Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.1, 2014

Fixed Point Theorems in Random Fuzzy Metric Space

Rajesh Shrivastava, Jagrati Singhal*
Professor & Head, Department of Mathematics, Govt. Science & Commerce College Benazir Bhopal, M.P., India
Department of Mathematics, Sagar Institute of Research & Technology Bhopal, M.P., India

1.Abstract

In the present paper some fixed point theorems in complete RandomFuzzy metric spaces are established which are generalizations of fuzzy metric spaces for various type mappings.

Key Words: Fuzzy metric spaces, Random fuzzy metric spaces, fixed point , Common fixed point .

AMS Subject classification 47H10, 54H25

2. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [39]. After him many authors have developed the theory of fuzzy sets and applications. Especially, Deng [9], Erceg [11], Kaleva and Seikkala [26]. Kramosil and Michalek [28] have introduced the concept of fuzzy metric spaces by generalizing the definition of probabilistic metric space. Many authors have also studied the fixed point theory in these fuzzy metric spaces are [1], [7], [13], [19], [21], [24], [25], [32] and for fuzzy mappings [2], [3], [4], [5], [22], [31].

In 1994, George and Veeramani [18] modified the definition of fuzzy metric spaces given by Kramosil and Michalek [28] in order to obtain Hausdroff topology in such spaces. Gregori and Sapena [20] in 2002 extended Banach fixed point theorem to fuzzy contraction mappings on complete fuzzy metric space in the sense of George and Veeramani [18].

It is remarkable that Sharma, Sharma and Iseki [34] studied for the first time contraction type mappings in 2-metric space. Wenzhi [38] and many others initiated the study of Probabilistic 2-metric spaces. As we know that 2-metric space is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area of function in Euclidean spaces. Now it is natural to expect 3-metric space which is suggested by the volume function. The method of introducing this is naturally different from 2-metric space theory from algebraic topology.

The concept of Fuzzy-random-variable was introduced as an analogous notion to random variable in order to extend statistical analysis to situations when the outcomes of some random experiment are fuzzy sets. But in contrary to the classical statistical methods no unique definition has been established before the work of Volker [37]. He presented set theoretical concept of fuzzy-random-variables using the method of general topology and drawing on results from topological measure theory and the theory of analytic spaces. No results in fixed point are introduced in random fuzzy spaces. In [17] authors Gupta, Dhagat, Shrivastava introduced the fuzzy random spaces and proved common fixed point theorem.

In the present chapter we will find some fixed point theorems in random fuzzy metric space, random fuzzy 2-metric space and random fuzzy 3-metric space through rational expressionmotivated by [17] and [8].

To start the main result we need some basic definitions.

2.2 Preliminaries:

2.2.1 Definitions

Definition2.2.1.1: (Kramosil and Michalek 1975)

A binary operation $\ast$: $[0,1]\times[0,1] \rightarrow [0,1]$ is a t-norm if it satisfies the following conditions :

(i) $\ast(1,a) = a$, $\ast(0,0) = 0$

(i) $\ast(a,b) = \ast(b,a)$

(ii) $\ast(c,d) \geq \ast(a,b)$ whenever $c \geq a$ and $d \geq b$

(iii) $*(*(a,b),c) = *(a, *(b,c))$ where a,b, c,d$\in [0,1]$

Definition 2.2.1.2: (Kramosil and Michalek 1975)

The 3-tuple $(X,M, *)$ is said to be a fuzzy metric space if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x,y,0) = 0$
(ii) $M(x,y,t) = 1$ for all $t > 0$ iff $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \le M(x,z,t + s)$,
(v) $M(x,y, .) : [0,\infty[\to[0,1]$ is left-continuous,

Where x, y, z $\in$ X and t, s > 0.

In order to introduce a Hausdroff topology on the fuzzy metric space, in (Kramosil and Michalek 1975) the following definition was introduced.

Definition 2.2.1.3: (George and Veermani 1994)

The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times]0,\infty[$ satisfying the following conditions :

(i) $M(x,y,t) > 0$
(ii) $M(x,y,t) = 1$ iff $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \le M(x,z,t + s)$,
(v) $M(x,y, .) :]0,\infty[\to[0,1]$ is continuous,

Where x, y, z $\in$ X and t, s > 0.

Definition 2.2.1.4: (George and Veermani 1994)

In a metric space (X, d) the 3-tuple (X, Md,*) where Md(x, y, t) = t / (t + d(x, y)) and a*b = ab is a fuzzy metric space. This Md is called the standard fuzzy metric space induced by d.

Definition 2.2.1.5: (Gregori and Sepene 2002)

Let $(X, M,*)$ be a fuzzy metric space. A mapping f: X→ X is fuzzy contractive if there exists $0 < k < 1$ such that

$$\frac{1}{M\,(f(x),f(y),t)} - 1 \le k\left(\frac{1}{M(x,y,t)} - 1\right)$$

For each x,y ϵ X and t > 0 .

Definition 2.2.1.6: (Gregori and Sepene 2002)

Let **(X,** M, $*$**)** be a fuzzy metric space . We will say that the sequence $\{x_n\}$ in X is fuzzy contractive if there exists k ϵ (0, 1) such that

$$\frac{1}{M(x_{n+1},x_{n+2},t)} - 1 \le k\left(\frac{1}{M(x_n,x_{n+1},t)} - 1\right) \quad \text{for all } t > 0 ,\ n \,\epsilon\, N.$$

We recall that a sequence $\{x_n\}$ in a metric space (X, d) is said to be contractive if there exist $0 < k < 1$ such that $d(x_{n+1}, xn_{+2}) \le kd(x_n, x_{n+1})$ for all n ϵ N.

Definition 2.2.1.7: (Kumar and Chugh 2001)

Let (X, τ) be a topological space. Let f and g be mappings from a topological space (X, τ) into itself. The mappings f and g are said to be compatible if the following conditions are satisfied**:**

(i) $fx = gx$, $x \in X$ Implies $fgx = gfx$,

(ii) The continuity of f at a point x in X implies $\lim gfx_n=fx$ whenever $\{x_n\}$ is a sequence in X such that $\lim gx_n=\lim fx_n=fx$ for some x in X.

Definition 2.2.1.8 : A binary operation $*$: [0,1] x [0,1] x [0,1]$\to$[0,1] is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a_1* b_1* c_1\leq a_2* b_2* c_2$ whenever $a_1\leq a_2$, $b_1\leq b_2$, $c_1\leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition 2.2.1.9 : The 3-tuple (X,M,$*$) is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in X^3 x $[0,\infty)$ satisfying the following conditions for all x, y, z, u $\in$ X and $t_1, t_2, t_3 > 0$.

(FM'-1) $M(x, y, z, 0) = 0$,

(FM'-2) $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,

(FM'-3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$,

(Symmetry about three variables)

(FM'-4) $M(x, y, z, t_1+t_2+t_3) \geq M(x, y, u, t_1)*M(x, u, z, t_2)* M(u, y, z, t_3)$

(This corresponds to tetrahedron inequality in 2-metric space)

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

(FM'-5) M(x, y, z, .): [0, 1)$\to$[0,1] is left continuous.

