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Analysis of Accuracy, Stability, Consistency and Convergence of an Explicit Iterative Algorithm

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Abstract

In this work, an analysis is carried out vis-à-vis an explicit iterative algorithm proposed by Qureshi *et al* (2013) for initial value problems in ordinary differential equations. The algorithm was constructed using the well – known Forward Euler's method and its variants. Discussion carries with it an investigation for stability, consistency and convergence of the proposed algorithm-properties essential for an iterative algorithm to be of any use. The proposed algorithm is found to be second order accurate, consistent, stable and convergent. The regions and intervals of absolute stability for Forward Euler method and its variants have also been compared with that of the proposed algorithm. Numerical implementations have been carried out using MATLAB version 8.1 (R2013a) in double precision arithmetic. Further, the computation of approximate solutions, absolute and maximum global errors provided in accompanying figures and tables reveal equivalency of the algorithm to other second order algorithms taken from the literature.

Keywords: Iterative Algorithm, Ordinary Differential Equations, Accuracy, Consistency, Convergence.

1. Introduction

Ordinary Differential Equations (ODEs) are ubiquitous whether it be Mathematics (Computational and Applied), Physics (Newton's second law and harmonic oscillations), Chemistry (chemical kinetics and chaos), Ecology (logistic and unlimited population growth), Weather Forecasting (Lorenz Model), Economics (Ramsey – Cass – Koopmans Model and Competitive Equilibrium Model), and Romance (Dynamical Love Model: The Romeo and Juliet Scenario); as discussed by number of scholars such as Strogatz (1994), Blanchard *et al* (2012), Sunday and Odekunle (2012), Sunday *et al* (2012), Obayomi and Olabode (2013), Qureshi *et al* (2013), Soomro *et al* (2013); and Jia and Sogabe (2013). Considerably, large class of linear models can easily be handled with analytical means whereas nonlinear models in terms of either scalar or vector ODEs have always posed new challenges for researchers working in various scientific domains as these models do not possess solutions in the form of elementary mathematical functions as discussed by Akanbi (2010), Chandio and Memon (2010), Soomro *et al* (2013); and Chapra and Canale (2010). It is because of this reason that researchers are engrossed – to date – for devising either new algorithms or otherwise improve the existing ones (explicit or implicit) by reducing the number of function evaluations per step, increasing order of accuracy and expanding regions of absolute stability. Significant contributions in the form of textbooks and scientific work came from almost every corner of the globe but no unique algorithm could be agreed upon (Palais and Robert, 2009; Chandio and Memon, 2010). Standard explicit iterative algorithms have been improved by OCHOCHE (2008), Chandio and Memon, (2010), Rabiei and Ismail (2011), Rabiei and Ismail (2012), Rabiei *et al* (2013); just to mention a few; whereas (Fatunla, 1976), (Ogunrinde and Fadugba, 2012), and (Ramos, 2007) are few members of a huge family of researchers who have proposed nonstandard algorithms to solve first order initial value problems (IVPs) in ODEs. Whether it be an improved version of some standard algorithm or a newly constructed one; in either case the algorithm so obtained is required to satisfy crucial and fundamental characteristics. These characteristics involve accuracy, convergence, and stability of the algorithm. For further details, see the work of Ma (2010), Ma (2010), Odekunle and Sunday (2012), Sunday *et al* (2012), and Rabiei *et al* (2013).

For working on the present paper, motivation ascended from the well acknowledged explicit first order Euler method and its variants. Though, explicit algorithms do not offer reasonable solutions when it comes to stiff problems but they are extensively being employed because of their simplicity and easy implementation as a computer code as evident from the remarks given by Hassan *et al* (2006), Edwards and Penny (2008), and Zill (2009). An explicit iterative algorithm is constructed by Qureshi *et al* (2013) to solve first order IVPs in ODEs of the form (1) and (2), as given below:

$$\left.\frac{dy}{dt}=f(t,y),\quad y(t_0)=y_0\right\} \tag{1}$$

and

$$\left.\begin{aligned}\frac{dy_1}{dt}&=f_1(t,y_1,y_2,\cdots,y_n),\quad y_1(t_0)=y_1\\ \frac{dy_2}{dt}&=f_2(t,y_1,y_2,\cdots,y_n),\quad y_2(t_0)=y_2\\ &\vdots\qquad\qquad\qquad\qquad\quad\vdots\\ &\vdots\qquad\qquad\qquad\qquad\quad\vdots\\ \frac{dy_n}{dt}&=f_n(t,y_1,y_2,\cdots,y_n),\quad y_n(t_0)=y_n\end{aligned}\right\} \tag{2}$$

An explicit iterative algorithm constructed by Qureshi *et al* (2013) is:

$$y_{i+1}=y_i+\Delta t\, f\left(t_i+\frac{\Delta t}{2},y_i+\frac{\Delta t}{2}\left(\frac{f(t_i,y_i)+f(t_{i+1},y_i+\Delta t\, f(t_i,y_i))}{2}\right)\right) \tag{3}$$

for $i=0,1,2,\cdots,n$.

