ultra-null(super-null) sequence.

(ii) The sequence $(x_k)$ is called ultra-Cauchy (or super-Cauchy), if for all $\varepsilon > 0$ there exists a $k_0$ positive integer such that $u(x_k - x_i) < \varepsilon$ for all $k, i \geq k_0$.

(iii) The sequence $(x_k)$ is called ultra-bounded (or super-bounded), if $u(x_k) \leq K$ for $K \geq 0$.

(iv) Let $(X, u)$ be ultra-normed space. If every ultra-Cauchy sequence in $(X, u)$ ultra-convergent to $x_0 \in X$ then $X$ called ultracomplete(or ultraBanach) space, [17].

Let $F$ be a non-Archimedian field. In this case set $w = \{(x_k): f: IN \to F, f(k) = x_k\}$ is called the sets

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of all sequences. The sets of ultra-bounded, ultra-convergent and ultra-null sequences are denoted by $l_\infty(F)$, $c(F)$ and $c_0(F)$, respectively and defined as follow,

$$l_\infty(F) = \left\{x = (x_k) \in w: \sup_{k\in\mathbb{N}} \langle |x_k| \rangle < \infty \right\},$$
$$c(F) = \left\{x = (x_k) \in w: x_k \overset{u}{\to} x_{0,}\ k \to \infty,\ x_0 \in F \right\},$$
$$c_0(F) = \left\{x = (x_k) \in w: x_k \overset{u}{\to} 0\ k \to \infty \right\}.$$

This shows that the spaces $l_\infty(F), c(F)$ and $c_0(F)$ are ultra-Banach spaces with the norm defined by

$$u_{l_\infty(F)}(x) = u_{c(F)}(x) = u_{c_0(F)}(x) = \sup_k \langle |x_k| \rangle,$$

where $|.|$ is satisfies the property $(a_3)'$. Some new type ultra-normed sequence spaces defined by Diagana [8-10] as follows:

$$l_\infty(F,\rho) = \left\{x = (x_k) \in w: \sup_{k\in\mathbb{N}} \langle |x_k| \rangle \rho_k < \infty \right\},$$
$$c(F,\rho) = \left\{x = (x_k) \in w: \lim_k \langle |x_k| \rangle \rho_k \text{exists} \right\},$$
$$c_0(F,\rho) = \left\{x = (x_k) \in w: \lim_k \langle |x_k| \rangle \rho_k = 0 \right\}$$

and these are ultra-Banachspaces with defined ultra-norm

$$u_{l_\infty(F,\rho)}(x) = u_{c(F,\rho)}(x) = u_{c_0(F,\rho)}(x) = \sup_k \langle |x_k| \rangle \rho_k.$$

since every ultra-normed space is a normed space, if take $\rho_k = 1$ for all $k$, then the spaces $l_\infty(F,\rho)$, $c(F,\rho)$ and $c_0(F,\rho)$ are reduced to, $l_\infty(F)$, $c(F)$ and $c_0(F)$ respectively. In other words, the spaces $l_\infty(F,\rho)$, $c(F,\rho)$ and $c_0(F,\rho)$ are large than the spaces $l_\infty(F)$, $c(F)$, $c_0(F)$, $l_\infty$, $c$ and $c_0$, where $l_\infty$, $c$ and $c_0$ denotes ordinary bounded, convergent and null sequence spaces in Achimedian field, respectively. Now, we will give a new definition.

**Definition 1.2** (Ultra-isometry) Let $X$ and $Y$ be vector space on non Archimedian field $F$ and the mapping $T: X \to Y$ be given. If, for all $x \in X$, $u(Tx)_Y = u(x)_X$ then the map $T$ is called ultra-isometry from $X$ to $Y$. In this case, the spaces $X$ and $Y$ are called as ultra-isometric spaces. In addition, if the mapping $T$ is bijective, then the spaces $X$ and $Y$ are called as ultra-isomorphic spaces and denoted by $X \cong Y$ [17].

## 2. Zweier Type Ultra-normed Spaces

Each linear subspace of $w$ is called a sequence space. Let $\lambda$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ (n, k = 0,1, 2, ...) be an infinite matrix of real or complex numbers $a_{nk}$. Then, we can say that $A$ defines a matrix mapping from $\lambda$ to $\mu$, and we denote it by writing $A: \lambda \to \mu$ if for every sequence $(x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$ the $A$- transform of $x$ is in $\mu$ where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{2.1}$$

By $(\lambda: \mu)$, we denote the class of matrices $A$ such that $A: \lambda \to \mu$ Thus, $A \in (\lambda : \mu)$, if and only if the series on the right side of (2.1) converges for each positive integer $n$ and every $(x_k) \in \lambda$ we have $Ax = \{(Ax)_n\} \in \mu$ for all $(x_k) \in \lambda$ .