Definition 2.2.1.10: Let (X, M,$*$) is a fuzzy 2-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$, if

$$lim_{n\to\infty} M(x_n, x, a, t) = 1$$

for all $a\in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$$

for all $a\in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.2.1.11: A function M is continuous in fuzzy 2-metric space iff whenever $x_n\to x$, $y_n\to y$, then

$$lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a,t)$$

for all $a\in X$ and $t > 0$.

Definition 2.2.1.12: Two mappings A and S on fuzzy 2-metric space X are weakly commuting iff

$$M (ASu, SAu, a, t) \geq M (Au, Su, a, t)$$

for all u, a $\in$ X and $t > 0$.

Definition 2.2.1.13: A binary operation $*$: $[0,1]^4 \to[0,1]$ is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a_1* b_1* c_1* d_1\leq a_2* b_2* c_2 * d_2$ whenever $a_1\leq a_2$, $b_1\leq b_2$, $c_1\leq c_2$ and $d_1\leq d_2$ for all a_1, a_2, b_1, b_2 , c_1, c_2 and d_1, d_2 are in [0,1].

Definition 2.2.1.14 : The 3-tuple (X, M, $*$) is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in X^4 x $[0,\infty)$ satisfying the following conditions : for all x, y, z, w, u $\in$X and $t_1, t_2, t_3, t_4 > 0$.

(FM''-1) $M(x, y, z, w, 0) = 0$,

(FM''-2) $M(x, y, z, w, t) = 1$ for all $t > 0$,

(only when the three simplex $\langle x, y, z, w \rangle$ degenerate)

(FM''-3) $M(x, y, z, w, t) = M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) = \ldots$

(FM''-4) $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2)$

$* M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$

(FM''-5) $M(x, y, z, w, .) : [0,1) \to [0,1]$ is left continuous.

Definition 2.2.1.15: Let $(X, M, *)$ be a fuzzy 3-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 3-metric space x is said to be convergent to a point $x \in X$, if
$$lim_{n\to\infty} M(x_n, x, a, b, t) = 1$$
for all $a, b \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if
$$lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$$
for all $a, b \in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.2.1.16: A function M is continuous in fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$

$$lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$$

for all $a, b \in X$ and $t > 0$.

Definition 2.2.1.17: Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, b, t)$$

for all $u, a, b \in X$ and $t > 0$.

Definition2.2.1.18: Throughout this paper $(\boldsymbol{\Omega}, \boldsymbol{\Sigma})$ denotes a measurable space. $\boldsymbol{\xi} : \boldsymbol{\Omega} \to \mathbf{X}$ is a measurable selector. X is any non empty set. $\star$ is continuous t-norm, **M** is a fuzzy set in $X^2 \times [0, \infty)$

A binary operation $*:[0,1]x[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian Topological monodies with unit 1 such that $a * b \geq c * d$ whenever

$a \geq c$ and $b \geq d$, For all $a, b, c, d, \in [0, 1]$

Example of t-norm are $a * b = a\,b$ and $a * b = \min\{a, b\}$

Definition2.2.1.18 (a): The 3-tuple $(X, M, \Omega\ *)$ is called a **Random fuzzy metric space**, if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set in $X^2 x [0,\infty)$

satisfying the following conditions: for all

$\xi x, \xi y, \xi z \in X$ and $s, t > 0$,

$$(RFM-1): M(\xi x, \xi y, 0) = 0$$
$$(RFM-2): M(\xi x, \xi y, t) = 1, \forall t \succ 0, \Leftrightarrow x = y$$
$$(RFM-3): M(\xi x, \xi y, t) = M(\xi y, \xi x, t)$$
$$(RFM-4): M(\xi x, \xi z, t+s) \geq M(\xi x, \xi y, t) * M(\xi z, \xi y, s)$$
$$(RFM-5): M(\xi x, \xi y, \xi a) : [0,1) \to [0,1]\, is\ left\ continuous$$

In what follows, (X, M, Ω ,*) will denote a random fuzzy metric space. Note that M (ξx, ξy, t) can be thought of as the degree of nearness between ξx and ξy with respect to t. We identify ξx = ξy with M (ξx, ξ y, t) = 1 for all t > 0 and M (ξx, ξy, t) = 0 with ∞.In the following example, we know that every metric induces a fuzzy metric.

Example Let (X, d) be a metric space.

Define a $*$b = a b, or ab =min {a, b}) and for all x, y, ∈ X and t > 0,

$$M\left(\xi x, \xi y, t\right) = \frac{t}{t + d\left(\xi x, \xi y\right)}$$

Then (X, M, Ω, *) is a fuzzy metric space. We call this random fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition2.2.1.18 (b): Let (X, M, Ω, *) is a random fuzzy metric space.

(i)A sequence {ξx_n} in X is said to be convergent to a point ξx ∈ X,

$$\lim_{n\to\infty} M(\xi x_n, \xi x, t) = 1$$

(ii) A sequence {ξx_n} in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(\xi x_{n+p}, \xi x_n, t) = 1, \forall t \succ 0 \, and \; p \succ 0$$

(iii) A random fuzzy metric space in which every Cauchy sequence is convergent is said to be

Complete.

Let (X.M,*) is a fuzzy metric space with the following condition.

(RFM-6) $\lim_{t\to\infty} M(\xi x, \xi y, t) = 1, \forall \xi x, \xi y \; \varepsilon \; X$

. **Definition2.2.1.18 (c)**: A function M is continuous in fuzzy metric space iff whenever

$$\xi x_n \to \xi x, \xi y_n \to \xi y \Rightarrow \lim_{n\to\infty} M(\xi x_n, \xi y_n, t) \to M(\xi x, \xi y, t)$$

Definition2.2.1.18 (d): Two mappings A and S on fuzzy metric space X are weakly commuting iff

M (ASξu, SAξu, t) ≥ M (Aξu, Sξu, t)

Some Basic Results 2.2.1.18 (e):

Lemma (i) [Motivated by 19] for all ξx, ξy, ∈ X, M(ξx, ξy) is non -decreasing.

Lemma (ii) Let {ξy_n} be a sequence in a random fuzzy metric space (X, M, Ω,*) with the condition

(RFM -6) If there exists a number q ∈ (0,1) such that

$$M(\xi y_{n+2}, \xi y_{n+1}, qt) \geq M(\xi y_{n+1}, \xi y_n, t) \;, \forall t \succ 0 \, and \; n = 1, 2, 3......,$$
$$then \; \left\{\xi y_n\right\} is \, a \; cauchy \, sequence \, in \; X.$$

Lemma (iii) [Motivated by 32] If, for all ξx, ξy ∈ X, t > 0 and for a number q ∈ (0,1),

$$M(\xi x,\xi y,qt)\geq M(\xi x,\xi y,t), then\, \xi x=\xi y$$

Lemmas 1, 2, 3 of **2.2.1.18 (e)**:) hold for random fuzzy 2-metric spaces and random fuzzy 3-metric spaces also.