2. Materials and Methods

2.1 Preliminaries

Given below are some key theorems and definitions needed to support rest of the discussion.

Theorem 01. A function $f(t,y)$ is said to satisfy *Lipschitz condition* in the variable y on a convex set $R=\{(t,y)|t_0\le t\le b,-\infty<y<\infty\}$ if a fixed constant $L>0$ exists such that

$$|f(t,y)-f(t,y^*)|\le L|y-y^*|$$

whenever (t,y) and $(t,y^*)\in R$. The fixed constant L is known as *Lipschitz constant* for $f(t,y)$.

Theorem 02. If $f(t,y)$ satisfies Lipschitz condition and is continuous on the convex set R then the IVP (1) will have a *unique* solution $y(t)$ on $[t_0,b]$.

Definition 01. An iterative algorithm is said to be *one – step* if it is of the form:

$$y_{i+1}=y_i+h\phi(t_i,y_i,y_{i+1};h)$$

making the algorithm explicit in the absence of the term y_{i+1} on the right hand side otherwise implicit, where ϕ is called incremental function.

Definition 02. The *order* of an explicit one – step iterative algorithm is the largest integer $p\ge 1$ such that

$$y(t_{i+1})-[y_i+h\phi(t_i,y_i;h)]=O(h^{p+1})$$

where $y(t_{i+1})$ is the exact solution to (1).

Definition 03. An iterative algorithm is said to be *convergent* if the numerical solution y_i (with round-off errors assumed to be zero) approaches the exact solution $y(t_i)$ as the step size approaches zero. Symbolically,

$$\max_{0\le i\le N}\left\|y(t_i)-y_i\right\|\to 0 \text{ as } \Delta t\to 0$$

What follows next is the investigation for order of accuracy, stability, consistency, and convergence of the proposed algorithm (3).

2.2 Order of Accuracy of the Proposed Algorithm

To determine order of accuracy of the proposed algorithm (3); it has been expanded using Taylor's theorem and equated to Taylor series expansion of the exact solution $y(t_{i+1})$. After some algebraic simplification, expansion of (3) turns out to be equal to the expansion of $y(t_{i+1})$ up to the term containing $(\Delta t)^2$ declaring the proposed algorithm to be second order accurate. Taylor series expansion of $y(t_{i+1})$ is given by:

$$y(t_{i+1})=y(t_i)+\Delta t\, y'(t_i)+\frac{(\Delta t)^2}{2!}y''(t_i)+\frac{(\Delta t)^3}{3!}y'''(t_i)+\cdots$$

Or, $$y_{i+1}=y_i+\Delta t\, f(t_i,y_i)+\frac{(\Delta t)^2}{2!}f'(t_i,y_i)+\frac{(\Delta t)^3}{3!}f''(t_i,y_i)+O(\Delta t)^4 \tag{4}$$

It is also known that:

$$f'(t,y(t))=f'=y''(t)=f_t+f_y(y'(t))=f_t+ff_y=F \text{ (say)} \tag{5}$$

$$f''(t,y(t))=f''=y'''(t)=f_{tt}+2ff_{ty}+f^2f_{yy}+f_y(f_t+ff_y)=G+f_yF \quad \text{(say)} \tag{6}$$

Hence, (4) can be rewritten as

$$y_{i+1}=y_i+\Delta tf+\frac{(\Delta t)^2}{2}F+\frac{(\Delta t)^3}{6}(G+f_yF)+O(\Delta t)^4 \tag{7}$$

Now, the second inner function of the proposed algorithm (3) is expanded as:

$$f(t_i+\Delta t,y_i+\Delta t\, f)=f+\Delta tf_t+\Delta tff_y+\frac{1}{2}\left[(\Delta t)^2 f_{tt}+2(\Delta t)^2 ff_{ty}+f^2f_{yy}\right]$$

$$=f+\Delta tF+\frac{(\Delta t)^2}{2}G$$

Substituting it into (3), we obtain

$$y_{i+1}=y_i+\Delta tf\left(t_i+\frac{\Delta t}{2},y_i+\frac{\Delta t}{4}\left(2f+\Delta tF+\frac{(\Delta t)^2}{2}G\right)\right)$$

