

RATIONAL TYPE CONTRACTION MAPPING IN $T-$ ORBITALLY COMPLETE METRIC SPACE

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Abstract

Our aim of this paper is to obtain some common fixed point theorems in $T-$ orbitally metric space satisfying different rational contractive conditions.

Key Words: T-orbital metric space, fixed point, common fixed point

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2. **Introduction:** One of the most popular generalizations of metric space is $T-$ orbitally metric space.

In 1997 Alber and Guerre- Delabriere [1] introduced the concept of weakly contractive map in Hilbert space and proved the existence of fixed point results. Rhoades [4] extended this concept in Banach space and established the existence of fixed points. Our aim of this paper is to obtain some common fixed point in $T-$ orbitally metric space satisfying different rational contractive conditions

3. Preliminaries:

Throughout this paper (X, d) is a metric space which we denote simply by X.

We denote, $R^+ = [0, \infty)$ is positive real number, N the set of natural number and R the set of real number. We write $\Phi = \{ \psi : R^+ \to R^+ \}$ where ψ setisfies following conditions;

i. ψ is continuous (ii) ψ is non decreasing(iii)$\psi(t) > 0$ for $t > 0$
ii. $\psi(0) = 0$

Definition 3.1.1 For any $x_0 \in X$; $O(x_0) = \{ T^n x_o ; n = 0,1,2,3 \dots \dots \}$ is said to the orbit of x_0 where, $T^0 = I$, is the identity map of X. $\overline{O(x_0)}$ represent the closer of $O(x_0)$.

A metric space X is said to be $T-$orbitally complete; if every Cauchy sequence Which is contained in $O(x)$ for all $x \in X$ converges to the point of X.

Here we note that every complete metric space is $T-$ orbitally complete for any T, but converges is not true.

Definition 3.1.2 Let A and S be the mapping from a metric space X into itself, then the mapping is said to weakly compatible if they are commute at their coincidence points, that is,

$$Ax = Sx \text{ implies that } ASx = SAx.$$

Definition 3.1.3 A self map $T: X \to X$ is said to be generalized weakly contractive map if there exists a $\psi \in \Phi$ such that,

$$d(Tx, Ty) \le d(x, y) - \psi\big(d(x, y)\big)$$

with $\lim_{t\to\infty} \psi(t) = 0$ for all $x, y \in X$.

In this paper we prove some more common fixed point results in $T-$ orbitally complete metric space. This paper is divided into two sections. In first section we prove some common fixed point theorems satisfying weakly rational symmetric contractive conditions and in section two we prove some common fixed point theorems satisfying integral type rational symmetric contractive conditions.

3.2 RATIONAL CONTRACTIVE CONDITION AND COMMON FIXED POINT THEOREM ON T − ORBITALLY COMPLETE METRIC SPACE

In this section we prove a common fixed point result satisfying rational contractive condition in T-orbitally complete metric space. We also give some corollaries which is equivalent to the proved theorem. In fact we prove following theorem.

Theorem 3.2.1 Let (X,d) be a $T-$ orbitally complete metric space, if A, B, S, T be the self mapping of X into itself such that;

3.2.1(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.2.1(ii) The pair (A,S) and (B,T) are weakly compatible and generalized weakly contractive map.

3.2.1(iii) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax,By) \leq k.M(x,y) - \psi(M(x,y))$$

$$\text{Where , } M(Ax,By) = \max\left\{\begin{matrix} \frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Ax,Sx)+d(By,Ty)}, \\ \frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,By)+d(Ax,Ty)}, \\ d(Ax,Sx), d(By,Ty), \\ d(Sx,By), d(Ax,Ty), \\ d(Sx,Ty) \end{matrix}\right\}.$$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof:- We suppose that, $x_0 \in X$ arbitrary and we choose a point $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1 \text{ and } y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \ldots\ldots\ldots$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 3.2.1(iii) we have,

$$d(y_{2n},y_{2n+1}) \leq k.M(Ax_{2n},Bx_{2n+1}) - \psi(M(Ax_{2n},Bx_{2n+1}))$$

$$d(y_{2n},y_{2n+1}) \leq k\max\left\{\begin{matrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bx_{2n+1},Tx_{2n+1})}{1+d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1})+d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Bx_{2n+1})+d(Ax_{2n},Tx_{2n+1})} \\ d(Ax_{2n},Sx_{2n}), d(Bx_{2n+1},Tx_{2n+1}), \\ d(Sx_{2n},Bx_{2n+1}), d(Ax_{2n},Tx_{2n+1}), \\ d(Sx_{2n},Tx_{2n+1}), \end{matrix}\right\}$$

$$-\psi\left(\max\left\{\begin{matrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bx_{2n+1},Tx_{2n+1})}{1+d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1})+d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Bx_{2n+1})+d(Ax_{2n},Tx_{2n+1})} \\ d(Ax_{2n},Sx_{2n}), d(Bx_{2n+1},Tx_{2n+1}), \\ d(Sx_{2n},Bx_{2n+1}), d(Ax_{2n},Tx_{2n+1}), \\ d(Sx_{2n},Tx_{2n+1}), \end{matrix}\right\}\right)$$

$$d(y_{2n},y_{2n+1}) \leq k \max\left\{\begin{array}{c}\frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})}\\ \frac{d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ \frac{d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ d(y_{2n-1},y_{2n}),\end{array}\right\}$$

$$-\psi\left(\max\left\{\begin{array}{c}\frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})},\\ \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}\\ d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),\\ d(y_{2n-1},y_{2n+1}),d(y_{2n},y_{2n}),\\ d(y_{2n-1},y_{2n}),\end{array}\right\}\right)$$

$$d(y_{2n},y_{2n+1}) \leq k \max\left\{\begin{array}{c}\frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})},\\ \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}\\ d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),\\ d(y_{2n-1},y_{2n+1}),d(y_{2n},y_{2n}),\\ d(y_{2n-1},y_{2n}),\end{array}\right\}$$

$$-\psi\left(\max\left\{\begin{array}{c}\frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})},\\ \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}\\ d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),\\ d(y_{2n-1},y_{2n+1}),d(y_{2n},y_{2n}),\\ d(y_{2n-1},y_{2n}),\end{array}\right\}\right)$$