Definition2.2.1.18 (f): A binary operation *: [0, 1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever

$a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition2.2.1.18 (g): The 3-tuple (X, M, Ω,*) is called a random fuzzy 2-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in X^3 x [0,∞) satisfying the followings

$$(RFM^{'}-1): M\left(\xi x,\xi y,\xi z,0\right)=0$$

$$(RFM^{'}-2): M\left(\xi x,\xi y,\xi z,t\right)=1, \forall t\succ 0, \Leftrightarrow x=y$$

$$(RFM^{'}-3): M\left(\xi x,\xi y,t\right)=M(\xi x,\xi z,\xi y,t)=M(\xi y,\xi z,\xi x,t), symmetry\, about\, three\, \mathrm{var}\, riable$$

$$(RFM^{'}-4): M\left(\xi x,\xi y,\xi z,t_{1,}\, t_{2,},t_{3,}\right)\geq M(\xi x,\xi y,\xi u,t_{1,})*M(\xi x,\xi u,\xi z,t_{2,})*M(\xi u,\xi y,\xi z,t_{3,})$$

$$(RFM^{'}-5): M(\xi x,\xi y,\xi z):\left[0,1\right)\rightarrow\left[0,1\right]\, is\, left\, continuous, \forall\, \xi x,\xi y,\xi z,\xi u\, \varepsilon\, X, t_1,t_2,t_3\succ 0$$

.

Definition2.2.1.18 (h): Let (X, M, Ω,*) be a random fuzzy 2-metric space. A sequence { ξx_n} in fuzzy 2-metric space X is said to be convergent to a point $\xi x \in X$,

$$\lim_{n\to\infty} M(\xi x_n,\xi x,\xi a,t)=1,\ for\ all\ \xi a\,\varepsilon\, X\ and\ t\succ 0$$

(2) A sequence { ξx_n} in random fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(\xi x_{n+p},\xi x_n,\xi a,t)=1,\ for\ all\ \xi a\,\varepsilon\, X\ and\ t,\, p\succ 0$$

(3) A random fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition2.2.1.18 (i): A function M is continuous in random fuzzy 2-metric space, iff whenever

For all $\xi a \in X$ and t > 0.

$$\xi x_n\rightarrow\xi x,\xi y_n\rightarrow\xi y, then\ \lim_{n\to\infty} M(\xi x_n,\xi y_n,\xi a,t)=M(\xi x,\xi y,\xi a,t),\forall\,\xi a\,\varepsilon\, X\ and\ t\succ 0$$

Definition2.2.1.18 (j): Two mappings A and S on random fuzzy 2-metri space X are weakly commuting iff

$$M(AS\xi u,SA\xi u,\xi a,t)\geq M(A\xi u,S\xi u,\xi a,t),\forall\,\xi u,\xi a\,\varepsilon\, Xand\, t\succ 0$$

Definition2.2.1.18 (k): A binary operation *: $[0, 1]^4 \rightarrow [0, 1]$ is called a continuous t-norm if

([0, 1], *) is an abelian topological monoid with unit 1 such that

$a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$ Whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2 are in [0,1].

Definition2.2.1.18 (l): The 3-tuple (X, M, Ω, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm monoid and M is a fuzzy set in X^4 x [0, ∞] satisfying the following conductions:

$$(RFM^{"}-1): M(\xi x, \xi y, \xi z, \xi w, 0) = 0$$

$$(RFM^{"}-2): M(\xi x, \xi y, \xi z, \xi w, t) = 1, \forall t \succ 0,$$

$$Only\ when\ the\ three simplex \langle x, y, z, w\rangle \deg enerate$$

$$(RFM^{"}-3): M(\xi x, \xi y, \xi z, \xi w, t) = M(\xi x, \xi w, \xi z, \xi y, t) = M(\xi z, \xi w, \xi x, \xi y, t) = ----$$

$$(RFM^{"}-4): M(\xi x, \xi y, \xi z, \xi w, t + t_2 + t_3) \geq M(\xi x, \xi y, \xi z, \xi u, t_{1,}) *$$

$$M(\xi x, \xi y, \xi u, \xi w, t_{2,}) * M(\xi x, \xi u, \xi z, \xi w, t_{3,}) * M(\xi u, \xi y, \xi z, \xi w, t_4)$$

$$(RFM^{"}-5): M(\xi x, \xi y, \xi z, \xi w): [0,1) \to [0,1]\ is\ left\ continuous,$$

$$\forall \xi x, \xi y, \xi z, \xi u, \xi w\ \varepsilon\ X, t_1, t_2, t_3, t_4 \succ 0$$

Definition2.2.1.18 (m): Let (X, M, **Ω**,*) be a Random fuzzy 3-metric space:

(1)A sequence {ξXn} in fuzzy 3-metric space X is said to be convergent to a point ξx ∈ X, if

$$\lim_{n\to\infty} M(\xi x_n, \xi x, \xi a, \xi b, t) = 1,\ for\ all\ \xi a, \xi b\ \varepsilon\ X\ and\ t \succ 0$$

(2)A sequence { ξ x_n} in random fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(\xi x_{n+p}, \xi x_n, \xi a, \xi b, t) = 1,\ for\ all\ \xi a, \xi b\ \varepsilon\ X\ and\ t, p \succ 0$$

(3)A random fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition2.2.1.18 (n): A function M is continuous in random fuzzy 3-metric space if

$$\xi x_n \to \xi x, \xi y_n \to \xi y, then \lim_{n\to\infty} M(\xi x_n, \xi y_n, \xi a, \xi b, t) = M(\xi x, \xi y, \xi a, t), \forall \xi a, \xi b\ \varepsilon\ X\ and\ t \succ 0$$

Definition2.2.1.18 (o): Two mappings A and S on random fuzzy 3-metric space X are weakly commuting iff,

$$M(AS\xi u, SA\xi u, \xi a, \xi b, t) \geq M(A\xi u, S\xi u, \xi a, \xi b, t)\ \forall u, a, b\ \varepsilon\ X and\ t \succ 0$$

2.2.2 Prepositions.

Preposition 2.2.2.1 (Gregori and Sepene 2002)

Let (X, d) be a metric space. The mapping f: X → X is a contractive (a contraction) on the metric space (X, d) with contractive constant k iff f is fuzzy contractive, with contractive constant k, on the standard fuzzy metric space (X, Md, ∗), induced by d.

Preposition 2.2.2.2 (Gregori and Sepene 2002)

Let (X, M, ∗) be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let T: X → X be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.

Preposition 2.2.2.3 (Gregori and Sepene 2002)

Let (X, Md, ∗) be the standard fuzzy metric space induced by the metric d on X. The sequence {x_n} in X is contractive in (X, d) iff {x_n} is fuzzy contractive in (X, Md,∗).

Preposition 2.2.2.1 and 2.2.2.3 imply that Preposition 2.2.2.2 is a generalization of Banach fixed point theorem to fuzzy metric spaces as defined by George and Veermani.

It is to be noted that all the prepositions are true for (RFM)

Now, we state and prove our main theorem as follows,

2.3 Main Results

Theorem 2.3.1 Let $(X, M, \Omega, *)$ be a complete Random fuzzy metric space and let $S, T: X \to X$ be two continuous self mappings , $\xi : \Omega \to X$ is a measurable selector satisfying the following conditions :

$$2M(S\xi x, T\xi y, t) \geq M\left(\xi x, S\xi x, \frac{t_1}{a}\right) + M\left(\xi y, T\xi y, \frac{t_1}{b}\right) \qquad 3.1.1$$

Where $t = t_1 + t_2;\ t_1, t_2 \geq 0; t > 0; a, b > 0$ with $b > a; 0 < a + b < 1$ and $\xi x, \xi y \in X$. Then S and T have a unique common fixed point.