Taylor expansion further gives:

$$y_{i+1}=y_i+\Delta tf+\frac{(\Delta t)^2}{2}F+\frac{(\Delta t)^3}{4}f_yF+\frac{(\Delta t)^3}{8}G \tag{8}$$

Comparison of (7) and (8) reveals that the right hand side terms are agree with each other up to the term containing $(\Delta t)^2$. Thus the proposed algorithm (3) is of second order accuracy; that is, local truncation error is $O(\Delta t)^3$.

2.3 Consistency of the Proposed Algorithm

As reported in Fatunla (1988), an iterative algorithm is said to be *consistent* if its incremental function with step size approaching zero, agrees with the IVP (1), that is,

$$\lim_{\Delta t \to 0} \phi(t_i, y_i; \Delta t) = f(t_i, y_i) \tag{9}$$

From (3), we have

$$\phi(t_i, y_i; \Delta t) = f\left(t_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2}\left(\frac{f(t_i, y_i) + f(t_{i+1}, y_i + \Delta t\, f(t_i, y_i))}{2}\right)\right)$$

$$\lim_{\Delta t \to 0} \phi(t_i, y_i; \Delta t) = \lim_{\Delta t \to 0} f\left(t_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2}\left(\frac{f(t_i, y_i) + f(t_{i+1}, y_i + \Delta t\, f(t_i, y_i))}{2}\right)\right)$$

$$\lim_{\Delta t \to 0} \phi(t_i, y_i; \Delta t) = f(t_i, y_i)$$

Thus, having satisfied (9), the proposed algorithm (3) is said to be consistent.

2.4 Stability and Convergence of the Proposed Algorithm

Theorem 03. (Lambert 1973, Fatunla 1988)

Suppose $y_i = y(t_i)$ and $z_i = z(t_i)$ be two different approximate solutions to IVP of the type (1), subject to initial conditions $y(t_0) = \alpha$ and $z(t_0) = \alpha^*$, respectively; such that $|\alpha - \alpha^*| < \varepsilon$, $\varepsilon > 0$. If the two iterates are produced using one – step explicit linear iterative algorithm then we obtain:

$$y_{i+1} = y_i + \Delta t \phi(t_i, y_i; \Delta t)$$

$$z_{i+1} = z_i + \Delta t \phi(t_i, z_i; \Delta t)$$

The following condition

$$|y_{i+1} - z_{i+1}| \le K|\alpha - \alpha^*|$$

is the necessary and sufficient condition for the proposed algorithm to be *stable* and *convergent*.

Proof

From (3), we have

$$y_{i+1} = y_i + \Delta t\, f\left(t_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{2}\left(\frac{f(t_i, y_i) + f(t_{i+1}, y_i + \Delta t\, f(t_i, y_i))}{2}\right)\right)$$

$$z_{i+1} = z_i + \Delta t f\left(t_i + \frac{\Delta t}{2}, z_i + \frac{\Delta t}{2}\left(\frac{f(t_i, z_i) + f(t_{i+1}, z_i + \Delta t\, f(t_i, z_i))}{2}\right)\right)$$

Hence

$$y_{i+1} - z_{i+1} = y_i - z_i + \Delta t\begin{bmatrix} f\left(t_i + \frac{\Delta t}{2}, y_i + \frac{\Delta t}{4}\left(f(t_i, y_i) + f(t_{i+1}, y_i + \Delta t\, f(t_i, y_i))\right)\right) \\ -f\left(t_i + \frac{\Delta t}{2}, z_i + \frac{\Delta t}{4}\left(f(t_i, z_i) + f(t_{i+1}, z_i + \Delta t\, f(t_i, z_i))\right)\right)\end{bmatrix} \tag{10}$$

Using Mean Value Theorem for a function of two variables with (η,ξ) being an intermediate point of any two points in R^2, we get

$$y_{i+1}-z_{i+1}=\left[1+\Delta t+\frac{(\Delta t)^2}{4}\left(f_y(\eta,\xi)+\Delta t f_y(\eta,\xi)+1\right)\right](y_i-z_i)$$