There arise three cases:

Case- 1 If we take $\max\left\{\begin{array}{c}d(y_{2n-1},y_{2n+1}),\\ d(y_{2n-1},y_{2n+1}),\\ d(y_{2n},y_{2n-1}),\\ d(y_{2n-1},y_{2n+1}),\\ 0,d(y_{2n-1},y_{2n})\end{array}\right\} = d(y_{2n-1},y_{2n})$

Then we have

$$d(y_{2n},y_{2n+1}) \leq k.d(y_{2n-1},y_{2n}) - \psi(d(y_{2n-1},y_{2n}))$$

Taking limit then we have, $\lim_{n\to\infty}\psi(d(y_{2n-1},y_{2n})) \to 0$ and hence

$$d(y_{2n},y_{2n+1}) \leq k.d(y_{2n-1},y_{2n}).$$

Case- 2 If we take $\max\left\{\begin{array}{c}d(y_{2n-1},y_{2n+1}),\\ d(y_{2n-1},y_{2n+1}),\\ d(y_{2n},y_{2n-1}),\\ d(y_{2n-1},y_{2n+1}),\\ 0,d(y_{2n-1},y_{2n})\end{array}\right\} = d(y_{2n+1},y_{2n})$

then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n}) - \psi(d(y_{2n+1}, y_{2n}))$$

taking limit then we have, $\lim_{n\to\infty} \psi(d(y_{2n-1}, y_{2n})) \to 0$ and hence

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n})$$

which contradiction.

Case- 3 If we take $\max\left\{\begin{matrix} d(y_{2n-1}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1}), \\ d(y_{2n}, y_{2n-1}), \\ d(y_{2n-1}, y_{2n+1}), \\ 0, d(y_{2n-1}, y_{2n}) \end{matrix}\right\} = 0$

then we have

$$d(y_{2n}, y_{2n+1}) \leq 0$$

which contradiction.

from the above all three cases we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

processing the same way we have

$$d(y_{2n}, y_{2n+1}) \leq k^{2n}.d(y_0, y_1)$$

or

$$d(y_n, y_{n+1}) \leq k^n.d(y_0, y_1)$$

for any $m > n$ we have

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (k^n + k^{n+1} + \dots\dots..+k^{m-1})d(y_0, y_1)$$

$$d(y_n, y_m) \leq \frac{k}{1-k} d(y_0, y_1).$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1}$$

$$= \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y\,.$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 3.2.1(iii),

$$d(Ax_{2n}, Bv) \leq k \max\left\{\begin{matrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bv,Tv)}{1+d(Ax_{2n},Sx_{2n})+d(Bv,Tv)}, \\ \frac{d^2(Sx_{2n},Bv)+d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Bv)+d(Ax_{2n},Tv)}, \\ d(Ax_{2n}, Sx_{2n}), d(Bv, Tv), \\ d(Sx_{2n}, Bv), d(Ax_{2n}, Tv), \\ d(Sx_{2n}, Tv) \end{matrix}\right\}$$

$$- \psi\left(\max\left\{ \begin{array}{c} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bv,Tv)}{1+d(Ax_{2n},Sx_{2n})+d(Bv,Tv)}, \\ \frac{d^2(Sx_{2n},Bv)+d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Bv)+d(Ax_{2n},Tv)}, \\ d(Ax_{2n},Sx_{2n}), d(Bv,Tv), \\ d(Sx_{2n},Bv), d(Ax_{2n},Tv), \\ d(Sx_{2n},Tv) \end{array} \right\} \right)$$

$$d(y,Bv) < k\,.d(y,Bv)$$

Which contradiction,

Hence $Bv = y = Tv$ and that $BTv = TBv$ implies that $By = Ty$.

Now we proof that $By = y$ for this again from 3.1.1(iii)

$$d(Ax_{2n},By) \leq k \max\left\{ \begin{array}{c} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(By,Ty)}{1+d(Ax_{2n},Sx_{2n})+d(By,Ty)}, \\ \frac{d^2(Sx_{2n},By)+d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},By)+d(Ax_{2n},Ty)} \\ d(Sx_{2n},By), d(Ax_{2n},Ty), \\ d(Ax_{2n},Sx_{2n}), d(By,Ty), \\ d(Sx_{2n},Ty) \end{array} \right\}$$

$$- \psi\left(\max\left\{ \begin{array}{c} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(By,Ty)}{1+d(Ax_{2n},Sx_{2n})+d(By,Ty)}, \\ \frac{d^2(Sx_{2n},By)+d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},By)+d(Ax_{2n},Ty)} \\ d(Sx_{2n},By), d(Ax_{2n},Ty), \\ d(Ax_{2n},Sx_{2n}), d(By,Ty), \\ d(Sx_{2n},Ty) \end{array} \right\} \right)$$

$$\lim_{n\to\infty} d(Ax_{2n},By) \leq k\, d(y,By)$$

$$By = y = Ty$$

Since $B(X) \subseteq S(X)$

for $w \in X$ such that $Sw = y$.

