

Some Fixed Point Result in Metric Spaces for Rational Expression

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Abstract:

In the present paper we establish some fixed point theorems in complete metric space taking rational expression. Our Result Generalize the result of many authors.

Key words: Fixed point, common fixed point, rational expressions

2. Introduction

In this paper some extension of well known Banach contraction theorem [1] has obtained in terms of a new symmetric rational expression. This celebrated principle has been generalized by many authors viz. Chu & Diaz[3] Sehgal[13], Holmes[8], Reich[12], Hardy and Rogers[7], Wong[15], Iseki[9], Sharma and Rajput[14], Gupta and Dass[6], Jaggi[10], Chatterjee[2], Fisher[5], Kannan[11], Ciric[4] and others.

In this Paper we shall establish some unique fixed point and common fixed point theorems, through new symmetric rational expressions.

3. Main Result

Theorem 3.1 Let T be a continuous self map, defined on a complete metric space X. Further, T satisfies the following condition;

$$d(Tx, Ty) < \alpha \max\left\{\frac{d(x,Tx)d(y,Ty)+d(x,Ty)d(y,Tx)}{d(x,y)}, \frac{d(x,Tx)d(x,Ty)+d(y,dy)d(y,Tx)}{d(x,y)}\right\}$$
$$+ \gamma[d(x,Tx)+d(y,Ty)]+\delta[d(y,Tx)+d(x,Ty)]+\eta d(x,y) \quad (3.1.1)$$

for all $x,y\epsilon X$, $x\neq y$ and for some $\alpha, \gamma, \delta, \eta\epsilon[0,1)$ with $(2\alpha+ 2\gamma+ 2\delta+\eta<1)$.

Then T has unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X and we define a sequence $\{x_n\}$ by means of iterates of T by setting

$T_{x_0}^n = x_n$, where n is a positive integer. If $x_n=x_{n+1}$ for some n, then x_n is a fixed point of T.

Taking $x_n\neq x_{n+1}$, for all n

Now

$d(x_{n+1},x_n)=d(Tx_n ,Tx_{n-1})$

$$\leq \alpha\max\left\{\frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})+d(x_n,Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_n,x_{n-1})},\frac{d(x_n,Tx_n)d(x_n,Tx_{n-1})+d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_n,x_{n-1})}\right\}$$

$$+\gamma\left[d\left(x_n,Tx_n\right)+d(x_{n-1},Tx_{n-1})\right]+\delta\left[d\left(x_n,Tx_{n-1}\right)+d(x_{n-1},Tx_n)\right]+\eta d(x_n,x_{n-1})$$

$$\leq \alpha\ \max\left\{\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)+d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})},\frac{d(x_n,x_{n+1})d(x_n,x_n)+d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})}\right\}$$

$$+\gamma\left[d\left(x_n,x_{n+1}\right)+d(x_{n-1},x_n)\right]+\delta\left[d\left(x_n,x_n\right)+d(x_{n-1},x_{n+1})\right]+\eta d(x_n,x_{n-1}).$$

$$\leq \alpha\max\left\{d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),\right\}+\gamma\left[d\left(x_n,x_{n+1}\right)+d(x_{n-1},x_n)\right]+\delta[d(x_{n-1},x_{n+1})]$$

$$+\eta d(x_n,x_{n-1}).$$

Case I.

If $d(x_n,x_{n+1}) > d(x_{n-1},x_{n+1})$

Then

$d(x_n,x_{n+1}) \leq (\alpha+\gamma+\delta) d(x_n,x_{n+1}) + (\alpha+\gamma+\delta+\eta) d(x_{n-1},x_n)$

$\therefore d(x_n,x_{n+1}) \leq \left(\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right) d(x_{n-1},x_n)$

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$$\left[\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right]^n\ d(x_0,x_1).$$

By the triangle inequality, we have for m>n

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m).$$

$$\leq (p^n + p^{n+1} + \cdots p^{m-1})\ d(x_0,Tx_0)$$

Where, $p=\left[\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right]<1$, Since $2\alpha+2\gamma+2\delta+\eta<1$.