Proof: Let x_0 be any point of X. We define a sequence $\{\xi x_n\}$ recurrently as follows

$$\xi x_{2n+1} = S\xi x_{2n},$$

$$\xi x_{2n+1} = T\xi x_{2n+1}, \qquad n = 0,1,2,3, \ldots$$

Putting $\xi x = \xi x_{2n}, \xi y = \xi x_{2n+1}, t_1 = (1-b)t, t_2 = bt$ in 3.1.1, We have for all $t > 0$ and $n = 0,1,2 \ldots$

$$2M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{(1-b)t}{a}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)$$

Or $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{c}\right)$ for $n = 0,1,2,3, \ldots$

Where $0 < c \left(= \frac{a}{1-b}\right) < 1 \quad (since\ a + b < 1)$

Which implies that when $M(\xi x_{2n+1}, \xi x_{2n+2}, t) = 0$

We must have $M(\xi x_{2n}, \xi x_{2n+1}, t/c) = 0$

If $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \neq 0$, then

$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M(\xi x_{2n}, \xi x_{2n+1}, t/c)$, where $0 < c < 1$ [3.1.2]

For any positivep, we must have

$$M(\xi x_{2n+1}, \xi x_{2n+p+1}, t) \geq M(\xi x_{2n}, \xi x_{2n+1}, t/p) * \ldots \text{p times} \ldots * M(\xi x_{2n+p+1}, \xi x_{2n+p+2}, t/p)$$

$$\geq M(\xi x_0, x\xi_1, t/pc^{2n+p+1}))$$

Since $0 < c <$, therefore, by definition [d.2]

$M(\xi x_{2n+1}, \xi x_{2n+p+1}, t) \to 1$ as $n \to \infty$.

Therefore, by definition [d.4], $\{\xi x_n\}$ is a Cauchy sequence and hence is convergent in the fuzzy metric space X.

Let $\xi x_n \to z$ as $n \to \infty$ [3.1.3]

Then $S\xi x_{2n} = \xi x_{2n+1} \to z$ as $n \to \infty$ [3.1.4]

Similarly $T\xi x_{2n+1} \to z$ as $n \to \infty$ [3.1.5]

Since S and T are continuous, therefore [3.1.3], [3.1.4] and [3.1.5] imply that

$Tz = Sz = z$ [3.1.6]

We next prove the uniqueness. If possible let u and v be two common fixed points of S and T with $u \neq v$. Then putting $t_1 = (1-b)t., t_2 = bt$ in 3.1.1 and proceeding as above and applying 3.1.4 and 3.1.5 in 3.1.2, for all $t > 0$, we get

$M(u,v,t) \geq M\left(u,v,\frac{t}{c}\right)$ where $0 < c < 1$ [3.1.7]

And subsequently, from 3.1.7, for all t>0, we must have

$M(u,v,t) \geq M\left(u,v,\frac{t}{c^2}\right) \geq \cdots \geq M\left(u,v,\frac{t}{c^n}\right) \to 1$ as $n \to \infty$

This shows, that u=v which proves the uniqueness.

Hence the proof is complete.

Theorem [3.2]: Let $(X, M, \Omega, *)$ be a complete Random fuzzy metric space and let $S, T: X \to X$ be two continuous self mapping satisfying the following condition.

$$[2M(S\xi x, T\xi y, t)]^2 \geq \left[M\left(\xi x, S\xi x, \frac{t_1}{e}\right) + M\left(\xi y, T\xi x, \frac{t_1}{a}\right)\right]\left[M\left(\xi y, S\xi y, \frac{t_2}{f}\right) + M\left(S\xi x, T\xi y, \frac{t_2}{b}\right)\right].$$

Where $t = t_1 + t_2;\ t_1, t_2 \geq 0; t > 0; a, b, e, f > 0; a + b = 1; 0 < e < a$ and $\xi x, \xi\xi y \in X$. Then S and T have a unique common fixed point.

Proof: Let ξx_0 be any point of X. We define a sequence a $\{\xi x_n\}$ recurrently as follows

$$\xi x_{2n+1} = S\xi x_{2n},$$

$$\xi x_{2n+1} = T\xi x_{2n+1}, \qquad n = 0,1,2,3, \ldots$$

Putting $\xi x = \xi x_{2n}, \xi y = \xi x_{2n+1}$, $t_1 = at$ and $t_2 = bt$ in [3.2.1] We have, for all $t > 0$ and $n = 0,1,2, \ldots$

$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{at}{e}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right].\left[M\left(\xi x_{2n+1}, \xi x_{2n+1}, \frac{bt}{f}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right].$

Let $k = e/a$, therefore $0 < k < 1$ and hence we have

$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right) + M(1, \xi x_{2n+2}, t)\right].[1 + M(\xi x_{2n+1}, \xi x_{2n+2}, t)].$

For all $t > 0$ *and* $n = 0,1,2,3, \ldots$ [3.2.2]

Which implies that when $M(\xi x_{2n+1}, \xi x_{2n+2}, t) = 0$, we must have $M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{k}\right) = 0$

If $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right)$, where $0 < k < 1$ [3.2.3]

For any positive integer p, we must have

$$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{p}\right) * \ldots \text{p times} \ldots * M\left(\xi x_{2n+p+1}, \xi x_{2n+p+2}, \frac{t}{p}\right)$$

$$\geq M\left(\xi x_0, \xi x_1, \frac{t}{(pk^{2n+1})}\right) * \ldots \text{p times} \ldots * M\left(\xi x_0, \xi x_1, \frac{t}{(pk^{2n+p+1})}\right)$$

Since $0 < k < 1$, therefore, by definition [d. 2]

$M(\xi x_{2n+1}, \xi x_{2n+p+1}, t) \to 1$ as $n \to \infty$

Therefore, by definition[d. 4], it implies that $\{\xi x_n\}$ is a Cauchy sequence and hence is convergent in the fuzzy metric space X.

Let $\xi x_n \to \xi z$ as $n \to \infty$ [3.2.4]

Then $S\xi x_{2n} = \xi x_{2n+1} \to \xi z$ as $n \to \infty$ [3.2.5]

Similarly $Tx_{2n+1} \to \xi z$ *as* $n \to \infty$ [3.2.6]

Since S and T are continuous, therefore [3.2.4] [3.2.5] [3.2.6] imply that

$$T\xi z = S\xi z = \xi z \tag{3.2.7}$$

We next prove the uniqueness. If possible let ξu and ξv be two common fixed points of S and T with u≠v. Then putting t_1=at, $t_2 = bt$ in [3.2.1] and proceeding as above and applying [3.2.5] and [3.2.6] in [3.2.2], for all t>0, we get

$$[2M(\xi u, \xi v, t)]^2 \geq \left[M\left(\xi u, \xi v, \frac{t}{k}\right) + M(\xi u, \xi v, t)\right].[1 + M(\xi u, \xi v, t)] \quad \text{where } 0<k<1 \quad [3.2.8]$$

Now, as shown in [3.2.3] for all t>0, in the similar manner, from [3.2.8],

We must have

$$M(\xi u, \xi v, t) \geq M\left(\xi u, \xi v, \frac{t}{k}\right) \quad \text{where } 0<k<1 \quad [3.2.9]$$

And subsequently, from [3.2.9] for all t>0, we must have

$$M(\xi u, \xi v, t) \geq M\left(\xi u, \xi v, \frac{t}{k^2}\right) \geq \cdots \geq M\left(\xi u, \xi v, \frac{t}{k^n}\right) \to 1 \text{ as } n \to \infty.$$

This shows, by definition that ξu=ξv which proves the uniqueness.