Now we show that $Aw = y$

$$d(Aw,By) \leq k \max\left\{ \begin{array}{c} \frac{d^2(Aw,Sw)+d^2(By,Ty)}{1+d(Aw,Sw)+d(By,Ty)}, \\ \frac{d^2(Sw,By)+d^2(Aw,Ty)}{1+d(Sw,By)+d(Aw,Ty)}, \\ d(Aw,Sw), d(By,Ty), \\ d(Sw,By), d(Aw,Ty) \\ d(Sw,Ty) \end{array} \right\}$$

$$- \psi\left(\max\left\{ \begin{array}{c} \frac{d^2(Aw,Sw)+d^2(By,Ty)}{1+d(Aw,Sw)+d(By,Ty)}, \\ \frac{d^2(Sw,By)+d^2(Aw,Ty)}{1+d(Sw,By)+d(Aw,Ty)}, \\ d(Aw,Sw), d(By,Ty), \\ d(Sw,By), d(Aw,Ty) \\ d(Sw,Ty) \end{array} \right\} \right)$$

It follows that, $d(Aw,y) \leq kd(Aw,y)$

which contradiction, $d(Aw, y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ for this again from 3.2.1(iii),

$$d(Aw, By) \leq k \max \begin{Bmatrix} \frac{d^2(Ay,Sy)+ d^2(By,Ty)}{1+d(Ay,Sy)+ d(By,Ty)}, \\ \frac{d^2(Sy,By)+d^2(Ay,Ty)}{1+d(Sy,By)+d(Ay,Ty)} \\ d(Ay, Sy), d(By, Ty), \\ d(Sy, By), d(Ay, Ty) \\ d(Sy, Ty) \end{Bmatrix}$$

$$- \psi \left(\max \begin{Bmatrix} \frac{d^2(Ay,Sy)+ d^2(By,Ty)}{1+d(Ay,Sy)+ d(By,Ty)}, \\ \frac{d^2(Sy,By)+d^2(Ay,Ty)}{1+d(Sy,By)+d(Ay,Ty)} \\ d(Ay, Sy), d(By, Ty), \\ d(Sy, By), d(Ay, Ty) \\ d(Sy, Ty) \end{Bmatrix} \right)$$

it follows that, $d(Ay, y) \leq k\, d(Ay, y)$

which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

that is y is common fixed point of A, B, S, T.

If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness We suppose that x, is another fixed point for A, B, S, T then, by using 3.2.1(iii) then we have

$$d(x, y) \leq k.\, d(x, y)$$

Which contradiction. so that $x = y$ and y is unique fixed point of A, B, S, T.

This complete the prove of the theorem.

Theorem 3.2.2 Let (X, d) be a $T-$ orbitally complete metric space, if A, B, S, T be the self mapping of X into itself such that;

3.2.2(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.2.2(ii) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive map.

3.2.2(iii) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \leq k \max\left\{\frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx, Ty)\right\}.$$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof We suppose that, $x_0 \in X$ arbitrary and we choose a poin t $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1 \text{ and } y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \ldots \ldots \ldots ..$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 3.2.2(iii) we have,

$$d(y_{2n}, y_{2n+1}) \leq k.M(Ax_{2n}, Bx_{2n+1}) - \psi(M(Ax_{2n}, Bx_{2n+1}))$$

$$d(y_{2n}, y_{2n+1}) \leq k \max \begin{Bmatrix} \frac{d(Ax_{2n}, Sx_{2n}).d(Bx_{2n+1}, Tx_{2n+1})}{1+d(Sx_{2n}, Tx_{2n+1})}, \\ \frac{d(Sx_{2n}, Bx_{2n+1}).d(Ax_{2n}, Tx_{2n+1})}{1+d(Sx_{2n}, Tx_{2n+1})}, \\ d(Sx_{2n}, Tx_{2n+1}) \end{Bmatrix}$$

$$d(y_{2n}, y_{2n+1}) \leq k \max \begin{Bmatrix} \frac{d(y_{2n}, y_{2n-1}).d(y_{2n+1}, y_{2n})}{1+d(y_{2n-1}, y_{2n})}, \\ \frac{d(y_{2n-1}, y_{2n+1}).d(y_{2n}, y_{2n})}{1+d(y_{2n-1}, y_{2n})}, \\ d(y_{2n-1}, y_{2n}) \end{Bmatrix}$$

$$d(y_{2n}, y_{2n+1}) \leq k \max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\}$$

There arise three cases:

Case- 1 If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n})$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

Case- 2 If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n+1}, y_{2n})$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n})$$

which contradiction.

Case- 3 If we take

$$\max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = 0$$

then we have

$$d(y_{2n}, y_{2n+1}) \leq 0$$

which contradiction.

From the above all three cases we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

Processing the same way we have

$$d(y_{2n}, y_{2n+1}) \leq k^{2n}.d(y_0, y_1)$$

Or $$d(y_n, y_{n+1}) \leq k^n.d(y_0, y_1)$$

For any $m > n$ we have

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (k^n + k^{n+1} + \dots\dots.. + k^{m-1}) d(y_0, y_1)$$

$$d(y_n, y_m) \leq \frac{k}{1-k} d(y_0, y_1) \ .$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1}$$

$$= \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y \ .$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 3.2.2(iii),

$$d(Ax_{2n}, Bv) \leq k \max \begin{Bmatrix} \frac{d(Ax_{2n},Sx_{2n}) \, d(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \\ \frac{d(Sx_{2n},Bv) d(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)}, \\ d(Sx_{2n}, Tv) \end{Bmatrix}$$

$$d(y, Bv) \leq k \max\{ d(Bv, y), d(y, Bv), 0\}$$

$$d(y, Bv) < k \, . d(y, Bv)$$

which contradiction.