Or
$$|y_{i+1}-z_{i+1}|\le\left[1+\Delta t+\frac{(\Delta t)^2}{4}\left(\sup_{(\eta,\xi)\in R^2} f_y(\eta,\xi)(1+\Delta t)+1\right)\right]|y_i-z_i|$$

Letting $L^*=\sup_{(\eta,\xi)\in R^2} f_y(\eta,\xi)$ and $K=1+\Delta t+\frac{(\Delta t)^2}{4}\left(L^*(1+\Delta t)+1\right)$ with $y(t_0)=\alpha$ and $z(t_0)=\alpha^*$, we get

$$|y_{i+1}-z_{i+1}|\le K|\alpha-\alpha^*|$$

This shows that the proposed algorithm (3) is stable and consistent, therefore convergent.

Finally, analysis of region of absolute stability for the proposed algorithm (3) is carried out using the scalar model problem:

$$\frac{dy}{dt}=\lambda y,\ \ y(t_0)=y_0 \text{ for } \lambda\in\mathbb{C}$$

We would like the numerical solution to decline when $\mathrm{Re}(\lambda)<0$ and the region of absolute stability is the complex $\lambda\Delta t$ plane where numerical solutions satisfy $|y_{i+1}|<|y_i|$.

When (3) is applied on the model problem, it yields

$$y_{i+1}=y_i\left[1+\lambda\Delta t+\frac{(\lambda\Delta t)^2}{2}+\frac{(\lambda\Delta t)^3}{4}\right]$$

Thus, the region of absolute stability is given as

$$\left|1+\lambda\Delta t+\frac{(\lambda\Delta t)^2}{2}+\frac{(\lambda\Delta t)^3}{4}\right|\le 1$$

Region of absolute stability of the proposed algorithm has been given in figure 1 along with stability regions for Forward Euler, Midpoint Euler, Improved Euler and MIME[1] methods whereas interval of absolute stability for proposed algorithm and other methods is found to be identical as shown in table I. It is also worth to be noted in figure 1 that regions of absolute stability for the proposed algorithm and MIME method are same.

[1] $y_{i+1}=y_i+\Delta t f\left(t_i+\frac{\Delta t}{2}, y_i+\frac{\Delta t}{2} f\left(t_i+\frac{\Delta t}{2}, y_i+\frac{\Delta t}{2} f(t_i,y_i)\right)\right)$; proposed by OCHOCHE (2008).

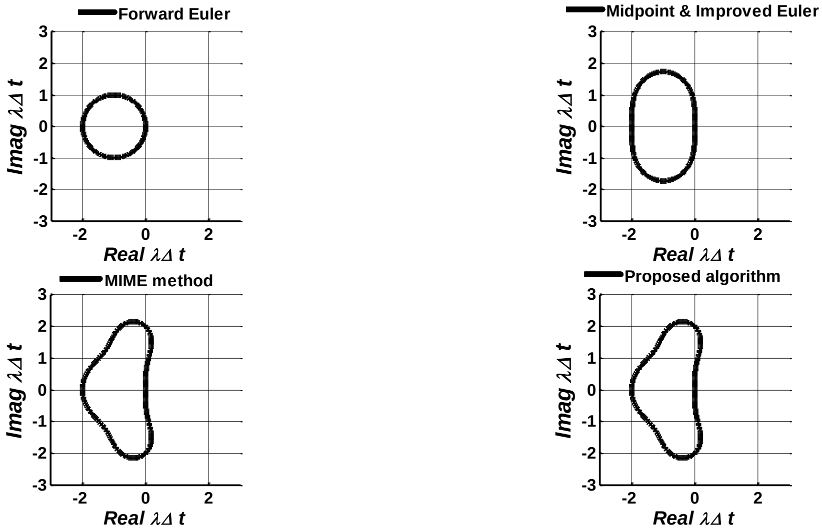

Fig 1. Regions of absolute stability (inside portion of the closed loops).

Table 1. Stability polynomials and Intervals of absolute stability

Algorithm	Stability Polynomial	Interval of absolute stability
Forward Euler	$1+\lambda\Delta t$	$(-2,0)$
Midpoint Euler	$1+\lambda\Delta t+\dfrac{(\lambda\Delta t)^2}{2}$	$(-2,0)$
Improved Euler	$1+\lambda\Delta t+\dfrac{(\lambda\Delta t)^2}{2}$	$(-2,0)$
MIME	$1+\lambda\Delta t+\dfrac{(\lambda\Delta t)^2}{2}+\dfrac{(\lambda\Delta t)^3}{4}$	$(-2,0)$
Proposed	$1+\lambda\Delta t+\dfrac{(\lambda\Delta t)^2}{2}+\dfrac{(\lambda\Delta t)^3}{4}$	$(-2,0)$

3. Numerical Implementation and Discussion

In this part of the work, performance of proposed algorithm has been checked against some 2nd order algorithms taken from literature. Numerical computations and graphical displays have been obtained using MATLAB version 8.1 (R2013a) in double precision arithmetic.

