

Fuzzy Laplace Transforms for Derivatives of Higher Orders

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Abstract

In this paper, we find the formula of fuzzy derivative of the third order and fourth order and find the fuzzy Laplace transforms for the fuzzy derivative of the above mentioned orders by using generalized H-differentiability.

Keywords: Fuzzy numbers, generalized H-differentiability**,** Fuzzy Laplace transform.

1. Introduction

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972) , it was followed up by Dubois and Prade (1982) , Puri and Ralescu (1983). Recently, Allahviranloo and Ahmadi (2010) have proposed the fuzzy Laplace transforms for solving first order FIVP under generalized H-differentiability. Salahshour and Allahviranloo (2013) have proposed fuzzy Laplace transform and its inverse, proof and discussion about some useful results and proposed equivalent integral forms for solving second order FIVP.

This paper is arranged as follows: Basic concepts are given in Section 2. In Sections 3 and 4, we find the formula, also fuzzy Laplace transforms for the fuzzy derivative of the third order and fourth order respectively. In Sections 5, we solve an example of the fourth order. In Sections 6, conclusions are drawn.

2. Basic Concepts

In this section, some necessary definitions and concepts are introduced:

Definition 2.1 (Allahviranloo et al. 2011) A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha),\ 0\le\alpha\le1$ which satisfy the following requirements:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(\alpha)\le\overline{u}(\alpha),\ 0\le\alpha\le1.$

A crisp number α is simply represented by $\underline{u}(\alpha)=\overline{u}(\alpha)=\alpha,\ 0\le\alpha\le1$. We recall that for $a<b<c$ which $a,b,c\in R$, the triangular fuzzy number $u=(a,b,c)$ determined by a,b,c is given such that $\underline{u}(\alpha)$ $=a+(b-a)\alpha$ and that $\overline{u}(\alpha)=c-(c-b)\alpha$ are the endpoints of the $\alpha-$level sets, for all $\alpha\in[0,1]$. The Hausdorff distance between fuzzy numbers is given by

$$d:E\times E\to R_+\bigcup\{0\},$$

$$d(u,v)=\sup_{\alpha\in[0,1]}\max\{|\underline{u}(\alpha)-\underline{v}(\alpha)|,|\overline{u}(\alpha)-\overline{v}(\alpha)|\},$$

where $u=(\underline{u}(\alpha),\overline{u}(\alpha)),v=(\underline{v}(\alpha),\overline{v}(\alpha))\subset R$ is utilized. Then, it is easy to see that d is a metric in E and has the following properties:

1. $d(u+w,v+w)=d(u,v),\quad \forall u,v,w\in E$
2. $d(ku,kv)=|k|d(u,v),\quad \forall k\in R,u,v\in E$,
3. $d(u+v,w+e)\le d(u,w)+d(v,e),\quad \forall u,v,w,e\in E.$
4. (E,d) is a complete metric space.

Definition 2.2 (Bede et al. 2006) Let $x,y\in E$. If there exists $z\in E$ such that $x=y+z$, then z is called the H-difference of x and y, and it is denoted by $x\ominus y$**.**

In this paper, the sign "Θ "stands always for H-difference, and let us remark that $x\ominus y\ne x+(-1)y$.

Definition 2.3 (Allahviranloo et al. 2011) Let $f:(a,b)\to E$ and $x_0\in(a,b)$. We say that f is strongly generalized differential at x_0 if there exists an element $f'(x_0)\in E$, such that

i. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0)$, $\exists f(x_0)\ominus f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h]=f'(x_0)$$

or

ii. For all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/-h]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/-h]=f'(x_0)$$

or

iii. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/-h]=f'(x_0)$$

or

iv. For all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0)\ominus f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/-h]=\lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h]=f'(x_0)$$

Definition 2.4 (Allahviranloo and Ahmadi 2010) Let $f(t)$ be continuous fuzzy-valued function. Suppose that $f(t)\,e^{-st}$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^\infty f(t)\,e^{-st}dt$ is called fuzzy Laplace transforms and is denoted as $L(f(t))=\int_0^\infty f(t)e^{-st}dt,\, s>0$. We have

$$\int_0^\infty f(t)\,\mathrm{e}^{-st}dt=\left(\int_0^\infty \underline{f}(t,r)\,\mathrm{e}^{-st}dt,\ \int_0^\infty \overline{f}(t,r)\,\mathrm{e}^{-st}dt\right).$$