The matrix domain $\lambda_A$ of an infinite matrix $A$ in a sequence space $\lambda$ is defined by

$$\lambda_A = \{x = (x_k) \in w: Ax \in \lambda\} \tag{2.2}$$

If we take $\lambda = c$ then $c_A$ is called convergence domain of $A$.

By using of the matrix domain of a particular limitation method so many sequences spaces have been

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built and published in many maths journals. By reviewing the literature, one can reach them easily ( for instance, see Altay and Başar [11-13], Kirişçi and Başar [14], Şengönül and Kayaduman [5], Şengönül [6]. Finally, the new technique for deducing certain topological properties, such as AB-, KB-, AD-properties, solidity and monotonicity etc., and determining the $\alpha-, \beta-$$ and $\gamma-$duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [11], [12]. [13].

**Definition 2.1.** Let $n, k \in \mathbb{N}$ and consider infinite matrix $Z = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} p, & n = k \\ 1-p, & n-1 = k \\ 0, & \text{others} \end{cases}, p \in \mathbb{R} - \{-1\}.$$

The infinite matrix $Z = (z_{nk})$ is called Zweier matrix, [9]. Let $x = (x_k)$ be a sequence and $Z$-transform of $x = (x_k)$ is defined as follows [6]:

$$y_k = (Zx)_k = px_k + (1-p)x_{k-1} \tag{2.3}$$

Now we will define some new type ultra-normed sequence spaces as follows:

$$\tilde{c}_0(Z, F) = \left\{x: px_k + (1-p)x_{k-1} \xrightarrow{u} x_0, k \to \infty\right\},$$
$$\tilde{c}(Z, F) = \left\{x: px_k + (1-p)x_{k-1} \xrightarrow{u} x_0, k \to \infty, x_0 \in F\right\},$$
$$\tilde{l}_\infty(Z, F) = \left\{x: \sup_k \langle |px_k + (1-p)x_{k-1}| \rangle < \infty\right\}.$$

It is clear that the sets $\tilde{c_0}(Z, F)$, $\tilde{c}(Z, F)$ and $\tilde{l}_\infty(Z, F)$ are defined with $Zx$ transforms in $c_0(F)$, $c(F)$ and $l_\infty(F)$, respectively. And these spaces are ultra (super) normed spaces defined by ultra (super) norm;

$$u_{\tilde{l}_\infty(Z,F)}(x) = \sup_k \langle |px_k + (1-p)x_{k-1}| \rangle \tag{2.4}$$

If we write $\tilde{\lambda}(Z, F) \in \{\tilde{c_0}(Z, F), \tilde{c}(Z, F), \tilde{l}_\infty(Z, F)\}$ and $\lambda(F) \in \{c_0(F), c(F), l_\infty(F)\}$ then we can give a theorem as follows:

**Theorem 2.2.** Let us consider transformation

$$\mathrm{Z}: \tilde{\lambda}(Z, F) \to \lambda(F),$$
$$x \to \mathrm{Zx} = \mathrm{y}, \quad \mathrm{y} = \mathrm{y}_k, \mathrm{y}_k = px_k + (1-p)x_{k-1}.$$

In this case, the map Z is ultra-isomorphism and the spaces $\tilde{\lambda}(Z, F)$ and $\lambda(F)$ are also ultra-isometric.

**Proof.** Let us suppose that $x, y \in \tilde{\lambda}(Z, F)$ and $\alpha \in F$. It is clear that $Z$ is linear. Furthermore, if $\mathrm{Zx} = \theta$, so $px_k + (1-p)x_{k-1} = \theta$ then $Z^{-1}(\mathrm{Zx}) = Z^{-1}\theta$ and we obtain that $x = \theta$. This result shows to us the transformation Z is one to one from $\tilde{\lambda}(F)$ to $\lambda(F)$. Since every element of the space $\tilde{\lambda}(F)$ is obtainied from transform of $x \in \tilde{\lambda}(F)$, it is clear that Z is onto. Secondly, we must show that the transform Z preserves ultra-norm between the spaces $\tilde{\lambda}(F)$ and $\lambda(F)$. For this, let us define the sequence $x_k = \sum_{j=0}^{k} (-1)^{k-j} \frac{(1-p)^{k-j}}{p^{k-j+1}} y_j$ , [15,17].

$$u(x) = \sup_k \langle |px_k + (1-p)x_{k-1}| \rangle$$
$$= \sup_k \left\langle \left| p \sum_{j=0}^{k} (-1)^{k-j} \frac{(1-p)^{k-j}}{p^{k-j+1}} y_j + (1-p) \sum_{j=0}^{k} (-1)^{k-j} \frac{(1-p)^{k-j}}{p^{k-j+1}} y_j \right| \right\rangle$$
$$= \sup_k \langle |y_k| \rangle = \sup_k \langle |Zx| \rangle = u(\mathrm{Zx}).$$

Then here the proof ends.

**Theorem 2.3.** The spaces $\tilde{c_0}(Z,F)$, $\tilde{c}(Z,F)$, $\tilde{l}_\infty(Z,F)$ are ultra-Banach spaces defined by ultra-norm (2.4).