This completes the proof of the theorem.

Theorem 3.3: Let $(X, M, \Omega,*)$ be a complete fuzzy metric space and let $S, T: X \to X$ be two continuous self mappings satisfying the following condition

$$[2M(S\xi x, T\xi y, t)]^3 \geq \left[M\left(\xi x, \xi y, \frac{t_1}{e}\right) + M\left(\xi y, S^2\xi x, \frac{t_1}{a}\right)\right].\left[M\left(\xi y, S\xi x, \frac{t_2}{f}\right) + M\left(S\xi x, T\xi y, \frac{t_2}{b}\right)\right]$$
$$.\left[M\left(T\xi y, S^2\xi x, \frac{t_3}{g}\right) + M\left(\xi y, T\xi y, \frac{t_3}{c}\right)\right] \quad [3.3.1]$$

Where

$t = t_1 + t_2 + t_3; t_1, t_2, t_3 \geq 0; t > 0; a, b, c, e, f, g > 0; a + b + c = 1; 0 < e < a \text{ and } \xi x, \xi\xi y \in X$. Then S and T have a unique common fixed point.

Proof: Let x_0 be any point of X. We define a sequence $\{\xi x_n\}$ recurrently as follows

$$\xi x_{2n+1} = S\xi x_{2n},$$
$$\xi x_{2n+1} = T\xi x_{2n+1}, \quad n = 0,1,2,3, \ldots$$

Putting $\xi x = \xi x_{2n}$,ξy=ξx_{2n+1}, $t_1 = \text{at}$ and $t_2 = \text{bt}$ $t_3 = \text{ct}$ in [3.3.1] we have, for all $t > 0$ and $n = 0,1,2, \ldots$

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^3 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{at}{e}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$
$$.\left[M\left(\xi x_{2n+1}, \xi x_{2n+1}, \frac{bt}{f}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$
$$.\left[M\left(\xi x_{2n+2}, \xi x_{2n+2}, \frac{ct}{g}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$

Let $k = e/a$, therefore $0 < k < 1$ and hence we have

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^3 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right) + M(\xi x_{2n+2}, \xi x_{2n+2}, t)\right]$$
$$.[1 + M(\xi x_{2n+1}, \xi x_{2n+2}, t)].[1 + M(\xi x_{2n+1}, \xi x_{2n+2}, t)]$$

For all $t > 0$ *and* $n = 0,1,2,3, \dots$ [3.3.2]

Which implies that when $M(\xi x_{2n+1}, \xi x_{2n+2}, t) = 0$, we must have $M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{k}\right) = 0$

If $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \neq 0$then from [3.3.2], it is obvious that

$$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right), \quad \text{where } 0 < k < 1 \qquad [3.2.3]$$

For any positive integer p, we have

$$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{p}\right) * \dots \text{p times} \dots * M\left(\xi x_{2n+p+1}, \xi x_{2n+p+2}, \frac{t}{p}\right)$$

$$\geq M\left(\xi x_0, \xi x_1, \frac{t}{(pk^{2n+1})}\right) * \dots \text{p times} \dots * M\left(\xi x_0, \xi x_1, \frac{t}{(pk^{2n+p+1})}\right)$$

Since $0 < k < 1$, therefore, by definition [d. 2] we have

$M\big(\xi x_{2n+1}, \xi x_{2n+p+1}, t\big) \to 1$ as $n \to \infty$

Therefore, by definition[d. 4], it implies that $\{\xi x_n\}$ is a Cauchy sequence and hence is convergent in the fuzzy metric space X.

Let $\xi x_n \to \xi z$ as $n \to \infty$ [3.3.4]

Then $S\xi x_{2n} = \xi x_{2n+1} \to \xi z$ as $n \to \infty$ [3.3.5]

Similarly $T\xi x_{2n+1} \to \xi z \text{ as } n \to \infty$ [3.3.6]

Since S and T are continuous, therefore [3.3.4] [3.3.5] [3.3.6] imply that

$$T\xi z = S\xi z = \xi z \qquad [3.3.7]$$

We next prove the uniqueness. If possible let u and v be two common fixed points of S and T with u≠v. Then putting t_1=at, $t_2 = bt$, $t_3 = ct$ in [3.3.1] and proceeding as above for all t>0, we get

$$[2M(\xi u, v, t)]^2 \geq \left[M\left(\xi u, \xi v, \frac{t}{k}\right) + M(\xi u, \xi v, t)\right].[1 + M(\xi u, \xi v, t)].[1 + M(\xi u, \xi v, t)]$$

where 0<k<1 [3.3.8]

Now, as shown inabove for all t>0, in the similar manner, from [3.3.8],

We must have

$$M(\xi u, \xi v, t) \geq M\left(\xi u, \xi v, \frac{t}{k}\right) \qquad \text{where } 0<k<1 \qquad [3.2.9]$$

And subsequently, from [3.3.9] we have

$$M(\xi u, \xi v, t) \geq M\left(\xi u, \xi v, \frac{t}{k^2}\right) \geq \dots \geq M\left(\xi u, \xi v, \frac{t}{k^n}\right) \to 1 \text{ as } n \to \infty.$$

This shows, by definition that ξu=ξv which proves the uniqueness.

Hence the theorem is proved.

Theorem [3.4]: Let $(X, M, \Omega, *)$ be a complete Random fuzzy metric space and let $R, S, T: X \to X$ be three continuous self mapping satisfying the following condition:

$$[2M(RS\xi x, ST\xi y, t)]^2 \left[M\left(\xi x, \xi y, \frac{t_1}{e}\right) + M\left(\xi y, ST\xi y, \frac{t_1}{a}\right)\right].\left[M\left(\xi y, RS\xi x, \frac{t_2}{f}\right) + M\left(RS\xi x, ST\xi y, \frac{t_2}{b}\right)\right].$$

[3.4.1]

Where $t = t_1 + t_2;\ t_1, t_2 \geq 0; t > 0; a, b, e, f > 0; a + b = 1; 0 < e < a$ and $\xi x, \xi y \in X$.

$\xi RS = \xi SR$ and $\xi ST = \xi TS$ [3.4.2]

Then R, S and T have a unique common fixed point.