Case II

If $d(x_{n-1},x_n) > d(x_n,x_{n+1})$

$d(x_n,x_{n+1}) \leq (\alpha+\gamma+\delta)\ d(x_{n-1},x_{n+1}) + (\alpha+\gamma+\delta+\eta) d(x_{n-1},x_n)$

$d(x_n,x_{n+1}) \leq \left(\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right) d(x_{n-1},x_n)$

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$$\left[\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right]^n\ d(x_0,x_1).$$

By the triangle inequality, we have for m>n

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots d(x_{m-1}, x_m).$$

$$\leq (\mathrm{q}^{\mathrm{n}} + \mathrm{q}^{\mathrm{n+1}} + \cdots \mathrm{q}^{\mathrm{m+1}}) d(x_0, Tx_0)$$

Where, $q = \left[\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right] < 1$, $2\alpha + 2\gamma + 2\delta + \eta$<1. Take k=p=q<1.

Therefore

$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, Tx_0)$ →0, as m, n→∞.

So, { x_n} is Cauchy sequence in X, so by completeness of X, there is a point uϵX such that x_n →0, as m, n→∞.

So, { x_n} is Cauchy sequence in X, so by completeness of X, there is a point uϵX such that x_n →u as n→∞.

Further, the continuity of T in X implies.

T(u)=T($\lim_{n\to\infty} x_n$)= $\lim_{n\to\infty} Tx_n$

=$\lim_{n\to\infty} x_{n+1}$)

=u.

Therefore, u is a fixed point of T in X.

Now if there is any other v($\neq u$)in X, such that T(v)=v, then.

d(u,v)=d(Tu,Tv)

$$\leq \alpha \, max \left\{\frac{d(u,Tu)d(v,Tv)+d(u,Tv)d(v,Tu)}{d(u,v)}, \frac{d(u,Tu)d(u,Tv)+d(v,Tv)d(v,Tu)}{d(u,v)}\right\} + \gamma[d(u,Tu) + d(v,Tv)] + \delta[d(u,Tv) + d(v,Tu)] + \eta d(u,v).$$

$$\leq \alpha \max\left\{\frac{d(u,u)d(v,v)+d(u,v)d(v,u)}{d(u,v)}, \frac{d(u,u)d(u,v)+d(v,v)d(v,u)}{d(u,v)}\right\} + \gamma[d(u,u) + d(v,v)] + \delta[d(u,v) + d(v,u)] + \eta d(u,v)]$$

$$\leq (\alpha + 2\delta + \eta) \, \mathrm{d(u,v)}$$

i.e. d(u,v) ≤($\alpha + 2\delta + \eta$)d(u,v).

Which is a contradiction because $\alpha + 2\delta + \eta < 1$.

Hence u is the unique fixed point of T.

Theorem 3.2: Let T be a self map defined on a complete metric space (X,d) such that (3.1.1) holds. If for some positive integer P, T^p is continuous, then T has a unique fixed point.

Proof: we define a sequence $\{ x_n\}$ as in theorem 1. Clearly it converges to some point uϵX. Therefore its subsequence $\{ x_{n_k}\}$, $(n_k = k_p)$ also converges to u.

Also,

$T^p u = T^p(\lim_{k\to\infty} x_{n_k})$

$=\lim_{k\to\infty}(T^p_{x_{n_k}})$

$=\lim_{k\to\infty} x_{n_k+1}$

=u

Therefore u is a fixed point of T^p.

Now, we show that, Tu=u.

Let m be the smallest positive integer such that

$T^m u = u$, but $T^q \neq u$ for q=1,2…m-1.