Also by using the definition of classical Laplace transform:

$$l[\underline{f}(t,r)]=\int_0^\infty \underline{f}(t,r)\,e^{-st}dt \text{ and } l[\overline{f}(t,r)]=\int_0^\infty \overline{f}(t,r)\,e^{-st}dt.$$

Then, we follow:

$$L(f(t))=(l(\underline{f}(t,r)),l(\overline{f}(t,r))).$$

Definition 2.5 (Salahshour and Allahviranloo 2013) A fuzzy-valued function f has exponential order p if there exist constants $M>0$ and p such that for some $t_0\ge 0$, $|f(t)|\le Me^{pt}\cdot\tilde{1},\ t\ge t_0$.

3. Fuzzy Laplace Transforms for the Third Order Derivative

In this section, we have the following results for third order derivative under generalized H-differentiability:

Theorem 3.1 Let $F(t), F'(t)$ and $F''(t)$ be differentiable fuzzy-valued functions, and if α-cut representation of $F(t)$ is denoted by $[F(t)]^\alpha=[f_\alpha(t),g_\alpha(t)]$, then

(a) Let $F(t), F'(t)$ and $F''(t)$ be (i)-differentiable, or
$F''(t)$ be (i)-differentiable and $F(t)$ and $F'(t)$ be (ii)-differentiable, or
$F'(t)$ be (i)-differentiable and $F(t)$ and $F''(t)$ be (ii)-differentiable, or
$F(t)$ be (i)-differentiable and $F'(t)$ and $F''(t)$ be (ii)-differentiable;

then $f_\alpha(t)$ and $g_\alpha(t)$ have first order, second order and third order derivatives and

$$[F'''(t)]^\alpha=[f_\alpha'''(t),g_\alpha'''(t)]$$

(b) Let $F'(t)$ and $F''(t)$ be (i)-differentiable and $F(t)$ be (ii)-differentiable, or
$F(t)$ and $F''(t)$ be (i)-differentiable and $F'(t)$ be (ii)-differentiable, or
$F(t)$ and $F'(t)$ be (i)-differentiable and $F''(t)$ be (ii)-differentiable, or

$F(t), F'(t)$ and $F''(t)$ be (ii)-differentiable; then $f_\alpha(t)$ and $g_\alpha(t)$ have first order, second order and third order derivatives and

$[F'''(t)]^\alpha = [g'''_\alpha(t), f'''_\alpha(t)]$

Proof Let $F(t), F'(t)$ and $F''(t)$ be (ii)-differentiable. Since $F(t)$ and $F'(t)$ be (ii)-differentiable, then we get **(**Salahshour and Allahviranloo 2013):

$[F''(t)]^\alpha = [f''_\alpha(t), g''_\alpha(t)]$.

Since $F''(t)$ is (ii)-differentiable then by ii of definition 2.3 we get:

$[F''(t) \ominus F''(t+h)]^\alpha = [f''_\alpha(t) - f''_\alpha(t+h), g''_\alpha(t) - g''_\alpha(t+h)]$,

$[F''(t-h) \ominus F''(t)]^\alpha = [f''_\alpha(t-h), -f''_\alpha(t), g''_\alpha(t-h) - g''_\alpha(t)]$.

and, multiplying by $\frac{1}{-h}$, $h > 0$ we get:

$$\frac{1}{-h}[F''(t) \ominus F''(t+h)]^\alpha = [\frac{g''_\alpha(t+h) - g''_\alpha(t)}{h}, \frac{f''_\alpha(t+h) - f''_\alpha(t)}{h}],$$

and

$$\frac{1}{-h}[F''(t-h) \ominus F''(t)]^\alpha = [\frac{g''_\alpha(t) - g''_\alpha(t-h)}{h}, \frac{f''_\alpha(t) - f''_\alpha(t-h)}{h}].$$

Finally, using $h \longrightarrow 0$ on both sides of aforementioned relation we get:

$[F'''(t)]^\alpha = [g'''_\alpha(t), f'''_\alpha(t)]$.

The other proofs are similar.

Theorem 3.2 Suppose that $g(t), g'(t)$ and $g''(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $g'''(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $g(t) = (\underline{g}(t,\alpha), \overline{g}(t,\alpha))$, then:

(1) If g, g' and g'' be (i)-differentiable, then:

$L(g'''(t)) = s^3 L(g(t)) \ominus s^2 g(0) \ominus sg'(0) \ominus g''(0).$

(2) If g' and g'' be (i)-differentiable and g be (ii)-differentiable, then:

$L(g'''(t)) = -s^2 g(0) \ominus (-s^3) L(g(t)) \ominus sg'(0) \ominus g''(0)$.