Hence $Bv = y = Tv$ and that $BTv = TBv$ implies that $By = Ty$.

Now we proof that $By = y$ for this again from 3.2.2(iii)

$$d(Ax_{2n}, By) \leq k \max \begin{Bmatrix} \frac{d(Sx_{2n},By).d(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}, \\ \frac{d(Ax_{2n},Sx_{2n}).d(By,Ty)}{1+d(Sx_{2n},Ty)}, \\ d(Sx_{2n}, Ty) \end{Bmatrix}$$

$$\lim_{n\to\infty} d(Ax_{2n}, By) \leq k \, d(y, By)$$

$$By = y = Ty \ .$$

Since $B(X) \subseteq S(X)$

for, $w \in X$ such that $Sw = y$

now we show that $Aw = y$

$$d(Aw, By) \leq k \max \begin{Bmatrix} \frac{d(Aw,Sw)d(By,Ty)}{1+d(Sw,Ty)}, \frac{d(Sw,By)d(Aw,Ty)}{1+d(Sw,Ty)}, \\ d(Sw, Ty) \end{Bmatrix}$$

It follows that, $d(Aw, y) \leq kd(Aw, y)$

Which contradiction, $d(Aw, y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ for this again from 3.2.2(iii),

$$d(Aw, By) \leq k \max \begin{Bmatrix} \frac{d(Ay,Sy) \, d(By,Ty)}{1+d(Sy,Ty)}, \frac{d(Sy,By)d(Ay,Ty)}{1+d(Sy,Ty)}, \\ d(Sy, Ty) \end{Bmatrix}$$

It follows that, $d(Ay, y) \leq k\, d(Ay, y)$

Which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

that is y is common fixed point of A, B, S, T.

If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness We suppose that x, is another fixed point for A, B, S, T then, by using 3.2.2(iii) then we have

$$d(x, y) \leq k. d(x, y)$$

which contradiction. so that $x = y$ and y is unique fixed point of A, B, S, T.

This complete the prove of the theorem.

If we omit the completeness of the space then we get following corollary.

Corollary 3.2.3 Let (X, d) be a $T-$ orbitally metric space, if A, B, S, T be the self mapping of X into itself such that;

3.2.3(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.2.3(ii) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive map.

3.2.3(iii) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \leq k. M(x, y) - \psi(M(x, y))$$

Where, $M(Ax, By) = \max \left\{ \begin{matrix} \frac{d^2(Ax,Sx) + d^2(By,Ty)}{1+d(Ax,Sx) + d(By,Ty)}, \\ \frac{d^2(Sx,By) + d^2(Ax,Ty)}{1+d(Sx,By) + d(Ax,Ty)}, \\ d(Ax, Sx), d(By, Ty), \\ d(Sx, By), d(Ax, Ty), \\ d(Sx, Ty) \end{matrix} \right\}.$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Corollary 3.2.4 Let (X, d) be a $T-$ orbitally complete metric space, if A, Bbe the self mapping of X into itself such that;

3.2.4(i) $A(X) \subseteq X$ and $B(X) \subseteq X$,

3.2.4(ii) The pair (A, B) weakly compatible and generalized weakly contractive map.

3.2.4(iii) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \leq k. M(x, y) - \psi(M(x, y))$$

Where, $M(Ax, By) = \max \left\{ \begin{matrix} \frac{d^2(Ax,x) + d^2(By,y)}{1+d(Ax,x) + d(By,y)}, \\ \frac{d^2(x,By) + d^2(Ax,y)}{1+d(x,By) + d(Ax,y)}, \\ d(Ax, x), d(By, y), \\ d(x, By), d(Ax, y), \\ d(x, y) \end{matrix} \right\}.$

Then A, B have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is enough if we take $S = T = I$ (identity mapping) in Theorem 3.2.1 then we get the result.

***Corollary* 3.2.5** Let (X,d) be a $T-$ *orbitally* complete metric space, if A, B be the self mapping of X into itself such that *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax, Ay) \leq k.M(x,y) - \psi(M(x,y))$$

Where , $M(Ax,Ay) = max\begin{Bmatrix} \frac{d^2(Ax,x)+d^2(Ay,y)}{1+d(Ax,x)+d(Ay,y)}, \\ \frac{d^2(x,Ay)+d^2(Ax,y)}{1+d(x,Ay)+d(Ax,y)}, \\ d(Ax,x), d(Ay,y), \\ d(x,Ay), d(Ax,y), \\ d(x,y) \end{Bmatrix}$.

Then A, B have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is enough if we take $A = B$ in Corollary 3.2.4 then we get the result.

***Corollary* 3.2.6** Let (X,d) be a $T-$ *orbitally* complete metric space, if A, B, S, T be the self mapping of X into itself such that;

3.2.6(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.2.6(ii) The pair (A,S) and (B,T) are weakly compatible and generalized weakly contractive map.

3.2.6(iii) *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax,By) \leq k.max\begin{Bmatrix} \frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Ax,Sx)+d(By,Ty)}, \\ \frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,By)+d(Ax,Ty)}, \\ d(Ax,Sx), d(By,Ty), \\ d(Sx,By), d(Ax,Ty), \\ d(Sx,Ty) \end{Bmatrix}$$

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof:- It is immediate to see that if we take $\psi(t) = 0$ in Theorem 3.2.1, then we get the result.