**Proof.** Since proofs are similar, we have only considered $\tilde{l}_\infty(Z,F)$. It is clear that ultra-norm defined by (2.4) satisfies the conditions $(n_1')$ , $(n_2')$ and $(n_3')$ .

Let suppose that $(x^n)$ be ultra-Cauchy in $\tilde{l}_\infty(Z,F)$. In this case, we see that there exists a positive integer $n_0$ such that

$$u_{\tilde{l}_\infty(Z,F)}({x_k}^m - {x_k}^n) = \sup_k \langle |(p{x^m}_k + (1-p){x^m}_{k-1}) - (p{x^n}_k + (1-p){x^n}_{k-1})| \rangle$$
$$= u_{l_\infty(Z,F,\rho)}(Z({x_k}^m - {x_k}^n)) < \varepsilon \qquad \text{for} \qquad m,n \geq n_0.$$

From her we can write

$$\langle |(p{x^m}_k + (1-p){x^m}_{k-1}) - (p{x^n}_k + (1-p){x^n}_{k-1})| \rangle = \langle |(Z({x_k}^m - {x_k}^n))\ | \rangle < \varepsilon.$$

This shows to us, the sequence $(Z({x_k}^m - {x_k}^n))$ is a ultra-Cauchy sequence in $l_\infty(F)$. If we take into account $l_\infty(F)$ is ultra-complete and the space $l_\infty(F)$and $\tilde{l}_\infty(Z,F)$ are linearly ultra-isometric, then we can easily claim that every ultra-Cauchy sequence in $\tilde{l}_\infty(Z,F)$ is convergent. Let us suppose that $x_k^n \overset{u}{\rightarrow} x_k$, $n \to \infty$, $k = 0,1,2,\dots$ . Now we will show that the sequence $x = (x_k) \in \tilde{l}_\infty(Z,F)$. We know that from Theorem 2.1, the mapping $Z$ is bijective from $\tilde{l}_\infty(Z,F)$ to $l_\infty(F)$. It is deduce that $x = (x_k) \in \tilde{l}_\infty(Z,F)$.

Consequently the space $(\tilde{l}_\infty(Z,F),u)$ is a ultra-Banach space.

**Theorem 2.4.** The inclusions

$(i_1)$ $\tilde{l}_\infty(Z,F) \subseteq l_\infty(F)$

$(i_2)$ $\tilde{c_0}(Z,F) \subseteq \tilde{c}(Z,F) \subseteq \tilde{l}_\infty(Z,F)$

$(i_3)$ $c_0 \subseteq \tilde{c_0}(Z,F)$ and $c \subseteq \tilde{c}(Z,F)$ are hold.

**Proof.** The proof of the $(i_2)$ and $(i_3)$ are clear so we will give a proof for only $(i_1)$.

$(i_1)$ If $x \in \tilde{l}_\infty(Z,F)$, then

$$u(x)_{\tilde{l}_\infty(Z,F)} = \sup_k \langle |px_k + (1-p)x_{k-1}| \rangle \leq \sup_k \{\max\{|px_k|, |(1-p)x_{k-1}|\}\}$$
$$= K \sup_k \{\max\{|x_k|, |x_{k-1}|\}\} = K u_{l_\infty(Z,F)}(x)$$

where $K = \max\{|p|, |1-p|\}$. This shows to us $x \in l_\infty(F)$. Similarly, we can easily prove that the inclusions $\tilde{c}(Z,F) \subseteq c(F)$ and $\tilde{c_0}(Z,F) \subseteq c_0(F)$ are hold.

## 3. Open Problems

**1**- Lets us suppose that $A = (a_{nk})$ be an infinite matrix and x= $(x_k)$ be sequence in $l_\infty(F)$ (or $c(F)$, $c_0(F)$). Then when does $(Ax) \in l_\infty(F)$ for all x= $(x_k) \in l_\infty(F)$? And other classes?

**2**-Similarly to the open problem 1, when does $(Ax) \in \tilde{l}_\infty(Z,F)$ for all $x = (x_k) \in \tilde{l}_\infty(Z,F)$?

**3**- If λ is super-Banach space then the sets

$$\lambda^\alpha = \{a = (a_k): ax \in cs(F) \text{ for all } x \in \lambda\}$$
$$\lambda^\beta = \{a = (a_k): ax \in bs(F) \text{ for all } x \in \lambda\}$$
$$\lambda^\gamma = \{a = (a_k): ax \in l(F) \text{ for all } x \in \lambda\}$$

are called $\alpha-$, $\beta-$ , and $\gamma-$duals of the super-Banach space $\lambda$.

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In this case, what is the $\alpha-, \beta-,$ and $\gamma-$duals of the spaces $l_\infty(F), c(F), c_0(F)$ other spaces?

## Conclusions

In this paper, we have obtained a new type ultra-Banach space is called Zweier ultra-Banach space and investigated some inclusisons theorems.

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