Consider the following autonomous first order IVP:

Problem 1: $\dfrac{dy}{dt}=\cos 2t+\sin 3t,\;\; y(0)=1;\quad t\in[0,10]$ (source: Burden & Faires (2010))

Exact solution: $y(t)=1/2\sin 2t-1/3\cos 3t+4/3$

It has been observed from figure 2 (a – d) that the proposed algorithm follows the pattern of the exact solution as being followed by other methods with considerably a large step size of 0.5 for problem 1. Further, maximum absolute global errors computed in table 2 reveal equivalency of the proposed algorithm with 2nd order Midpoint Euler and MIME methods whereas Improved Euler contains a bit higher amount of errors. It is obvious from table 2 that by further decreasing step size, the numerical solution obtained by the proposed algorithm will approach to the exact solution.

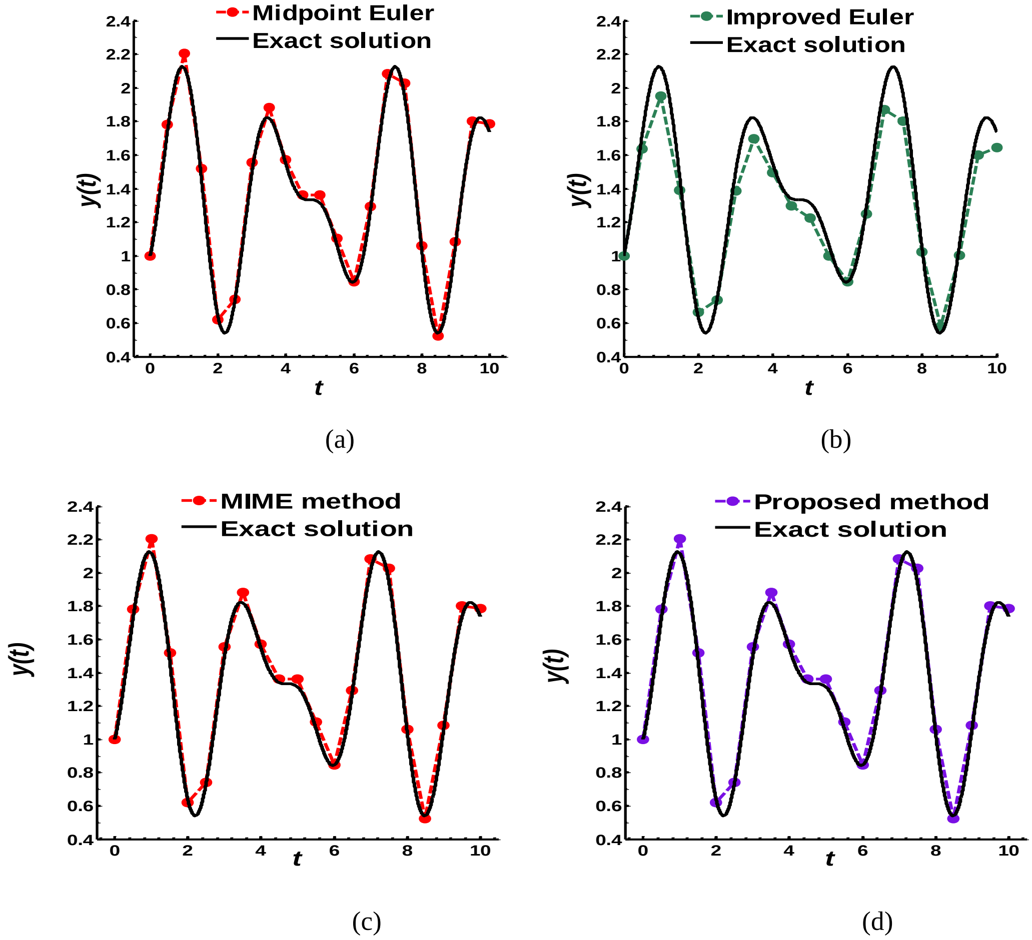

Fig 2. Comparison of proposed algorithm with two standard methods and one nonstandard MIME method.