3.3 COMMON FIXED POINT THEOREM ON $T-$ *ORBITALLY* COMPLETE METRIC SPACE SATISFYING INTEGRAL TYPE CONTRACTION CONDITION.

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [3] in 1922. In the general setting of complete metric space, this theorem runs as the follows,

Theorem 3.3.1 : Let (X,d) be a complete metric space, $c \in (0,1)$ and $f: X \to X$ be a mapping such that for each $x, y \in X$,

$$d(fx, fy) \leq cd(x,y) \qquad 3.3.1(a)$$

Then f has a unique fixed point $a \in X$, such that for each $x \in X$, $lim_{n\to\infty} f^n x = a$.

In 2002, Branciari [2], obtained the following theorem.

Theorem 3.3.2: Let (X,d) be a complete metric space, $\alpha \in (0,1)$ and let $T: X \to X$, be a mapping such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \xi(t)\, dt \leq \int_0^{d(x,y)} \xi(t)\, dt$$

Where $\xi : [0,+\infty] \to [0,+\infty]$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty]$, non negative, and such that, $for\ all\ \varepsilon > 0$, $\int_0^{\varepsilon} \xi(t)\ dt > 0$ Then, T has unique fixed point $z \in X$ such that for each $x \in X$, $T^n x \to z$ as $n \to \infty$.

Our aim of this section to extend the results of $section$ 3.2 for integral type mapping. In fact we prove following results.

$\boldsymbol{Theorem}$ **3.3.3** Let (X,d) be a $T-orbitally$ complete metric space, if A, B, S, T be the self mapping of X into itself such that;

3.3.3(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.3.3(ii) The pair (A,S) and (B,T) are weakly compatible and generalized weakly contractive map.

3.3.3(iii) $for\ all\ x,y \in \overline{O(x_0)}\ and\ k \in [0,1),\ we\ define,$

$$\int_0^{d(Ax,By)} \xi(t)\ dt \le \int_0^{k\,max\left\{\begin{array}{c}\frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)},\\ \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx,Ty)\end{array}\right\}} \xi(t)\ dt$$

Where $\xi : [0,+\infty] \to [0,+\infty]$ is a lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty]$, non negative, and such that, $\forall\ \varepsilon > 0$, $\int_0^{\varepsilon} \xi(t)\ dt > 0$ .Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof We suppose that, $x_0 \in X$ arbitrary and we choose a point t $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1\ and\ \ y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1}\ and\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \dots \dots \dots$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 3.3.3(iii) we have,

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

Or
$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)\ dt = \int_0^{d(Ax_{2n},Bx_{2n+1})} \xi(t)\ dt$$

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)\ dt \le \int_0^{k\,max\left\{\begin{array}{c}\frac{d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}\\ ,\frac{d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})},\\ d(Sx_{2n},Tx_{2n+1})\end{array}\right\}} \xi(t)\ dt$$

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)\ dt \le \int_0^{k\ max\left\{\begin{array}{c}\frac{d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ \frac{d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})},\\ d(y_{2n-1},y_{2n})\end{array}\right\}} \xi(t)\ dt$$

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)\ dt \le \int_0^{k\ max\{d(y_{2n+1},y_{2n}),0,d(y_{2n-1},y_{2n})\}} \xi(t)\ dt$$

Since ξ is a lebesgue integrable mapping, so we have

$$d(y_{2n}, y_{2n+1}) \le k\ max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\}$$

There arise three cases:

Case- 1 If we take $max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n-1}, y_{2n})$

Then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

Case- 2 If we take $max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = d(y_{2n+1}, y_{2n})$

Then we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n})$$

Which contradiction.

Case- 3 If we take $max\{d(y_{2n+1}, y_{2n}), 0, d(y_{2n-1}, y_{2n})\} = 0$

Then we have

$$d(y_{2n}, y_{2n+1}) \leq 0$$

Which contradiction.

From the above all three cases we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

Processing the same way we have

$$d(y_{2n}, y_{2n+1}) \leq k^{2n}.d(y_0, y_1)$$

Or

$$d(y_n, y_{n+1}) \leq k^n.d(y_0, y_1)$$

For any $m > n$ we have

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots \ldots . + d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (k^n + k^{n+1} + \ldots \ldots . . + k^{m-1})d(y_0, y_1)$$

$$d(y_n, y_m) \leq \frac{k}{1-k} d(y_0, y_1)$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$lim_{n\to\infty} y_n = lim_{n\to\infty} Ax_{2n} = lim_{n\to\infty} Tx_{2n+1}$$

$$= lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Sx_{2n+2} = y$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 3.3.3(iii),

$$\int_0^{d(Ax_{2n},Bv)} \xi(t)\ dt \leq \int_0^{k\ max\left\{\begin{array}{c}\frac{d(Ax_{2n},Sx_{2n})\ d(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \\ \frac{d(Sx_{2n},Bv)d(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)}, \\ d(Sx_{2n},Tv)\end{array}\right\}} \xi(t)\ dt$$

$$\int_0^{d(y,Bv)} \xi(t)\ dt \leq \int_0^{k\ max\{d(Bv,y),d(y,Bv),0\}} \xi(t)\ dt$$

$$\int_0^{d(y,Bv)} \xi(t)\ dt \leq \int_0^{k\ .d(y,Bv)} \xi(t)\ dt$$

Which implies

$$d(y,Bv) \leq k\,.\,d(y,Bv)$$

This is a contradiction,

Hence $Bv = y = Tv$ *and that* $BTv = TBv$ *implies that* $By = Ty$.