Proof: Let ξx_0 be any point of X. We define a sequence $\{\xi x_n\}$ recurrently as follows

$$\xi x_{2n+1} = RS\xi x_{2n},$$

$$\xi x_{2n+1} = ST\xi x_{2n+1}, \qquad n = 0,1,2,3, \dots$$

Putting $\xi x = \xi x_{2n}, \xi y = \xi x_{2n+1}$, $t_1 = at$ and $t_2 = bt$ in [3.4.1] we have, for all $t > 0$ and $n = 0,1,2, \dots$

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{at}{e}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$

$$.\left[M\left(\xi x_{2n+1}, \xi x_{2n+1}, \frac{bt}{f}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$

Let $k = e/a$, therefore $0 < k < 1$ and hence we have

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right) + M(\xi x_{2n+2}, \xi x_{2n+2}, t)\right].[1 + M(\xi x_{2n+1}, \xi x_{2n+2}, t)]$$

For all $t > 0$ *and* $n = 0,1,2,3, \dots$ [3.4.3]

Which implies that when $M(\xi x_{2n+1}, \xi x_{2n+2}, t) = 0$, we must have $M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right) = 0$

If $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \neq 0$then from[3.4.3], it is obvious that

$$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right), \qquad \text{where } 0 < k < 1 \qquad [3.3.4]$$

Similarly,

$M(\xi x_{2n+1}, \xi x_{2n+3}, t) \geq M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{k_1}\right)$ for $n = 0,1,2 \dots$

Where $0 < k_1 < 1$ [3.4.5]

We can continue this process, which in turn implies that $\{\xi x_n\}$ is a Cauchy sequence and hence is convergent.

Let $\xi x_n \to \xi u$ as $n \to \infty$ then for $t > 0$, we have

$M(\xi u, ST\xi u, t) \geq M(\xi u, \xi x_{2n+1}, t_3) * M(\xi x_{2n+1}, ST\xi u, t_4)$, where $t_3 + t_4 = t$

Or $M(u, STu, t) \geq M(u, \xi x_{2n+1}, t_3) * M(RS\xi x_{2n}, STu, t_4)$ [3.4.6]

And hence, from[3.4.6], we have

$$[2M(\xi u, ST\xi u, t)]^2 \geq [M(\xi u, \xi x_{2n+1}, t_3) * 2M(RS\xi x_{2n}, ST\xi u, t_4)]^2$$

Now from[3.4.1], we have

$$[2M(\xi u, ST\xi u, t)]^2 \geq [M(\xi u, \xi x_{2n+1}, t_3)]^2 * \left\{\left[M\left(\xi u, \xi x_{2n}, \frac{t_5}{e}\right) + M\left(\xi u, ST\xi u, \frac{t_5}{a}\right)\right]\right.$$

$$\left..\left[M\left(u, RS\xi x_{2n}, \frac{t_6}{f}\right) + M(RS\xi x_{2n}, ST\xi u, \frac{t_6}{b})\right]\right\}$$

Where $t_5 + t_6 = t_4$ and $t_5, t_6 \geq 0$ [3.4.7]

Since $RS\xi x_{2n} = \xi x_{2n+1}$ and $\xi x_n \to u$ as $n \to \infty$, therefore making $n \to \infty$, from[3.4.7], we get

$[2M(\xi u, ST\xi u, t)]^2 \geq 1 * [1 + M(\xi u, ST\xi u, \frac{t_5}{a})].[1 + M(\xi u, ST\xi u, \frac{t_6}{b})]$

Therefore as shown in [3.4.4] and [3.4.5], we must have in succession

$M(\xi u, ST\xi u, t) \geq M(\xi u, ST\xi u, t/k^n)$ for $n = 0,1,2 \ldots$ [3.4.8]

As $0 < k < 1$, therefore making $n \to \infty$, from [3.4.8] we have

$M(\xi u, ST\xi u, t) = 1$ for all $t > 0$

Therefore by definition [d. 2] and [d. 3], we have

$ST\xi u = \xi u$ [3.4.9]

Similarly $RS\xi u = \xi u$ [3.4.10]

Now $M(R\xi u, \xi u, t) = M(R(RS\xi u), ST\xi u, t)$

$= M(RS(R\xi u), ST\xi u, t)$

(since, from[3.4.2], $\xi RS = \xi SR$)

Now from [3.4.1], we have

$$[2M(RS(R\xi u), ST\xi u, t)]^2 \geq \left[M\left(R\xi u, \xi u, \frac{t_1}{e}\right) + M\left(\xi u, ST\xi u, \frac{t_1}{a}\right)\right]$$

$$.\left[M\left(\xi u, RS(R\xi u), \frac{t_2}{f}\right) + M\left(RS(R\xi u), ST\xi u, \frac{t_2}{b}\right)\right] \quad [3.4.11]$$

Repeating the same as we have done in [3.4.3] , in [3.4.11], we get

$$[2M(RS(R\xi u), ST\xi u, t)]^2 \geq \left[M\left(RS(R\xi u), ST\xi u, \frac{t}{k}\right) + 1\right].[M(ST\xi u, RS(R\xi u), t) + M(RS(R\xi u), ST\xi u, t)]$$

Where 0<k<1 (applying) [3.4.9]&[3.4.10] or $M(RS(R\xi u), ST\xi u, t) \geq M\left(RS(R\xi u), ST\xi u, \frac{t}{k}\right)$

Which in turn implies that

$M(RS(R\xi u), ST\xi u, t) \geq M(RS(R\xi u), ST\xi u, t^n)$

Making $n \to \infty$, we get

$M(RS(R\xi u), ST\xi u, t) = 1$ for all t>0, (as 0<k<1)

i.e. $M(R\xi u, \xi u, t) = 1$ for all t>0

i.e. $R\xi u = \xi u$

In the similar manner, we can have

$S\xi u = T\xi u = \xi u$

$\therefore$ u is the common fixed point of R,S and T.

Now to prove the uniqueness of the fixed point, If u_1 and u_2 are two common fixed points of R,S and T with $\xi u_1 \neq \xi u_2$, then using [3.4.1] and proceeding as above, we have, $M(u_1, u_2, t) \geq 1$ for all t>0. That is $u_1 = u_2$.

This completes the proof of the theorem.

Theorem [3.5][Deduction]: Let $(X, M, \Omega,*)$ be a complete fuzzy metric space and let $R, S, T: X \to X$ are three continuous self mappings satisfying the following conditions:

$$2M(S\xi x, T\xi y, t)]^3 \geq \left[M\left(\xi x, \xi y, \frac{t_1}{e}\right) + M\left(\xi y, ST\xi x, \frac{t_1}{a}\right)\right].\left[M\left(\xi y, RSx, \frac{t_2}{f}\right) + M\left(RSx, ST\xi y, \frac{t_2}{b}\right)\right]$$

$$.\left[M\left(RS\xi y, ST\xi y, \frac{t_3}{g}\right) + M\left(\xi y, RS\xi y, \frac{t_3}{c}\right)\right] \qquad [3.5.1]$$

Where

$t = t_1 + t_2 + t_3; t_1, t_2, t_3 \geq 0; t > 0; a, b, c, e, f, g > 0; a + b + c = 1; 0 < e < a \text{ and } \xi x, \xi y \in X.$

$RS = SR$ and $ST = TS$ [3.5.2]

Then R, S and T have a unique common fixed point if $RS = ST$.