If m>1, then by (3.1.1)

$d(Tu, u)$=$d(Tu, T^q_u)$= $d(Tu, T(T^{m-1}_u))$

$$\leq \alpha max\left\{\frac{d(u,Tu)d(T^{m-1}_u,T^m_u)+d(u,T^m_u)d(T^{m-1}_u,Tu)}{d(u,T^{m-1}_u)}, \frac{d(u,Tu)d(u,T^m_u)+d(T^{m-1}u,T^m_u)d(T^{m-1}_u,Tu)}{d(u,T^{m-1}_u)}\right\} + \gamma[d(u,Tu) + d(T^{m-1}_u,T^m_u)] + \delta[d(u,T^m_u) + d(T^{m-1}_u,Tu)] + \eta d(u,T^{m-1}_u).$$

$d(Tu, u) \leq \alpha\ max\{(d(u,Tu) + 0), (0 + d(T^{m-1}_u, u) + d(u,Tu))\}$

$+\gamma[d(u,Tu) + d(T^{m-1}_u,T^m_u)] + \delta[d(u,T^m_u) + d(T^{m-1}_u,Tu)] + \eta d(u,T^{m-1}_u).$

[Since by triangle inequality d(T^{m-1}_u,Tu)$\leq d(T^{m-1}_u, u) + d(u,Tu)$ and since

$(d(T^{m-1}_u, \text{Tu}) + \text{d(u, Tu)} \geq \text{d(u, Tu)}$

$\leq \alpha[d(T^{m-1}_u, u) + d(u,Tu)] + \gamma[d(u,Tu) + d(T^{m-1}_u,T^m_u)] + \delta[d(u,T^m_u) + d(T^{m-1}_u,Tu)] + \eta d(u,T^{m-1}_u).$

$\leq \alpha[d(T^{m-1}_u, u) + d(u,Tu)] + \gamma[d(u,Tu) + d(T^{m-1}_u,T^m_u)] + \delta[d(u,T^m_u) + d(T^{m-1}_u,Tu)] + \eta d(u,T^{m-1}_u).$

$d(u,Tu)$≤($\alpha + \gamma + \delta$)d(u,Tu)+ ($\alpha + \gamma + \delta + \eta$)d(u, T^{m-1}_u).

$(1 - \alpha - \gamma - \delta)$d(u,Tu) ≤ $(\alpha + \gamma + \delta + \eta)$d(u, T^{m-1}_u)

Which implies

$d(u,Tu)$≤kd(u, T^{m-1}_u) where $k = \left(\frac{(\alpha+\gamma+\delta+\eta)}{(1-\alpha-\gamma-\delta)}\right)$<1

Since,

$2\alpha + 2\gamma + 2\delta + \eta$<1, thus we write,

$d(u,Tu) \leq K^m d(u,Tu)$, Since $K^m < 1$

Therefore

$d(u,Tu) < d(u,Tu)$, Which contradicts.

Hence $Tu = u$ i.e. u is a fixed point of T. The uniqueness of u follows as in theorem 1.

We further generalize the result of theorem 1, in which T is neither continuous nor satisfies (3.1.1). In what follows T^m, for some positive integer m, satisfying the same rational expression and continuous still T has unique fixed point. In fact we prove.

Theorem 3.3.: Let T be a self-map, defined on a complete metric space (X,d) such that for some positive integer m satisfy the condition.(3.3.1)

$$d(T^m x, T^m y) \leq \alpha max\left\{\frac{d(x,T^m x)d(y,T^m y)+d(x,T^m y)d(y,T^m y)}{d(x,y)}, \frac{d(x,T^m x)d(x,T^m y)+d(x,T^m y)d(y,T^m x)}{d(x,y)}\right\} + \gamma[d(x,T^m x) + d(y,T^m y)] + \delta[d(x,T^m y) + d(y,T^m x)] + \eta d(x,y).$$

For all x,y$\in$X, x$\neq$y and for $\alpha,\gamma,\delta,\eta \geq 0$ with $2\alpha + 2\gamma + 2\delta + \eta < 1$ If T^m is continuous then T has a unique fixed point.

Proof. By theorem 3.2, we assume that T^m has unique fixed point also

$Tu=T(T^m u)=T^m(Tu)$.