Now we proof that $By = y$ for this again from 3.3.3(iii)

$$\int_0^{d(Ax_{2n},By)} \xi(t)\ dt \leq \int_0^{k\max\left\{\begin{array}{c}\frac{d(Sx_{2n},By).d(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}, \\ \frac{d(Ax_{2n},Sx_{2n}).d(By,Ty)}{1+d(Sx_{2n},Ty)}, \\ d(Sx_{2n},Ty)\end{array}\right\}} \xi(t)\ dt$$

$$lim_{n\to\infty} d(Ax_{2n},By) \leq k\, d(y,By)$$

$$By = y = Ty$$

Since $B(X) \subseteq S(X)$

For, $w \in X$ such that $Sw = y$

Now we show that $Aw = y$

$$\int_0^{d(Aw,By)} \xi(t)\ dt \leq \int_0^{k\max\left\{\begin{array}{c}\frac{d(Aw,Sw)d(By,Ty)}{1+d(Sw,Ty)}, \\ \frac{d(Sw,By)d(Aw,Ty)}{1+d(Sw,Ty)}, \\ d(Sw,Ty)\end{array}\right\}} \xi(t)\ dt$$

It follows that, $d(Aw,y) \leq kd(Aw,y)$

Which contradiction, $d(Aw,y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ *for this a*gain from 3.3.3(iii),

$$\int_0^{d(Aw,By)} \xi(t)\ dt \leq \int_0^{k\max\left\{\begin{array}{c}\frac{d(Ay,Sy)\ d(By,Ty)}{1+d(Sy,Ty)}, \\ \frac{d(Sy,By)d(Ay,Ty)}{1+d(Sy,Ty)}, \\ d(Sy,Ty)\end{array}\right\}} \xi(t)\ dt$$

It follows that, $d(Ay,y) \leq k\, d(Ay,y)$

Which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

y is common fixed point of A,B,S,T.

If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness We suppose that x, is another fixed point for A, B, S, T then, by using 3.3.3(iii) then we have

$$d(x,y) \leq k.d(x,y)$$

which contradiction. *so that* $x = y$

y is unique fixed point of A, B, S, T.

This complete the prove of the theorem.

If we take $\xi(t)$ be an identity mapping in Theorem 3.3.3, then we get Theorem 3.2.2.

***Corollary* 3.3.4** Let (X,d) be a $T-$ *orbitally* complete metric space, if A *and* B be the self mappings of X into itself such that;

3.3.4(i) $A(X) \subseteq X$ and $B(X) \subseteq X$,

3.3.4(ii) $for\ all\ x,y \in \overline{O(x_0)}\ and\ k \in [0,1),\ we\ define,$

$$\int_0^{d(Ax,By)} \xi(t)\ dt \leq \int_0^{k\ max\left\{\frac{d(Ax,x)\,.d(By,y)}{1+d(x,y)},\frac{d(x,By).d(Ax,y)}{1+d(x,y)},d(Sx,Ty)\right\}} \xi(t)\ dt$$

Where $\xi : [0\,,+\infty] \rightarrow [0\,,+\infty]$ is a lebesgue integrable mapping which is summable on each compact subset of $[0\,,+\infty]$, non negative, and such that, $\forall\ \varepsilon > 0,\ \int_0^{\varepsilon} \xi(t)\ dt > 0$ .Then A *and* B have unique fixed point in $\overline{O(x_0)}$.

Proof To prove of the above corollary it is enough if we take $S = T = I$ (identity mapping) on X.

3.4 RATIONAL CONTRACTIVE CONDITION and COMMON FIXED POINT THEOREM ON $T-$ *ORBITALLY* COMPLETE METRIC SPACE

In this section we prove a common fixed point result satisfying rational contractive condition in T-orbitally complete metric space. We also give some corollaries which is equivalent to the proved theorem. In fact we prove following theorem.

***Theorem* 3.4.1** Let (X,d) be a $T-$ *orbitally* complete metric space, if $A\,,B,S\,,T$ be the self mapping of X into itself such that;

3.4.1(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.4.1 (ii) The pair (A,S) and (B,T) are weakly compatible and generalized weakly contractive map.

3.4.1 (iii) $for\ all\ x,y \in \overline{O(x_0)}\ and\ k \in [0,1),\ we\ define,$

$$d^2(Ax,By) \leq k.M(x,y) - \psi(M(x,y))$$

Where , $M(Ax,By) = max \left\{ \begin{array}{c} \frac{d^3(Ax,Sx).d(By,Ty)}{1+d(Ax,Sx)\,.d(By,Ty)}, \\ \frac{d^2(Sx,By).d^2(Ax,Ty)}{1+d(Sx,By).d(Ax,Ty)}, \\ d^2(Ax,Sx), d^2(By,Ty)\,, \\ d^2(Sx,By), d^2(Ax,Ty), \\ d^2(Sx,Ty) \end{array} \right\}.$

Then $A\,,B,S,T$ have unique fixed point in $\overline{O(x_0)}$.

Proof We suppose that, $x_0 \in X$ arbitrary and we choose a poin t $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1\ \ and\ \ y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1}\ and\ \ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \ldots \ldots \ldots$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 3.4.1 (iii) we have,

$$d^2(y_{2n}, y_{2n+1}) \leq k.M(Ax_{2n}, Bx_{2n+1}) - \psi(M(Ax_{2n}, Bx_{2n+1}))$$

$$d^2(y_{2n},y_{2n+1}) \le k\ max\begin{Bmatrix} \frac{d^3(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1}).d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})} \\ d^2(Ax_{2n},Sx_{2n}), \\ d^2(Bx_{2n+1},Tx_{2n+1}), \\ d^2(Sx_{2n},Bx_{2n+1}), \\ d^2(Ax_{2n},Tx_{2n+1}), \\ d^2(Sx_{2n},Tx_{2n+1}), \end{Bmatrix}$$