Theorem[3.6]: Let $(X, M, \Omega,*)$ be a complete fuzzy metric space and let $P, Q, R, S: X \to X$ are four continuous self mapping satisfying the following conditions:

$$[2M(PQ\xi x, RS\xi y, t)]^2 \geq \left[M\left(\xi x, \xi y, \frac{t_1}{e}\right) + M\left(\xi y, RS\xi y, \frac{t_1}{a}\right)\right].[M\left(\xi y, PQ\xi x, \frac{t_2}{f}\right) + M(PQ\xi x, RS\xi y, \frac{t_2}{b})]$$

Where $t_1, t_2 \geq 0; t = t_1 + t_2, t > 0; a, b, e, f, > 0; a + b = 1; 0 < e < a \text{ and } \xi x, \xi y \in X.$

$PQ = QP$ and $RS = SR$ [3.5.2]

Then P, Q, R and S have a unique common fixed point.

Proof: Let $\xi x_0 \in X$. We construct a sequence $\{\xi x_n\}$ recurrently as follows:

$$\xi x_{2n+1} = PQ\xi x_{2n},$$

$$\xi x_{2n+1} = RS\xi x_{2n+1}, \qquad n = 0,1,2,3, \ldots$$

Putting $\xi x = \xi x_{2n}, \xi y = \xi x_{2n+1}$, $t_1 = at$ and $t_2 = bt$ in [3.6.1] we have, for all $t > 0$ and $n = 0,1,2, \ldots$

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{at}{e}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$

$$.\left[M\left(\xi x_{2n+1}, \xi x_{2n+1}, \frac{bt}{f}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right]$$

Let $k = e/a$, therefore $0 < k < 1$ and hence we have

$$[2M(\xi x_{2n+1}, \xi x_{2n+2}, t)]^2 \geq \left[M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right) + M(\xi x_{2n+1}, \xi x_{2n+2}, t)\right].[1 + M(\xi x_{2n+1}, \xi x_{2n+2}, t)]$$

For all $t > 0$ *and* $n = 0,1,2,3, \ldots$ [3.6.3]

Which implies that when $M(\xi x_{2n+1}, \xi x_{2n+2}, t) = 0$, we must have $M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{k}\right) = 0$

If $M(\xi x_{2n+1}, \xi x_{2n+2}, t) \neq 0$then from [3.6.3], it is obvious that$M(\xi x_{2n+1}, \xi x_{2n+2}, t) \geq M\left(\xi x_{2n}, \xi x_{2n+1}, \frac{t}{k}\right)$, where $0 < k < 1$ [3.6.4]

Similarly,

$$M(\xi x_{2n+2}, \xi x_{2n+3}, t) \geq M\left(\xi x_{2n+1}, \xi x_{2n+2}, \frac{t}{k_1}\right) \qquad \text{for } n = 0,1,2, \ldots$$

Where $0 < k_1 < 1$ [3.6.5]

We can continue this process, which in turn implies that $\{\xi x_n\}$ is a Cauchy sequence and hence it is convergent.

Let $\xi x_n \to \xi u$ as $n \to \infty$, then for $t > 0$, we have

$$M(\xi u, RS\xi u, t) \geq M(\xi u, \xi x_{2n+1}, t_3) * M(\xi x_{2n+1}, RS\xi u, t_4), \text{where } t_3 + t_{4=}t$$

Or $M(\xi u, RS\xi u, t) \geq M[(\xi u, \xi x_{2n+1}, t_3) * M(PQ\xi x_{2n}, RS\xi u, t_4)]^2$ [3.6.6]

And hence, from [3.6.6], we have

$[2M(\xi u, RS\xi u, t)]^2 \geq [M(\xi u, \xi x_{2n+1}, t_3)] * 2M(PQ\xi x_{2n}, RS\xi u, t_4)]^2$

Now from[3.6.1], we have

$$[2M(\xi u, RS\xi u, t)]^2 \geq [M(\xi u, \xi x_{2n+1}, t_3)]^2 * \{\left[M\left(\xi x_{2n}, \xi u, \frac{t_5}{e}\right) + M\left(\xi u, RS\xi u, \frac{t_5}{a}\right)\right]$$
$$. M\left(\xi u, PQ\xi x_{2n}, \frac{t_6}{f}\right) + M(PQ\xi x_{2n}, RS\xi u, \frac{t_6}{b})\}$$

Where $t_5 + t_6 = t = t_4$ and $t_5, t_6 \geq 0$ [3.6.7]

Since $PQ\xi x_{2n} = \xi x_{2n+1}$ and $\xi x_{2n} \to \xi u$ as $n \to \infty$, therefore making $n \to \infty$, from [3.6.7], we get

$[2M(\xi u, RS\xi u, t)]^2 \geq 1 * \{\left[1 + M\left(\xi u, RS\xi u, \frac{t_5}{a}\right)\right]. [1 + M(\xi u, RS\xi u, \frac{t_6}{b})$

Therefore as shown in [3.6.4] and [3.6.5], we must have in succession

$$M(\xi u, RS\xi u, t) \geq M\left(\xi u, RS\xi u, \frac{t}{k^n}\right) \qquad \text{for } n = 0,1,2 \ldots \qquad [3.6.8]$$

As $0 < k < 1$, *therefore making* $n \to \infty$, from[3.6.8], we have

$M(\xi u, RS\xi u, t) = 1$ for all $t > 0$

Therefore by definition [d. 2]and [d. 3], we have

$RS\xi u = \xi u$ [3.6.9]

Similarly, $PQ\xi u = \xi u$ [3.6.10]

Now $M(P\xi u, \xi u, t) = M(P(PQ\xi u), RS\xi u, t)$

$= M(PQ(P\xi u), RS\xi u, t)$

(Since, from[3.6.2], $\xi PQ = \xi QP$)

Now from [3.6.1], we have

$$[2M(PQ(P\xi u), RS\xi u, t)]^2 \geq \left[M\left(P\xi u, \xi u, \frac{t_1}{e}\right) + M\left(\xi u, RS\xi u, \frac{t_1}{a}\right)\right]$$
$$. \left[M\left(\xi u, PQ(P\xi u), \frac{t_2}{f}\right) + M\left(PQ(P\xi u), RS\xi u, \frac{t_2}{f}\right)\right] \qquad [3.6.11]$$

Repeating the same as we have done in [7.6.3],in [7.6.11], we get

$$[2M(PQ(P\xi u), RS\xi u, t)]^2 \geq \left[M\left(PQ(P\xi u), RS\xi u, \frac{t}{k}\right) + 1\right]. [M(RS\xi u, PQ(P\xi u), t) + M(PQ(P\xi u), RS\xi u, t)]$$

Where $0 < k < 1$ (applying[3.6.9]&[3.6.10])

Or $M(PQ(P\xi u), RS\xi u, t) \geq M(PQ(P\xi u), RS\xi u, t/k^n)$

Making $n \to \infty$, we get

$$M(PQ(P\xi u), RS\xi u, t) = 1 \text{ for all } t > 0, (as\ 0 < k < 1)$$

i.e. $M(P\xi u, \xi u, t) = 1$ for all $t > 0$

i.e. $P\xi u = \xi u$

In the similar manner we can have

$$Q\xi u = R\xi u = S\xi u = \xi u$$

∴ u is the common fixed point of P,Q,R and S. Now to prove the uniqueness of the fixed point, let u_1 and u_2 are two common fixed points of P,Q,R and S with $u_1 \neq u_2$, then using 3.6.1 and proceeding as above, we have, $M(u_1, u_2, t) \geq 1$ for all t>0, that is $u_1 = u_2$.

This completes the proof of the theorem.