Which implies Tu=u, Further since a fixed point of T is also a fixed point T^m & T^m has a unique fixed point u, it follows that u is the unique fixed point of T.

Theorem 3.4 Let T_1 and T_2 be two self maps defined on a complete metric space (X,d) satisfying the condition;

$$d(Tx,Ty) < \alpha \max\left\{\frac{d(x,T_1x)d(y,T_2y)+d(x,T_2y)d(y,T_1x)}{d(x,y)}, \frac{d(x,T_1x)d(x,T_2y)+d(y,T_2y)d(y,T_1x)}{d(x,y)}\right\}$$
$$+\gamma[d(x,T_1x)+d(y,T_2y)]+\delta[d(y,T_2x)+d(x,T_1y)]+\eta d(x,y).$$
(3.4.1)

for all x,y$\in$X and for some $\alpha,\gamma,\delta,\eta \geq 0$ and ($2\alpha + 2\gamma + 2\delta + \eta < 1$) (3.4.2)

T_1 ,T_2 are continuous on X. (3.4.2)

There exist an $x_0 \in X$ such that in the sequence $\{x_n\}$ where,

$$x_n = \begin{cases} T_1 x_{n-1}, \text{where n is even} \\ T_2 x_{n-1}, \text{where n is odd} \end{cases}$$

$x_n \neq x_{n+1}$ for all n.

then T_1, T_2 have a unique common fixed point.

Proof: we have

$d(x_{2n}, x_{2n+1})=d(T_1x_{2n}, T_2x_{2n+1})$

$$\leq \alpha \max\left\{\frac{d(x_{2n-1},T_1x_{2n-1})d(x_{2n},T_2x_{2n})+d(x_{2n-1},T_2x_{2n})d(x_{2n},T_1x_{2n-1})}{d(x,y)}, \frac{d(x_{2n-1},T_1x_{2n-1})d(x_{2n-1},T_2x_{2n})+d(x_{2n},T_2y)d(x_{2n},T_1x_{2n-1})}{d(x_{2n-1},x_{2n})}\right\}$$

$+\gamma[d(x_{2n-1}, T_1x_{2n-1})+d(x_{2n}, T_2x_{2n})]+\delta[d(x_{2n-1}, T_2x_{2n})+d(x_{2n}, T_1x_{2n-1})]+\eta d(x_{2n-1}, x_{2n})$.

$$\leq \alpha \max\left\{\frac{d(x_{2n-1},x_{2n})d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n+1})d(x_{2n},x_{2n})}{d(x_{2n-1},x_{2n})}, \frac{d(x_{2n-1},x_{2n})d(x_{2n-1},x_{2n+1})+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n})}{d(x_{2n-1},x_{2n})}\right\}$$

$+\gamma[d(x_{2n-1}, x_{2n})+d(x_{2n}, x_{2n+1})]+\delta[d(x_{2n-1}, T_2x_{2n+1})+d(x_{2n}, x_{2n})]+\eta d(x_{2n-1}, x_{2n})$.

$\leq \alpha \max\{d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n+1})\}$
$+\gamma[d(x_{2n-1}, x_{2n})+d(x_{2n}, x_{2n+1})]+\delta[d(x_{2n-1}, T_2x_{2n+1})+d(x_{2n}, x_{2n})]+\eta d(x_{2n-1}, x_{2n})$.

$\leq \alpha \max\{d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\}$
$+\gamma[d(x_{2n-1}, x_{2n})+d(x_{2n}, x_{2n+1})]+\delta[d(x_{2n-1}, T_2x_{2n+1})+d(x_{2n}, x_{2n})]+\eta d(x_{2n-1}, x_{2n})$.

$\leq \alpha \{d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})\}$
$+\gamma[d(x_{2n-1}, x_{2n})+d(x_{2n}, x_{2n+1})]+\delta[d(x_{2n-1}, T_2x_{2n+1})+0]+\eta d(x_{2n-1}, x_{2n})$.