$$-\psi\left(max\begin{Bmatrix} \frac{d^3(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1}).d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})} \\ d^2(Ax_{2n},Sx_{2n}), d^2(Bx_{2n+1},Tx_{2n+1}), \\ d^2(Sx_{2n},Bx_{2n+1}), d^2(Ax_{2n},Tx_{2n+1}), \\ d^2(Sx_{2n},Tx_{2n+1}), \end{Bmatrix}\right)$$

$$d^2(y_{2n},y_{2n+1}) \le k\ max\begin{Bmatrix} \frac{d^3(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}, \\ \frac{d^2(y_{2n-1},y_{2n+1}).d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})} \\ d^2(y_{2n},y_{2n-1}), d^2(y_{2n+1},y_{2n}), \\ d^2(y_{2n-1},y_{2n+1}), d^2(y_{2n},y_{2n}), \\ d^2(y_{2n-1},y_{2n}), \end{Bmatrix}$$

$$-\psi\left(max\begin{Bmatrix} \frac{d^3(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}, \\ \frac{d^2(y_{2n-1},y_{2n+1}).d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})} \\ d^2(y_{2n},y_{2n-1}), d^2(y_{2n+1},y_{2n}), \\ d^2(y_{2n-1},y_{2n+1}), d^2(y_{2n},y_{2n}), \\ d^2(y_{2n-1},y_{2n}), \end{Bmatrix}\right)$$

$$d^2(y_{2n},y_{2n+1}) \le k\ max\begin{Bmatrix} \frac{d^3(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}, \\ \frac{d^2(y_{2n-1},y_{2n+1}).d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})} \\ d^2(y_{2n},y_{2n-1}), d^2(y_{2n+1},y_{2n}), \\ d^2(y_{2n-1},y_{2n+1}), d^2(y_{2n},y_{2n}), \\ d^2(y_{2n-1},y_{2n}), \end{Bmatrix}$$

$$-\psi\left(max\begin{Bmatrix} \frac{d^3(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}, \\ \frac{d^2(y_{2n-1},y_{2n+1}).d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})} \\ d^2(y_{2n},y_{2n-1}), d^2(y_{2n+1},y_{2n}), \\ d^2(y_{2n-1},y_{2n+1}), d^2(y_{2n},y_{2n}), \\ d^2(y_{2n-1},y_{2n}), \end{Bmatrix}\right)$$

There arise three cases:

Case- 1 If we take $max\begin{Bmatrix} d^2(y_{2n},y_{2n-1}), \\ 0, d^2(y_{2n},y_{2n-1}), \\ d^2(y_{2n-1},y_{2n+1}), \\ 0, d^2(y_{2n-1},y_{2n}) \end{Bmatrix} = d^2(y_{2n-1},y_{2n})$

Then we have

$$d^2(y_{2n}, y_{2n+1}) \leq k.d^2(y_{2n-1}, y_{2n}) - \psi\big(d^2(y_{2n-1}, y_{2n})\big)$$

Taking limit then we have, $lim_{n\to\infty}\, \psi\big(d(y_{2n-1}, y_{2n})\big) \to 0$ and hence

$$d^2(y_{2n}, y_{2n+1}) \leq k.d^2(y_{2n-1}, y_{2n}).$$

Case- 2 If we take $max \begin{Bmatrix} d^2(y_{2n}, y_{2n-1}), \\ 0, d^2(y_{2n}, y_{2n-1}), \\ d^2(y_{2n-1}, y_{2n+1}), \\ 0, d^2(y_{2n-1}, y_{2n}) \end{Bmatrix} = d^2(y_{2n+1}, y_{2n})$

then we have

$$d^2(y_{2n}, y_{2n+1}) \leq k.d^2(y_{2n+1}, y_{2n}) - \psi\big(d^2(y_{2n+1}, y_{2n})\big)$$

taking limit then we have, $lim_{n\to\infty}\, \psi\big(d(y_{2n-1}, y_{2n})\big) \to 0$ and hence

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n+1}, y_{2n})$$

which contradiction.

Case- 3 If we take $max \begin{Bmatrix} d^2(y_{2n}, y_{2n-1}), \\ 0, d^2(y_{2n}, y_{2n-1}), \\ d^2(y_{2n-1}, y_{2n+1}), \\ 0, d^2(y_{2n-1}, y_{2n}) \end{Bmatrix} = 0$

then we have

$$d^2(y_{2n}, y_{2n+1}) \leq 0$$

which contradiction.

from the above all three cases we have

$$d(y_{2n}, y_{2n+1}) \leq k.d(y_{2n-1}, y_{2n})$$

processing the same way we have

$$d(y_{2n}, y_{2n+1}) \leq k^{2n}.d(y_0, y_1)$$

or

$$d(y_n, y_{n+1}) \leq \mathrm{k}^n.d(y_0, y_1)$$

for any $m > n$ we have

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots \ldots . + d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (k^n + k^{n+1} + \ldots \ldots . . + k^{m-1}) d(y_0, y_1)$$

$$d(y_n, y_m) \leq \frac{k}{1-k}\, d(y_0, y_1).$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$lim_{n\to\infty}\, y_n = lim_{n\to\infty}\, Ax_{2n} = lim_{n\to\infty}\, Tx_{2n+1}$$

$$= lim_{\mathrm{n}\to\infty}\, Bx_{2n+1} = lim_{n\to\infty}\, Sx_{2n+2} = y.$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 3.4.1 (iii),