(3) If g and g'' be (i)-differentiable and g' is (ii)-differentiable, then:

(4) $L(g'''(t)) = -s^2 g(0) \ominus (-s^3) L(g(t)) - sg'(0) \ominus g''(0)$.

(5) If g and g' are (i)-differentiable and g'' is (ii)-differentiable, then:

$L(g'''(t)) = -s^2 g(0) \ominus (-s^3) L(g(t)) - sg'(0) - g''(0)$.

(6) If g'' are (i)-differentiable and g and g' be (ii)-differentiable, then:

$L(g'''(t)) = s^3 L(g(t)) \ominus s^2 g(0) - sg'(0) \ominus g''(0)$.

(7) If g' be (i)-differentiable and g and g'' be (ii)-differentiable, then:

$L(g'''(t)) = s^3 L(g(t)) \ominus s^2 g(0) - sg'(0) - g''(0)$.

(8) If g is (i)-differentiable and g' and g'' be (ii)-differentiable, then:

$L(g'''(t)) = s^3 L(g(t)) \ominus s^2 g(0) \ominus sg'(0) - g''(0)$

(9) If g, g' and g'' be (ii)-differentiable, then:

$L(g'''(t)) = -s^2 g(0) \ominus (-s^3) L(g(t)) \ominus sg'(0) - g''(0)$.

Proof: First, we state the notations carefully as follows: $\underline{g}'$, $\underline{g}''$ and $\underline{g}'''$ are the lower endpoints function's derivatives, $\overline{g}'$, $\overline{g}''$ and $\overline{g}'''$ are the upper endpoints function's derivatives. Also $\underline{g'}, \underline{g''}$ and $\underline{g'''}$ are the lower endpoints of the derivatives, $\overline{g}', \overline{g}''$ and $\overline{g}'''$ are the upper endpoints of the derivatives: For arbitrary fixed $\alpha \in [0,1]$ we have $g(t) = (\underline{g}(t,\alpha), \overline{g}(t,\alpha))$.

Now, we prove (3) as follows: Since $g(t)$, $g''(t)$ are (i)-differentiable and $g'(t)$ is (ii)-differentiable. Since $g(t)$ is (i)-differentiable and $g'(t)$ is (ii)-differentiable then by theorem 3.1(b) we get:

$$g'''(t) = (\overline{g}'''(t,\alpha), \underline{g}'''(t,\alpha)).$$

Therefore, we get:

$$\underline{g}'''(t,\alpha) = \overline{g}'''(t,\alpha),\ \overline{g}'''(t,\alpha) = \underline{g}'''(t,\alpha). \tag{1}$$

Then, from (1) we get:

$$\begin{aligned} L(g'''(t)) &= L(\underline{g}'''(t,\alpha), \overline{g}'''(t,\alpha)) \\ &= (l(\overline{g}'''(t,\alpha)), l(\underline{g}'''(t,\alpha))) \cdot \end{aligned} \tag{2}$$

Since $g(t)$ is (i)-differentiable and $g'(t)$ is (ii)-differentiable, we get:

$$\begin{aligned} &\underline{g}'(0,\alpha) = \underline{g}'(0,\alpha),\ \overline{g}'(0,\alpha) = \overline{g}'(0,\alpha), \\ &\underline{g}''(0,\alpha) = \overline{g}''(0,\alpha),\ \overline{g}''(0,\alpha) = \underline{g}''(0,\alpha). \end{aligned} \tag{3}$$

We know from the ordinary differential equations that:

$$\begin{aligned} l(\underline{g}'''(t,\alpha)) &= s^3 l(\underline{g}(t,\alpha)) - s^2\underline{g}(0,\alpha) - s\underline{g}'(0,\alpha) - \underline{g}''(0,\alpha), \\ l(\overline{g}'''(t,\alpha)) &= s^3 l(\overline{g}(t,\alpha)) - s^2\overline{g}(0,\alpha) - s\overline{g}'(0,\alpha) - \overline{g}''(0,\alpha). \end{aligned} \tag{4}$$

By using (3) and (4), equation (2) becomes:

$$\begin{aligned} L(g'''(t)) &= (s^3 l(\overline{g}(t,\alpha) - s^2\overline{g}(0,\alpha) - s\overline{g}'(0,\alpha) - \underline{g}''(0,\alpha), s^3 l(\underline{g}(t,\alpha) - s^2\underline{g}(0,\alpha) - s\underline{g}'(0,\alpha) \\ &\quad - \overline{g}''(0,\alpha)) \\ &= -s^2 g(0) \ominus (-s^3) L(g(t)) - sg'(0) \ominus g''(0). \end{aligned}$$

The other proofs are similar.