$= (\alpha+ \gamma+ \delta)\, d(x_{2n+1}, x_{2n}) +(\alpha+ \gamma+ \delta+\eta)\, d(x_{2n-1}, x_{2n})$

Therefore

$d(x_{2n}, x_{2n+1})\leq (\alpha+ \gamma+ \delta)\, d(x_{2n}, x_{2n+1})+ (\alpha+ \gamma+ \delta+\eta)d(x_{2n-1}, x_{2n}))$

which implies.

$d(x_{2n}, x_{2n+1})\leq \left(\frac{(\alpha+\gamma+\delta+\eta)}{(1-\alpha-\gamma-\delta)}\right) d(x_{2n-1}, x_{2n})$

i.e. $d(x_{2n}, x_{2n+1})\leq k^{2n}d(x_0,x_1)$.

When $k=\left(\frac{\alpha+\gamma+\delta+\eta}{1-\alpha-\gamma-\delta}\right)<1$

Since $2\alpha+2\gamma+2\delta+\eta<1$

$d(x_{2n+1}, x_{2n+2})\leq k^{2n+1} d(x_0,x_1)$.

Now it can be easily seen that $\{x_n\}$ is a Cauchy sequence.

Let $x_n\to u$, then the subsequence $\{x_{n_p}\}$ also converges to u for $n_p=2p$.

Now, $T_1 T_2(u)= T_1 T_2(\lim_{p\to\infty} x_{n_p})$

$= \lim_{p\to\infty} x_{n_p+1}$

$= u$.

We now show that $T_2u\neq u$

If $T_2u \neq u$, then

$d(u, T_2u)=d(T_1 T_2u, T_2u)$

$$\leq \alpha \max\left\{\frac{d(T_2u,T_1T_2u)d(u,T_2u)+d(T_2u,T_2u)d(u,T_1T_2u)}{d(T_2u,u)}, \frac{d(T_2u,T_1T_2u)d(T_2u,T_2u)+d(u,T_2u)d(u,T_1T_2u)}{d(T_2u,u)}\right\}$$

$$+\gamma[d(T_2u, T_1T_2u)+d(u, T_2u)]+\delta[d(T_2u, T_2u)+d(u, T_1T_2u)]+\eta d(T_2u, u).$$

Therefore

$d(u, T_2u) \leq \alpha \max\{d(u, T_2u),0\}+\gamma[2d(u, T_2u)]+0+\eta\, d(T_2u, u).$

Therefore

$d(u, T_2u) \leq (\alpha+2\gamma+\eta)d(u, T_2u)$

$< d(u, T_2u)$

Which is contradiction, since $\alpha+\beta+2\gamma+2\delta+\eta<1$

So, $\alpha+2\gamma+\eta<1$

Hence we have

$T_2u=u$

Now

$T_1T_2u= T_1u=u$

Thus u is the common fixed point of T_1 and T_2 .

For the **uniqueness**, if possible let $v \neq u$, $v \in X$, such that

$T_1v= T_2v=v$

So $d(u,v)=d(T_1u, T_2v)$

$$\leq \alpha \max\left\{\frac{d(u,T_1u)d(v,T_2v)+d(u,T_2u)d(v,T_1u)}{d(u,v)}, \frac{d(u,T_1u)d(u,T_2u)+d(v,T_1u)d(v,T_2v)}{d(u,v)}\right\}$$

$$+\gamma[d(u, T_1u)+ d(v, T_2v)]+\delta[d(u, T_2u)+d(v, T_1u)]+\eta d(u, v).$$

$\leq (\alpha+2\delta+\eta)d(v,u)$

Therefore

$d(u,v) \leq (\alpha+2\delta+\eta)d(u,v)$

$<d(u,v)$

Which is a contradiction, because $\alpha+\beta+2\gamma+2\delta+\eta<1$ and so we have $\alpha+2\delta+\eta<1$.

Hence, we have u=v.

This completes the proof of the theorem.

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