Table 2. Maximum absolute global errors with varying step sizes for problem 1.

‘ Δt ’	**Maximum Absolute Global Errors**			
	Midpoint Euler	**Improved Euler**	**MIME**	**Proposed**
0.1	1.537340214559269e-03	3.071662959107331e-03	1.537340214559269e-03	1.537340214559269e-03
0.2	6.185439200536636e-03	1.232238868990373e-02	6.185439200536636e-03	6.185439200536636e-03
0.4	2.508549427699337e-02	4.939081787710430e-02	2.508549427699337e-02	2.508549427699337e-02
0.5	4.062188173624455e-02	7.924420556674233e-02	4.062188173624455e-02	4.062188173624455e-02
1.0	1.982161478984918e-01	3.567045590319764e-01	1.982161478984918e-01	1.982161478984918e-01

As a second example, a nonlinear first order IVP is considered:

Problem 2: $\dfrac{dy}{dt} = 1 + y^3,\ y(0) = 0;\ t \in [0,1.2]$ (see Ramos (2007))

Implicit solution: $\sqrt{3}\pi + 6\sqrt{3}\tan^{-1}\left(\dfrac{2y-1}{\sqrt{3}}\right) + 6\ln(1+y) - 3\left[6t + \ln\left(1-y+y^2\right)\right] = 0$

For the problem 2, explicit expression for its solution is not available therefore the value at $t = 1.2$ has been obtained using Newton's iterative scheme as $y(t = 1.2) \approx 7.368587110472472$. This output is used as a reference point for calculating absolute errors at end point of the given interval. These computed errors and time taken by CPU to execute the results are shown in table 3 for different integration steps whereas figure 3 shows approximate solution obtained by the proposed algorithm (3) for 400 (say) integration steps. Table 3 shows inverse relation between integration steps and absolute error computed; that is, increment in integration steps results reduction in corresponding errors. It is also observed from third column of table 3 that error can further be decreased at the cost of more computational effort.

Table 3. Absolute errors at $t = 1.2$ and CPU time for problem 2.

Integration Steps	Absolute error	Computer Time (in microseconds)
200	7.921486078109474e-02	2110
400	3.079290602821860e-02	5647
600	1.572245833341501e-02	6575
800	9.463829887513597e-03	9584
1000	6.304863390318438e-03	11826
2000	1.705975830714301e-03	18457

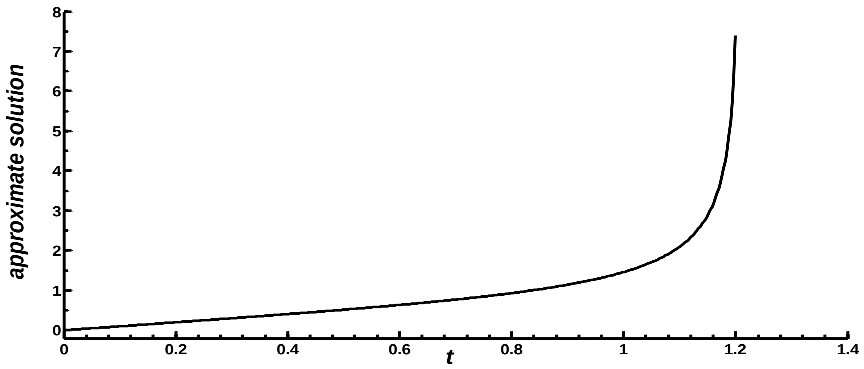

Figure 3. Approximate solution to problem 2 using proposed algorithm.

4. Conclusion

In the current work, an attempt has been made to discuss and examine accuracy, consistency, convergence and stability of an explicit iterative algorithm proposed by Qureshi *et al* (2013). The algorithm is found to be 2nd order accurate, consistent, convergent and stable and thereby can be used for finding approximate solution to the problems of the type (1) and (2). Region of absolute stability drawn for the proposed algorithm seems to be completely agree with that of 2nd order MIME method as indicated in figure 1. Furthermore, it is also worth to be noted from table 1 that intervals of absolute stability for all algorithms under consideration are alike. Numerical examples provided show reasonable performance of the proposed algorithm.

5. Future Work

In future, propagation of errors and error norms of the proposed algorithm would be discussed and compared with existing literature for Forward Euler and its variants. Numerical examples would be given to study dexterity of the proposed algorithm with respect to exact solution, local and global truncation errors and error norms.

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