4. Fuzzy Laplace Transforms for the Fourth Order Derivative

In this section, we have the following results for fourth order derivative under generalized H-differentiability:

Theorem 4.1 Let $F(t), F'(t), F''(t)$ and $F'''(t)$ be differentiable fuzzy-valued functions, and if α -cut representation of $F(t)$ is denoted by $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, then:

(a) Let $F(t), F'(t), F''(t)$ and $F'''(t)$ be (i)-differentiable, or

$F''(t)$ and $F'''(t)$ be (i)-differentiable and $F(t)$ and $F'(t)$ be (ii)-differentiable, or

$F'(t)$ and $F'''(t)$ be (i)-differentiable and $F(t)$ and $F''(t)$ be (ii)-differentiable, or

$F'(t)$ and $F''(t)$ be (i)-differentiable and $F(t)$ and $F'''(t)$ be (ii)-differentiable, or

$F(t)$ and $F'''(t)$ be (i)-differentiable and $F'(t)$ and $F''(t)$ be (ii)-differentiable, or

$F(t)$ and $F''(t)$ be (i)-differentiable and $F'(t)$ and $F'''(t)$ be (ii)-differentiable, or

$F(t)$ and $F'(t)$ be (i)-differentiable and $F''(t)$ and $F'''(t)$ be (ii)-differentiable, or

$F(t), F'(t), F''(t)$ and $F'''(t)$ be (ii)-differentiable; then $f_\alpha(t)$ and $g_\alpha(t)$ have first order, second order, third order and fourth order derivatives and $[F^{(4)}(t)]^\alpha = [f_\alpha^{(4)}(t), g_\alpha^{(4)}(t)]$

(b)Let $F'(t), F''(t)$ and $F'''(t)$ be(i)-differentiable and $F(t)$ be(ii)-differentiable, or

$F(t), F''(t)$ and $F'''(t)$ be (i)-differentiable and $F'(t)$ be (ii)-differentiable, or

$F(t), F'(t)$ and $F'''(t)$ be (i)-differentiable and $F''(t)$ be (ii)-differentiable, or

$F(t), F'(t)$ and $F''(t)$ be(i)-differentiable and $F'''(t)$ be (ii)-differentiable, or

$F'''(t)$ be (i)-differentiable and $F(t), F'(t)$ and $F''(t)$ be (ii)-differentiable, or

$F''(t)$ be (i)-differentiable and $F(t), F'(t)$ and $F'''(t)$ be (ii)-differentiable, or

$F'(t)$ be (i)-differentiable and $F(t), F''(t)$ and $F'''(t)$ be (ii)-differentiable, or

$F(t)$ be (i)-differentiable and $F'(t), F''(t)$ and $F'''(t)$ be (ii)-differentiable; then

$f_\alpha(t)$ and $g_\alpha(t)$ have first order, second order, third order and fourth order derivatives and

$[F^{(4)}(t)]^\alpha = [g_\alpha^{(4)}(t), f_\alpha^{(4)}(t)]$

Proof: The proofs as in proof of theorem 3.1.

Theorem 4.2 Suppose that $g(t), g'(t), g''(t)$ and $g'''(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $g^{(4)}(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $g(t) = (\underline{g}(t,\alpha), \overline{g}(t,\alpha))$, then

(1) If g, g', g'' and g''' be (i)- differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) \ominus s^2 g'(0) \ominus sg''(0) \ominus g'''(0).$$

(2) If g', g'' and g''' be (i)-differentiable and g is (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) \ominus s^2 g'(0) \ominus sg''(0) \ominus g'''(0).$$

(3) If g, g'' and g''' be (i)-differentiable and g' is (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) - s^2 g'(0) \ominus sg''(0) \ominus g'''(0).$$

(4) If g, g' and g''' be (i)-differentiable and g'' is (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) - s^2 g'(0) - sg''(0) \ominus g'''(0).$$

(5) If g, g' and g'' be (i)-differentiable and g''' is (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) - s^2 g'(0) - sg''(0) - g'''(0).$$