Theorem[3.7] [Deduction]: Let $(X, M, \Omega, *)$ be a complete random fuzzy metric space and let, $P, Q, R, S: X \to X$ are four continuous self mappings satisfying the following conditions:

$$[2M(PQ\xi x, RS\xi y, t)]^3 \geq \left[M\left(\xi x, \xi y, \frac{t_1}{e}\right) + M\left(\xi y, RS\xi y, \frac{t_1}{a}\right)\right]$$

$$.\left[M\left(\xi y, PQ\xi x, \frac{t_2}{f}\right) + M(PQ\xi x, RS\xi y, \frac{t_2}{b})\right].\left[M\left(PQ\xi y, RS\xi y, \frac{t_3}{g}\right) + M(\xi y, PQ\xi y, \frac{t_3}{c})\right]$$

Where $t_1, t_2, t_3 \geq 0; t = t_1 + t_2 + t_3, t > 0; a, b, c, e, f, g > 0; a + b + c = 1; 0 < e < a\ and\ \xi x, \xi y \in X$.

$$\xi PQ = \xi QP \text{ and } \xi RS = \xi SR$$

Then P,Q,R and S have a unique common fixed point if ξPQ=ξRS.

References

1. Badard, R. (1984): Fixed point theorems for fuzzy numbers, Fuzzy sets and systems, 13, 291-302.
2. Bose, B. K., Sahani, D.(1987): Fuzzy mappings and fixed point theorems, Fuzzy sets and systems, 21, 53-58.
3. Branciari, A. (2002): A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Sci, 29 no.9, 531-536.
4. Butnariu, D. (1982): Fixed point for fuzzy mappings, Fuzzy sets and systems, 7, 191-207.
5. Chang, S.S. (1985): Fixed point theorems for fuzzy mappings, Fuzzy sets and systems, 17, 181-187.
6. Chang, S.S., Cho, Y. J., Lee, B. S., Lee, G. M.(1997): Fixed degree and fixed point theorems for fuzzy mappings, Fuzzy sets and systems, 87(3), 325-334.
7. Chang, S.S., Cho, Y. J., Lee, B. S., Jung, J. S., Kang, S. M.(1997): Coincidence point and minimization theorems in fuzzy metric spaces, Fuzzy sets and systems, 88(1), 119-128.
8. Dubey, R.P., Bhardwaj ,R.K., Tiwari, Neeta, Tiwari, Ankur [2013]"Fixed Point Theorems in Random Fuzzy Metric Space ThroughRational Expression" Computer Engineering and Intelligent Systems Vol.4, No.9,14-32
9. Deng, Z. (1982): Fuzzy pseudo-metric space, J. Math. Anal. Appl., 86, 74-95.
10. Dey, D., Ganguly, A., and Saha, M., (2011) Fixed point theorems for mappings under general contractive condition of integral type, Bulletin of Mathematical Analysis and Applications, 3(1) 27-34.
11. Ekland, I., Gahler, S. (1988): Basic notions for fuzzy topology, Fuzzy sets and systems, 26, 333-356.
12. Erceg, M.A. (1979): Metric space in fuzzy set theory, J. Math. Anal. Appl., 69, 205-230.
13. Fang, J. X. (1992): On fixed point theorems in fuzzy metric spaces, Fuzzy sets and systems, 46, 106-113.
14. Gahler, S. (1983): 2-Metric space and its topological structure, Math. Nachr., 26, 115-148.
15. Gahler, S. (1964): Linear 2-Metric space, Math. Nachr., 28, 1-43.

16. Gahler, S. (1969): 2-Banach space, Math. Nachr., 42, 335-347.
17. Gupta, R, Dhagat, V., Shrivastava, R.,(2010) Fixed point theorem in fuzzy random spaces, International J. contemp. Math. Sciences, Vol 5, , No. 39, pp.1943-1949.
18. George, A., and Veermani, P. (1994): On some results in fuzzy metric spaces, Fuzzy sets and Systems 64,395.
19. Grabiec, M. (1988): Fixed points in fuzzy metric space, Fuzzy sets and systems, 27, 385-389.
20. Gregori, V., and Sepena, A., (2002): On fixed point theorems in fuzzy metric spaces, Fuzzy sets and Systems 125,245.
21. Hadzic, O. (1989): Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces, Fuzzy sets and systems, 29, 115-125.
22. Heilpern, S. (1981): Fuzzy mappings and fixed point theorems, J. Math. Anal. Appl., 83, 566-569.
23. H. Aydi (2012): A. Fixed point theorem for a contractive condition of integral type involving altering distence. Int. nonlinear Anal.Appl.3 42-53.
24. Jung, J. S., Cho, Y. J., Kim, J.K. (1994): Minimization theorems for fixed point theorems in fuzzy metric spaces and applications, Fuzzy sets and systems, 61, 199-207.
25. Jung, J. S., Cho, Y. J., Chang, S. S., Kang, S. M. (1996): Coincidence theorems for set-valued mappings and Ekland’s variational principle in fuzzy metric spaces, Fuzzy sets and systems, 79, 239-250.
26. Kaleva, O., Seikkala, S. (1984): On fuzzy metric spaces, Fuzzy Sets and Systems, 12, 215-229.
27. Kaleva, O. (1985): The completion of fuzzy metric spaces, J. Math. Anal. Appl., 109, 194-198.
28. Kramosil, J. and Michalek, J. (1975): Fuzzy metric and statistical metric spaces, Kymbernetica, 11,330.
29. Kumar, S., Chugh, R., and Kumar, R., (2007) Fixed point theorem for compatible mappings satisfying a contractive condition of integrable type, Soochow Journal Math. 33(2) 181-185.
30. Kumar, Sanjay and Chugh, Renu, (2001): Common fixed point for three mappings under semi-compatibility condition, The Mathematics Student, 70, 1-4,133.
31. Lee, B. S., Cho, Y. J., Jung, J.S. (1966): Fixed point theorems for fuzzy mappings and applications, Comm. Korean Math. Sci., 11, 89-108.
32. Mishra, S.N., Sharma, N., Singh, S. L. (1994): Common fixed points of maps on fuzzy metric spaces, Internet. J. Math. & Math. Sci., 17, 253-258.
33. Rhoades, B.E., (2003) Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, Int. J. Math. Sci. 63, 4007-4013.
34. Sharma, P.L., Sharma, B.K., Iseki, K. (1976): Contractive type mapping on 2-metric space, Math. Japonica, 21, 67-70.
35. Sharma, Sushil (2002) : On Fuzzy metric space, Southeast Asian Bulletin of Mathematics 26: 133-145
36. Tamilarasi, A and Thangaraj, P. (2003): Common fixed point for three operator, The Journal of fuzzy Mathematics, 11, 3,717.
37. Volker, Kratsckhmer, “A unified approach to fuzzy-random-variables” Seminar notes in Statistik and Oonometrie Fachbereich Wirtschaftswissenschaft, Universitat des Saarlandes Saarbrucken, Germany, 1998, pp. 1-17.
38. Wenzhi, z. (1987): Probabilistic 2-metric spaces, J. Math. Research Expo, 2, 241-245.
39. Zadeh, L.A. (1965): Fuzzy Sets, Inform. And Control, 8, 338-353.