$$d^2(Ax_{2n}, Bv) \leq k \max \left\{ \begin{array}{c} \frac{d^3(Ax_{2n}, Sx_{2n}).d(Bv,Tv)}{1+d(Ax_{2n}, Sx_{2n}).d(Bv,Tv)}, \\ \frac{d^2(Sx_{2n}, Bv).d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Bv).d(Ax_{2n},Tv)}, \\ d^2(Ax_{2n}, Sx_{2n}), d^2(Bv,Tv), \\ d^2(Sx_{2n}, Bv), d^2(Ax_{2n}, Tv), \\ d^2(Sx_{2n}, Tv) \end{array} \right\}$$

$$- \psi \left(\max \left\{ \begin{array}{c} \frac{d^3(Ax_{2n}, Sx_{2n}).d(Bv,Tv)}{1+d(Ax_{2n}, Sx_{2n}).d(Bv,Tv)}, \\ \frac{d^2(Sx_{2n}, Bv).d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Bv).d(Ax_{2n},Tv)}, \\ d^2(Ax_{2n}, Sx_{2n}), d^2(Bv,Tv), \\ d^2(Sx_{2n}, Bv), d^2(Ax_{2n}, Tv), \\ d^2(Sx_{2n}, Tv) \end{array} \right\} \right)$$

$$d^2(y, Bv) < k \,. d^2(y, Bv)$$

Which contradiction,

Hence $Bv = y = Tv$ *and that* $BTv = TBv$ *implies that* $By = Ty$.

Now we proof that $By = y$ for this again from 3.4.1 (iii)

$$d^2(Ax_{2n}, By) \leq k \max \left\{ \begin{array}{c} \frac{d^3(Ax_{2n}, Sx_{2n}).d^2(By,Ty)}{1+d(Ax_{2n}, Sx_{2n}).d(By,Ty)}, \\ \frac{d^2(Sx_{2n}, By).d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},By).d(Ax_{2n},Ty)} \\ d^2(Sx_{2n}, By), d^2(Ax_{2n},Ty), \\ d^2(Ax_{2n}, Sx_{2n}), d^2(By, Ty), \\ d^2(Sx_{2n}, Ty) \end{array} \right\}$$

$$- \psi \left(\max \left\{ \begin{array}{c} \frac{d^3(Ax_{2n}, Sx_{2n}).d(By,Ty)}{1+d(Ax_{2n}, Sx_{2n}).d(By,Ty)}, \\ \frac{d^2(Sx_{2n}, By).d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},By).d(Ax_{2n},Ty)} \\ d^2(Sx_{2n}, By), d^2(Ax_{2n},Ty), \\ d^2(Ax_{2n}, Sx_{2n}), d^2(By, Ty), \\ d^2(Sx_{2n}, Ty) \end{array} \right\} \right)$$

$$\lim_{n\to\infty} d^2(Ax_{2n}, By) \leq k\, d^2(y, By)$$

$By = y = Ty$

Since $B(X) \subseteq S(X)$

for $w \in X$ such that $Sw = y$.

Now we show that $Aw = y$

$$d^2(Aw,By) \leq k\ max\begin{Bmatrix} \frac{d^3(Aw,Sw).d(By,Ty)}{1+d(Aw,Sw).d(By,Ty)}, \\ \frac{d^2(Sw,By).d^2(Aw,Ty)}{1+d(Sw,By),d(Aw,Ty)}, \\ d^2(Aw,Sw), d^2(By,Ty), \\ d^2(Sw,By), d^2(Aw,Ty) \\ d^2(Sw,Ty) \end{Bmatrix}$$

$$-\psi\left(max\begin{Bmatrix} \frac{d^2(Aw,Sw).d(By,Ty)}{1+d(Aw,Sw).d(By,Ty)}, \\ \frac{d^2(Sw,By).d^2(Aw,Ty)}{1+d(Sw,By).d(Aw,Ty)}, \\ d^2(Aw,Sw), d^2(By,Ty), \\ d^2(Sw,By), d^2(Aw,Ty) \\ d^2(Sw,Ty) \end{Bmatrix}\right)$$

It follows that, $d^2(Aw,y) \leq kd^2(Aw,y)$

which contradiction, $d(Aw,y) > 0$ thus $Aw = y = Sw$

Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.

Now we show that, $Ay = y$ for $this$ again from 3.4.1(iii),

$$d^2(Aw,By) \leq k\,max\begin{Bmatrix} \frac{d^3(Ay,Sy).d(By,Ty)}{1+d(Ay,Sy).d(By,Ty)}, \\ \frac{d^2(Sy,By).d^2(Ay,Ty)}{1+d(Sy,By).d(Ay,Ty)} \\ d^2(Ay,Sy), d^2(By,Ty), \\ d^2(Sy,By), d^2(Ay,Ty) \\ d^2(Sy,Ty) \end{Bmatrix}$$

$$-\psi\left(max\begin{Bmatrix} \frac{d^3(Ay,Sy).d(By,Ty)}{1+d(Ay,Sy).d(By,Ty)}, \\ \frac{d^2(Sy,By).d^2(Ay,Ty)}{1+d(Sy,By).d(Ay,Ty)} \\ d^2(Ay,Sy), d^2(By,Ty), \\ d^2(Sy,By), d^2(Ay,Ty) \\ d^2(Sy,Ty) \end{Bmatrix}\right)$$

it follows that, $d(Ay,y) \leq k\,d(Ay,y)$

which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

that is y is common fixed point of A, B, S, T.

If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness We suppose that x, is another fixed point for A, B, S, T then, by using 3.4.1 (iii) then we have

$$d(x,y) \leq k.\,d(x,y)$$

This is a contradiction. So that $x = y$ and y is unique fixed point of A, B, S, T. This complete the prove of the theorem.

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