(6) If g'' and g''' be (i)-differentiable and g and g' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) - s^2 g'(0) \ominus sg''(0) \ominus g'''(0).$$

(7) If g' and g''' be (i)-differentiable and g and g'' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) - s^2 g'(0) - sg''(0) \ominus g'''(0).$$

(8) If g' and g'' be (i)-differentiable and g and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) - s^2 g'(0) - sg''(0) - g'''(0).$$

(9) If g and g''' be (i)-differentiable and g' and g'' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) \ominus s^2 g'(0) - sg''(0) \ominus g'''(0).$$

(10)If g and g'' be (i)-differentiable and g' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) \ominus s^2 g'(0) - sg''(0) - g'''(0).$$

(11) If g and g' be (i)-differentiable and g'' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) \ominus s^2 g'(0) \ominus sg''(0) - g'''(0).$$

(12) If g''' be (i)-differentiable and g, g' and g'' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) \ominus s^2 g'(0) - sg''(0) \ominus g'''(0).$$

(13) If g'' be (i)-differentiable and g, g' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) \ominus s^2 g'(0) - sg''(0) - g'''(0).$$

(14) If g' be (i)-differentiable and g, g'' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) \ominus s^2 g'(0) \ominus sg''(0) - g'''(0).$$

(15) If g be (i)-differentiable and g', g'' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = -s^3 g(0) \ominus (-s^4) L(g(t)) - s^2 g'(0) \ominus sg''(0) - g'''(0).$$

(16) If g, g', g'' and g''' be (ii)-differentiable, then:

$$L(g^{(4)}(t)) = s^4 L(g(t)) \ominus s^3 g(0) - s^2 g'(0) \ominus sg''(0) - g'''(0).$$

Proof: The proofs as in proof of theorem 3.2.

5.An Illustrative Example

Consider the following fourth order FIVP:

$$y^{(4)}(t) = y(t),$$
$$y(0) = y'(0) = y''(0) = y'''(0) = (r-1, 1-r).$$

We shall consider the same cases given in theorem 4.2 respectively, as follows:

Case 1 By using relation (1) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)\,e^{t},$$

$$\overline{y}(t,r)=(1-r)\,e^{t}.$$

This is the same result which was found by Tapaswini and Chakraverty (2013).

Case 2 By using relation (2) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\overline{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 3 By using relation (3) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\overline{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 4 By using relation (4) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\overline{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 5 By using relation (5) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}),$$

$$\overline{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}).$$

Case 6 By using relation (6) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cosh t-\sin t),$$

$$\overline{y}(t,r)=(1-r)(\cosh t-\sin t).$$

Case 7 By using relation (7) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos t-\sin t),$$

$$\overline{y}(t,r)=(1-r)(\cos t-\sin t).$$

Case 8 By using relation (8) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos t-\sinh t),$$

$$\overline{y}(t,r)=(1-r)(\cos t-\sinh t).$$

Case 9 By using relation (9) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\sinh t+\cos t),$$

$$\overline{y}(t,r)=(1-r)(\sinh t+\cos t).$$

Case 10 By using relation (10) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos t+\sin t),$$

$$\overline{y}(t,r)=(1-r)(\cos t+\sin t).$$

Case 11 By using relation (11) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cosh t+\sin t),$$

$$\overline{y}(t,r)=(1-r)(\cosh t+\sin t).$$

Case 12 By using relation (12) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}-\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}),$$

$$\bar{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}-\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}).$$

Case 13 By using relation (13) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\bar{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}+\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 14 By using relation (14) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\bar{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sqrt{2}\cos\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 15 By using relation (15) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}),$$

$$\bar{y}(t,r)=(1-r)(\cos\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}+\sqrt{2}\sin\frac{t}{\sqrt{2}}\cosh\frac{t}{\sqrt{2}}-\sin\frac{t}{\sqrt{2}}\sinh\frac{t}{\sqrt{2}}).$$

Case 16 By using relation (16) of theorem 4.2, then we get the r -cut representation of solution as following:

$$\underline{y}(t,r)=(r-1)\,e^{-t},$$

$$\bar{y}(t,r)=(1-r)\,e^{-t}.$$

6. Conclusions

The formula of fuzzy derivative of the third and fourth orders are found. In addition, fuzzy Laplace transforms for the fuzzy derivatives of the same orders are found. We used fuzzy Laplace transforms in solving FIVP of the fourth order, and multiple solutions are provided for this FIVP.

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